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Systematic Design of Optimal Performance Weight and Controller in Mixed-μ Synthesis

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Abstract: In this paper we investigate the problem of synthesizing a controller that maximizes the level of robust performance for a plant subject to both complex and real uncertainties. The technique here developed uses the mixed-μ synthesis approach but, unlike the classical D,G-K iteration, it maximizes the size of the performance weights that in the $\mathcal{H}_\infty$ framework capture desired closed loop performance. This optimization is restricted by the constraint that there exists an internally stabilizing controller that achieves robust performance with respect to the maximized weights. Thus, performance weights and a controller that achieves an optimized level of robust performance are synthesized together in a systematic way. The designer is only required to specify the plant uncertain set and some frequency dependent functions, dubbed optimization directionalities, that reflect, in a qualitative way, the desired performance requirements. It is pointed out that choosing this directionality function is much easier than choosing the performance weights directly so that the design of good performance weights is greatly simplified.

1. INTRODUCTION

Over the past decade, modern robust control theory has revolutionized multi-variable controller design. In particular, μ-synthesis (Doyle [1985], Balas et al. [1998]) has been widely applied to design complex multi-input-multi-output control systems with a guaranteed level of robust stability and performance. In general, this technique first requires the specification of weighting functions to reflect desired performance and robustness requirements. Then the control synthesis is recast into a weighted optimization problem to find a controller that attempts to achieve the level of robust stability and performance required by the specified weights. The definition of appropriate weighting functions is by no means a trivial task and it is often a result of a tedious trial and error procedure (Lanzon and Cantoni [2003]).

Papers by Lanzon [2005a] and Lanzon and Cantoni [2003] extend skewed-μ (see Fan and Tits [1992]) ideas to recast the selection of appropriate weights into an optimization problem.[1] The optimization technique proposed in Lanzon [2005a] and in Lanzon and Cantoni [2003], automatically synthesizes both the weighting functions and the controller resolving also the possible inconsistency between the desired specifications. However, this technique has been developed to handle only complex structured singular value problems.

[1] Related weight optimization work appeared in Lanzon [2005b] in the $\mathcal{H}_\infty$ loop-shaping framework.

The main objective of the work here described is to extend the ideas in Lanzon [2005a] and Lanzon and Cantoni [2003] to mixed μ problems (see Young [2001]). This will generalize the robust performance design associated with the previously developed techniques thus being able to handle also real parametric uncertainties in addition to complex perturbations. Note that the proofs are deleted here for the sake of brevity and will be published elsewhere.

Notation. Let $\overline{\mathbb{R}}_+$ denote the non negative real numbers, $\overline{\mathbb{C}}_+$ denote the closed right half complex plane and $\mathbb{C}^{m\times n}$ denote complex matrices of dimension $m \times n$. The maximum singular value of a matrix $A \in \mathbb{C}^{m\times n}$ is denoted by $\overline{\sigma}(A)$. A^T (resp. A^*) is the transpose (resp. complex conjugate transpose) of $A \in \mathbb{C}^{m\times n}$ and $||A||_F$ denotes the Frobenius norm of the matrix A. The $k\times k$ identity matrix and zero matrix are denoted by I_k, and O_k respectively and $\otimes$ denotes the Kronecker product. A real rational matrix function $\Sigma(s)$ of a complex variable s is such that $\Sigma(s) \in \mathcal{RH}_\infty$ if it is bounded and analytic in the open complex right half plane. The adjoint system of $\Sigma(s)$ is defined by $\Sigma^\sim(s) = \Sigma(-s)^T$. The $||\cdot||_\infty$ norm of a $m \times n$ matrix function $\Sigma(s)$ is defined by $||\Sigma||_\infty := \sup_\omega \overline{\sigma}(\Sigma(j\omega))$.

Finally, diag$[A, B]$ with $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ denotes the $(m + p) \times (n + q)$ block diagonal complex matrix composed of A and B.

2. PROBLEM STATEMENT

Most linear time invariant closed loop systems subject to perturbations can be redrawn into the form depicted in Figure 1(a), where $\Sigma(s)$, commonly referred to as the generalized plant, is partitioned consistently with the interconnection. In Figure 1(a), $\Delta(s)$ represents a stable structured perturbation with r inputs and r outputs. The structure of the perturbation is defined by the set

$$\begin{aligned}\mathbf{\Delta} := \{\Delta = \mathrm{diag}[I_{n_1} \otimes \Delta_1, .., I_{n_g} \otimes \Delta_g, I_{n_{g+1}} \otimes \Delta_{g+1}, .., \\ I_{n_{g+d}} \otimes \Delta_{g+d}] : \Delta_i = \Delta_i^T \in \mathbb{R}^{k_i \times k_i}\ \forall i \in \{1, .., g\} \\ \text{and } \Delta_i \in \mathbb{C}^{k_i \times k_i}\ \forall i \in \{g+1, .., g+d\}\}.\end{aligned} \tag{1}$$

where $\sum_{i=1}^{g+d} n_i k_i = r$. $\Delta(s)$ is assumed to belong to the set $\mathbf{B\Delta} := \{\Delta(s) \in \mathcal{RH}_\infty : \Delta(s_0) \in \mathbf{\Delta}\ \forall s_0 \in \overline{\mathbb{C}}_+, ||\Delta||_\infty \leq 1\}$. $K(s)$ is a controller with q inputs and p outputs belonging to the set of controllers that internally stabilize the generalized plant Σ (denoted by $\mathcal{K}$). The system is subject to n exogenous disturbances and the performance is measured in terms of the n error signals. Note that both the uncertainty blocks and the performance blocks (see below) can be assumed to be square without loss of generality, as if they were not square, one could always make them square by adding dummy input or output channels to the generalized plant. The closed loop requirements for the system performances are included in the design by means of the diagonal frequency dependent performance weights matrix $W \in \mathcal{W} := \{\mathrm{diag}_{i=1}^n[w_i] :\ w_i \in \mathcal{RH}_\infty\}$. The system achieves robust performance in the presence of uncertainty if the following condition, written in terms of the supremum over frequency of the structured singular value, denoted by μ, holds

$$\sup_\omega \mu_{\mathbf{\Delta}_T} [\mathrm{diag}[I_r, W(j\omega)]\mathcal{F}_l(\Sigma, K)(j\omega)] < 1, \tag{2}$$

where $\mathbf{\Delta}_T := \{\Delta_T = \mathrm{diag}(\Delta, \Delta_p) : \Delta_p \in \mathbb{C}^{n\times n}, \Delta \in \mathbf{\Delta}\}$ denotes the augmented uncertainty structure introduced to consider the robust performance problem (see Figure 1(b)). A classical μ-synthesis problem, with given performance and robustness specifications captured via given weights, involves the search for a controller that minimizes the left hand side of (2), i.e., a controller that maximizes the size of the smallest possible perturbation, $\Delta_T \in \mathbf{\Delta}_T$, that causes the loss of robust performance of the system. Following the idea in Lanzon [2005a] and Lanzon and

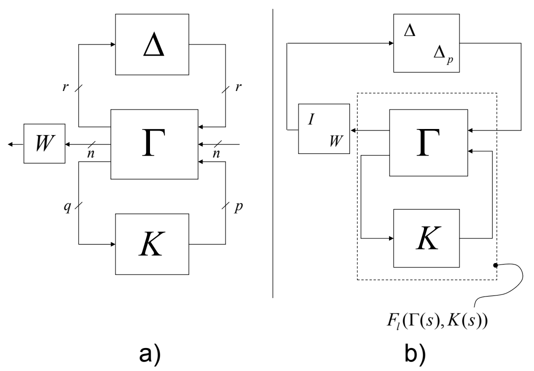

Fig. 1. Generalized block interconnection for synthesis and analysis.

Cantoni [2003], we will take a different approach to what is standard practice in robust control literature. In particular the problem addressed in this paper can be stated in words as:

Given a generalized plant Σ subject to structured mixed real and complex uncertainty, maximize in some sense the size of the performance weights W subject to the requirement that there exists an internally stabilizing controller K that achieves robust performance with respect to the maximized weights.

This statement can be mathematically reformulated as:

$$\begin{aligned}\max_{W\in\mathcal{W}} J(W) \ \text{ subject to} \\ \min_{K\in\mathcal{K}} \sup_\omega \mu_{\mathbf{\Delta}_T} [\mathrm{diag}[I_r, W(j\omega)]\mathcal{F}_l(\Sigma, K)(j\omega)] < 1\end{aligned} \tag{3}$$

for some cost function $J(\cdot)$. In the remainder of the paper the optimization problem in (3) is manipulated to allow for the definition of an efficient solution algorithm.

3. TECHNICAL PRELIMINARIES

In this section we recall some important facts about mixed-μ and we introduce some additional technical results that will form the theoretical background for the synthesis algorithm described in the forthcoming sections. The solution to the problem in (3) requires the evaluation of the mixed structured singular value. However it has been demonstrated in Braatz et al. [1994] that the computation of the exact value for the structured singular value is non-polynomial hard so that for common applications upper and lower bounds of μ need to be adopted. A convenient upper bound on μ when both real and complex uncertainties are present is given in Young [2001] in the form of a convex constraint. Such a constraint involves matrix scalings G and D allowed to vary in sets $\mathcal{D}$ and $\mathcal{G}$ that depend on the structure of the perturbation matrix i.e.:

$$\begin{aligned}\mathcal{D} = \{D = \mathrm{diag}[D_1 \otimes I_{k_1}, .., D_g \otimes I_{k_g}, D_{g+1} \otimes I_{k_{g+1}}, .., \\ D_{g+d} \otimes I_{k_{g+d}}] :\ 0 < D_i = D_i^* \in \mathbb{C}^{n_i \times n_i}\} \\ \mathcal{G} = \{G = \mathrm{diag}[G_1 \otimes I_{k_1}, .., G_g \otimes I_{k_g}, 0, ..., 0] : \\ G_i = G_i^* \in \mathbb{C}^{n_i\times n_i}\}\end{aligned} \tag{4}$$

Then the following lemma from Zhou and Doyle [1999] defines an upper bound on the structured singular value:

Lemma 1. Zhou and Doyle [1999] Let $M \in \mathbb{C}^{r\times r}$ and $\Delta \in \mathbf{\Delta}$. Then

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D\in\mathcal{D}, G\in\mathcal{G}} \min_\beta \{\beta : M^*DM + j(GM - M^*G) - \beta^2 D \leq 0\} \tag{5}$$

The following lemma introduces an alternative characterization of the upper bound on the mixed structured singular value.

Lemma 2. Given a complex matrix $M \in \mathbb{C}^{r\times r}$, $D \in \mathcal{D}$, $G \in \mathcal{G}$, $\beta > 0$ and $\gamma \in [0, 1]$, then

$$\overline{\sigma}\left(\left(\frac{DMD^{-1}}{\beta} - jG\right)\left(I + G^2\right)^{-\frac{1}{2}}\right) \leq \gamma \tag{6}$$

if and only if

$$\Omega(M, \hat{G}, \hat{D}, \beta, \gamma) :=$$
$$\begin{bmatrix} M^*\hat{D}M + j(\hat{G}M - M^*\hat{G}) - (\beta\gamma)^2\hat{D} & \sqrt{1-\gamma^2}\hat{G} \\ \sqrt{1-\gamma^2}\hat{G} & -\hat{D} \end{bmatrix} \leq 0 \tag{7}$$

where $\hat{D} = D^*D \in \mathcal{D}$ and $\hat{G} = \beta D^*GD \in \mathcal{G}$.

Note that the inequality in (7) is quasi-convex in $\hat{D}$, $\hat{G}$ and γ^2 (or β^2). The following corollary, part of which has first appeared in Zhou and Doyle [1999], provides a relation between β and γ and an upper bound on the mixed structured singular value.

Corollary 3. Given $M \in \mathbb{C}^{r\times r}$ and $\Delta \in \mathbf{\Delta}$. Then $\mu_{\mathbf{\Delta}}(M) \leq \inf_{D\in\mathcal{D}, G\in\mathcal{G}} \min_{\beta>0} \{\beta : \Omega(M, G, D, \beta, 1) \leq 0\}$. Furthermore, if $\exists D \in \mathcal{D}$, $G \in \mathcal{G}$, $\beta > 0$ and $\gamma \in [0, 1]$ such that $\Omega(M, G, D, \beta, \gamma) \leq 0$, then $\mu_{\mathbf{\Delta}}(M) \leq \gamma\beta$.

The following lemma will be useful in the remainder of this paper.

Lemma 4. Given $M \in \mathbb{C}^{r\times r}$, $D \in \mathcal{D}$ and $G \in \mathcal{G}$, let

$$\beta_* = \min_{\beta>0} \{\beta : \ \Omega(M, G, D, \beta, 1) \leq 0\}. \tag{8}$$

Given also $\epsilon \geq 0$, let

$$\gamma_* = \min_{\gamma\in\overline{\mathbb{R}}_+} \{\gamma : \ \Omega(M, G, D, \beta_*(1 + \epsilon), \gamma) \leq 0\}.$$

Then $\gamma_* \in [\frac{1}{1+\epsilon}, 1]$.

Note that if ϵ is chosen as $\epsilon = 0$, then γ_* must equal unity. Consequently if β is optimal at β_*, no additional minimization is gained by minimizing γ as it cannot reduce below $\gamma_* = 1$, of course unless β is increased above β_* in which case γ_* will be less than unity.

4. PERFORMANCE WEIGHT AND CONTROLLER OPTIMIZATION

In this section, we manipulate the optimization problem in (3) to make it computationally feasible. In particular, first the structure of the objective function $J(W)$ is defined in a way such that it will be possible for the designer to introduce preferences about the closed loop behavior of the system in the optimization. Then we manipulate the constraint in (3) to define the search space in a convenient way.

4.1 Objective Function Definition

The objective function in (3) must be able to capture the performance preferences of the design that in common practice are reflected as gain requirements on the closed loop transfer functions. These gain requirements are usually handled by penalizing each output of the closed loop system with a weight, $w_i(j\omega)$, whose magnitude reflects the inverse of the desired specification. The objective function in (3) shall then represent a cumulative measure across frequency that reflects qualitatively the desired inverse performance weights shape.

Following the work in Lanzon [2005a], let $[\omega_L, \omega_H]$ be a synthesis frequency range and $\upsilon_i(j\omega)$ be n given stable minimum phase transfer functions. Let us define

$$J(W) = \frac{1}{\int_{\log\omega_L}^{\log\omega_H} \sum_{i=1}^{n} \frac{1}{|w_i(j\omega)/\upsilon_i(j\omega)|^2} d(\log\omega)}. \tag{9}$$

The direction of steepest ascent in maximizing the function in (9) over any one weight $w_i(j\omega)$ at any one frequency ω in the frequency interval $[\omega_L, \omega_H]$ corresponds to the smallest ratio $|w_i(j\omega)/\upsilon_i(j\omega)|$. Consequently, the functions $\upsilon_i(j\omega)$ are called optimization directionalities because they can be specified so that they qualitatively direct the maximization where desired. Therefore $|\upsilon_i(j\omega)|$ should be set at a large value (resp. small) at frequencies and in channel directions where the magnitude of the performance weight $w_i(j\omega)$ is required to be large (resp. small) in order to capture the desired performance objectives. Defining an optimization directionality matrix as $\Upsilon(j\omega) := \mathrm{diag}(\upsilon_1(j\omega), .., \upsilon_n(j\omega))$, then (similar to Lanzon [2005a]) the cost function in (3) can be defined as:

$$J(W) = \frac{1}{||\Upsilon W^{-1}||^2_{[\omega_L,\omega_H]}},$$

where

$$||X||_{[\omega_L,\omega_H]} := \sqrt{\int_{\log_{10}\omega_L}^{\log_{10}\omega_H} ||X(j\omega)||_F^2 d(\log\omega)}.$$

Note that only the argument of the optimization is of interest. Therefore the maximization of the cost can be replaced by the minimization of the reciprocal of $J(W)$ as will be seen in the next subsection.

4.2 Search Space Definition

In every optimization problem a crucial issue is the definition of the search space. First of all note that, since $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(M^T)$, the optimization problem in (3) can be equivalently rewritten in terms of the dual system

$$\begin{gathered} \min_{W\in\mathcal{W}} ||\Upsilon W^{-1}||^2_{[\omega_L,\omega_H]} \quad \text{subject to} \\ \min_{K\in\mathcal{K}} \sup_{\omega} \mu_{\mathbf{\Delta}_T} \left[\mathcal{F}_l(\Sigma, K)(j\omega)^T \mathrm{diag}[I_r, W(j\omega)]\right] < 1 \end{gathered} \tag{10}$$

so that the inverses of the performance weights will appear in subsequent manipulations independently to form a convex constraint.

Now, in order to define an efficient solution algorithm, the robust performance constraint written in terms of $\mu_{\mathbf{\Delta}_T}$ will be replaced with a convex upper bound as the one defined in (5) and in (7). Note that by adding the fictitious uncertainty block $\Delta_p \in \mathbb{C}^{n\times n}$ to handle robust performance problems, the scaling matrices associated to the augmented uncertainty structure $\mathbf{\Delta}_T$ are $\mathrm{diag}[D, I_n]$, $D \in \mathcal{D}$ and $\mathrm{diag}[G, 0_n]$, $G \in \mathcal{G}$ where the last entry in the $D-$scales has been normalized to unity. The following lemma provides an equivalent reformulation of the upper bound on $\mu_{\mathbf{\Delta}_T}\left[\mathcal{F}_l(\Sigma, K)(j\omega)^T \mathrm{diag}[I_r, W(j\omega)]\right]$.

Lemma 5. Given a closed loop system $\mathcal{F}_l(\Sigma, K) \in \mathcal{RH}_\infty$ and performance weights $W \in \mathcal{W}$. Then, $\forall\ \omega\ \exists\ D_\omega \in \mathcal{D}, G_\omega \in \mathcal{G}, \gamma_\omega \in [0, 1]$ and $\beta_\omega > 0$ such that

$$\begin{aligned} &\Omega(\mathcal{F}_l(\Sigma, K)(j\omega)^T \mathrm{diag}[I_r, W(j\omega)], \mathrm{diag}[G_\omega, 0_n], \\ &\mathrm{diag}[D_\omega, I_n], \beta_\omega, \gamma_\omega) \leq 0 \end{aligned} \tag{11}$$

if and only if $\forall\,\omega\ \exists\ D_\omega \in \mathcal{D}, G_\omega \in \mathcal{G}, \gamma_\omega \in [0, 1]$ and $\beta_\omega > 0$ such that

$$\begin{aligned} &\Omega(\mathcal{F}_l(\Sigma, K)(j\omega)^T, \mathrm{diag}[G_\omega, 0_n], \mathrm{diag}[D_\omega, I_n], \beta_\omega, \gamma_\omega) \leq \\ &\mathrm{diag}[0_r, (\beta_\omega\gamma_\omega)^2(W_\omega - I_n), 0_{r+n}] \end{aligned}$$

where $W_\omega = [W(j\omega)^*W(j\omega)]^{-1}$.

With the result from lemma 5, and using the aforementioned upper bound of μ, the optimization problem in (3) is replaced by the following one:

$$\min_{W\in\mathcal{W}} ||\Upsilon W^{-1}||^2_{[\omega_L,\omega_H]} \text{ such that}$$
$$\forall\ \omega\ \exists D_\omega \in \mathcal{D}, G_\omega \in \mathcal{G} \text{ and } \beta_\omega \in (0,1) \text{ satisfying}$$
$$\Omega(\mathcal{F}_l(\Sigma, K)(j\omega)^T, \text{diag}[G_\omega, 0_n], \text{diag}[D_\omega, I_n], \beta_\omega, 1) \leq \text{diag}[0_r, \beta_\omega^2(W_\omega - I_n), 0_{r+n}]. \tag{12}$$

Note that, when K is held fixed, the search space in (12) is characterized by a set of LMI constraints, uncoupled at each ω, and simultaneously quasi-convex in D_ω, G_ω, W_ω and β_ω. Hence, with K fixed in the inequality constraint, the minimization of the integral appearing in the cost function in (12) is equivalent to the minimization of the integrand on the continuum of frequencies. Therefore, under these assumptions, the cost function in (12) can be replaced at each ω by $||\Upsilon(j\omega)W(j\omega)^{-1}||_F^2 = \text{trace}(\Upsilon_\omega \text{W}_\omega)$, where we define the diagonal positive matrix $\Upsilon_\omega := \frac{\Upsilon(j\omega)^*\Upsilon(j\omega)}{\omega}$ noting that the division by ω is necessary to take account of the logarithmic scale appearing in the cost function.

5. SOLUTION ALGORITHM

In this section, we describe an algorithm which we will introduce to search for optimized values of K_*, D_{ω_*}, G_{ω_*}, W_{ω_*} that solve optimization problem (12). In particular we exploit the fact that with K fixed the search space and the cost function are convex in the decision variables, to propose an iterative solution algorithm.

Inputs to the algorithm:

- Generalized plant Σ partitioned consistently with Figure 1 i.e.
$$\Sigma(s) = \left[\begin{array}{c|cc|c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right]$$
with (A, B_3) stabilizable, (A, C_3) detectable and $D_{33} = 0$ (without loss of generality),
- Optimization directionality matrix $\Upsilon(j\omega)$.

Algorithm:

- *Step 0:* First design a robustly stabilizing controller K_0 for the truncated generalized system which corresponds to deleted performance channels and deleted exogenous disturbances in $\Sigma(s)$, i.e.:
$$\hat{\Sigma}(s) = \left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right]$$
Choose $\beta_0 \in [\sup_\omega \mu_{\boldsymbol{\Delta}}(\mathcal{F}_l(\hat{\Sigma}, K_0)(j\omega)), 1)$ which represents an achieved level of robust stability. Fix the minimum number of iterations desired for convergence as N. Define
$$\epsilon := \left(\frac{1}{\beta_0}\right)^{\frac{1}{N}} - 1. \tag{13}$$
- *Step 1:* Set $i = i + 1$ and solve the following convex optimization problem *at each frequency* ω on a frequency grid:
$$\min_{W_\omega} \text{trace}(\Upsilon_\omega W_\omega) \text{ such that}$$
$$\exists D_\omega \in \mathcal{D}, G_\omega \in \mathcal{G} \text{ satisfying}$$
$$\Omega(\mathcal{F}_l(\Sigma, K_{i-1})(j\omega)^T, \text{diag}[G_\omega, 0_n], \text{diag}[D_\omega, I_n], \beta_{i-1}, 1) \leq \text{diag}[0_r, \beta_{i-1}^2(W_\omega - I_n), 0_{r+n}].$$
Set the resulting minimizing argument to W_{ω_*}.
- *Step 2:* If $i < N$, set $\beta_i = (1 + \epsilon)\beta_{i-1}$ else set $\beta_i = 1$. Then compute optimal pointwise D_ω and G_ω scaling matrices by solving the following quasi-convex optimization problem *at each frequency* ω:
$$\min_{D_\omega\in\mathcal{D}, G_\omega\in\mathcal{G}} \gamma_\omega \text{ subject to}$$
$$\Omega(\mathcal{F}_l(\Sigma, K_{i-1})(j\omega)^T, \text{diag}[G_\omega, 0_n], \text{diag}[D_\omega, I_n], \beta_i, \gamma_\omega) \leq \text{diag}[0_r, (\beta_i\gamma_\omega)^2(W_{\omega_*} - I_n), 0_{r+n}]. \tag{14}$$
By virtue of lemma 4 the solution to the above problem can be efficiently computed by means of a bisection search on γ_ω in the interval $[\frac{1}{1+\epsilon}, 1]$. Let the pointwise minimizing arguments and the pointwise solution to (14) be D_{ω_*}, G_{ω_*} and γ_{ω_*} respectively. An upper bound on $\mu_{\boldsymbol{\Delta}_T}(\mathcal{F}_l(\Sigma, K_{i-1})^T \text{diag}[I_r, W])$ associated with $D_{\omega_*}, G_{\omega_*}$ and W_{ω_*} is then $\beta_i\gamma_{\omega_*}$.
- *Step 3:* Find stable minimum phase transfer function matrices $W(s)$ and $D(s)$ so that $W(j\omega)^\sim W(j\omega) \approx W_{\omega_*}^{-1}$ and $D(j\omega)^\sim D(j\omega) \approx D_{\omega_*}$. Furthermore find a transfer function matrix $G(s)$ such that $G(j\omega) \approx j\frac{1}{\beta_i}D(j\omega)^{-*}G_{\omega_*}D(j\omega)^{-1}$. Then, following Zhou and Doyle [1999], find right coprime factors of $G(s)$, i.e. find G_N and G_M satisfying
$$G(s) = G_N G_M^{-1}\ ,\ G_N, G_M \in \mathcal{RH}_\infty$$
and
$$G_M^\sim(s)G_M(s) + G_N^\sim(s)G_N(s) = I_r.$$
Finally build the augmented generalized plant:
$$\Sigma_{DGW} = \text{diag}[D(s), W(s), I_p]\frac{\Sigma(s)}{\beta_i}\text{diag}[D^{-1}(s)G_M(s), I_{n+q}] + -\text{diag}[G_N(s), 0_{n+q}].$$
If the fitting is accurate enough to ensure that $||\mathcal{F}_l(\Sigma_{DGW}, K_{i-1})||_\infty \leq 1$, the existence of a controller that achieves robust performance with respect to the optimal performance weights in the next Step 4 is guaranteed. Note that "space" for fitting was created in Step 2 when β_{i-1} was increased by $(1+\epsilon)$ and when γ_ω was minimized to be less than unity in (14).
- *Step 4:* Find the controller K_i that minimizes $||\mathcal{F}_l(\Sigma_{DGW}, K_i)||_\infty$ via $\mathcal{H}_\infty$ synthesis. The new closed loop system is such that $||\mathcal{F}_l(\Sigma_{DGW}, K_i)||_\infty \leq \gamma$. If $\gamma <$ (1−tolerance), then, according to corollary 3, the peak of the structured singular value over frequency is strictly reduced so that, in the next iteration, a higher level of performance (through performance weight maximization) can be achieved. Then return to Step 1. If $\gamma \nleq (1 - \text{tolerance})$, then exit.

Outputs from the algorithm:

- Pointwise in frequency magnitudes of the inverses of the performance weights in W_ω,
- Pointwise in frequency magnitudes of D and G scalings in D_ω and G_ω,

- Internally stabilizing controller that achieves robust performance with respect to the synthesized weights in K_{i-1}.

In the procedure described above a scaled μ problem is solved in Step 1 of each iteration (i.e. for $i < N$, we choose $\beta = \beta_{i-1} < 1$). We consequently ask ourselves at Step 1 what is the highest robust performance level given a fixed controller K_{i-1} and a fixed upper bound on μ given by β_{i-1}. By virtue of lemma 4 with $\epsilon = 0$, at the end of Step 1 it is not possible to further decrease γ_ω. In order to account for the approximations that will be introduced by the fitting, we relax the μ upper bound from β_{i-1} to $\beta_i = (1+\epsilon)\beta_{i-1}$. Then, following Young [2001], in Step 2 we solve the optimization problem in (14) to evaluate D_ω and G_ω scaling matrices which minimize a (scaled) singular value with $\beta = \beta_i$ fixed across frequency. These optimal pointwise D_ω and G_ω then are continuous functions of ω that focus γ_ω to the frequency region where K_i has to put most work. Note that the parameter ϵ is also an indicator of the level of accuracy required for the fitting to be done in Step 3, i.e. the larger is the parameter ϵ the smaller can be the order of the transfer function used to fit the scaling matrices. However, by the relation in (13), a high value of ϵ implies a low number of desired iterations for convergence. Then, the scalings and the weights are absorbed in the generalized plant and an optimal controller is designed via $\mathcal{H}_\infty$ synthesis techniques. Note that, at each iteration, a new controller is synthesized that achieves a level of performance that is certainly no worse than the previous controller.

6. NUMERICAL EXAMPLE

In this section, we present a simple numerical example to illustrate the algorithm proposed in section 5. The example comprises two different test cases that highlight how the design algorithm is capable to resolve the possible inconsistencies between the closed loop specifications in different situations of interest. Let us consider the uncertain plant set

$$P = \frac{\kappa(s-z)}{s^2} \tag{15}$$

where we set $\kappa = 5(1 + 0.5\delta)$, $\delta \in [-1, 1]$. The location of the right half plane zero z is set alternatively to $z = 10$ rad/sec and to $z = 0.8$ rad/sec in the two different test cases. We consider as design set up the typical S/T

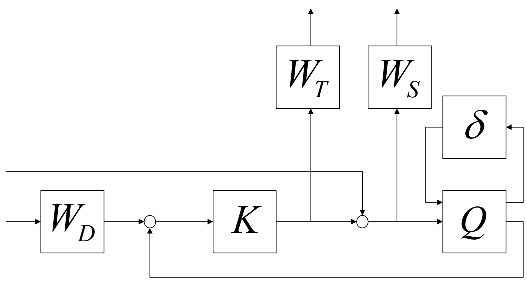

Fig. 2. Design set up.

mixed shaping scheme shown in Figure 2 (where Q is the generalized plant obtained after the real parametric uncertainty is extracted from the plant in (15)) with an additional input (associated to input weight W_D) to fulfill the assumptions needed for the $\mathcal{H}_\infty$ design. W_S and W_T denote the weighting functions that shape the sensitivity and complementary sensitivity functions respectively. These weighting functions will be automatically designed by the algorithm together with an internally stabilizing controller. W_D is chosen as static and small so that it does not affect the $\mathcal{H}_\infty$ norm of the whole system. The optimization directionalities chosen in this example are shown in Figure 3. We require the optimization algorithm

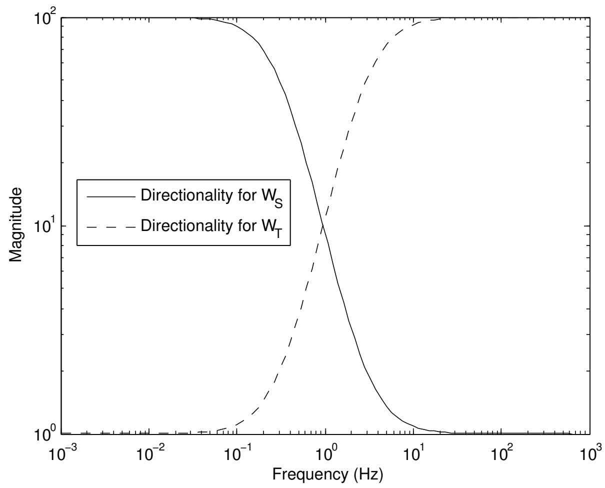

Fig. 3. Desired directionality Υ for the optimization.

to maximize with the same preference the magnitude of W_S at low frequencies and the magnitude of W_T at high frequencies. Moreover both the directionality functions present a reduction of one order of magnitude with respect to their maximum values at the frequency of 1 rad/sec that can therefore be considered as the desired bandwidth of the closed loop system. We set $N = 4$ so that convergence of the algorithm is attained at least after 4 iterations in both the two test cases.

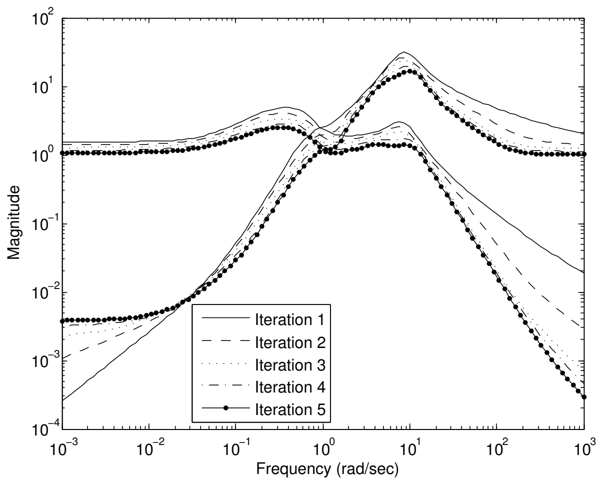

Fig. 4. Inverse performance weights $|W_S(j\omega)|^{-1}$ and $|W_T(j\omega)|^{-1}$ for $z = 10$ rad/sec at each iteration.

In Figure 4, we show $|W_S(j\omega)^{-1}|$ and $|W_T(j\omega)^{-1}|$ computed after every iteration for the test case with $z = 10$ rad/sec. In this example the algorithm converged after

5 iterations. Figure 4 shows how the algorithm reduces the magnitude of the inverse performance weights in the appropriate frequency regions as iterations proceed. In

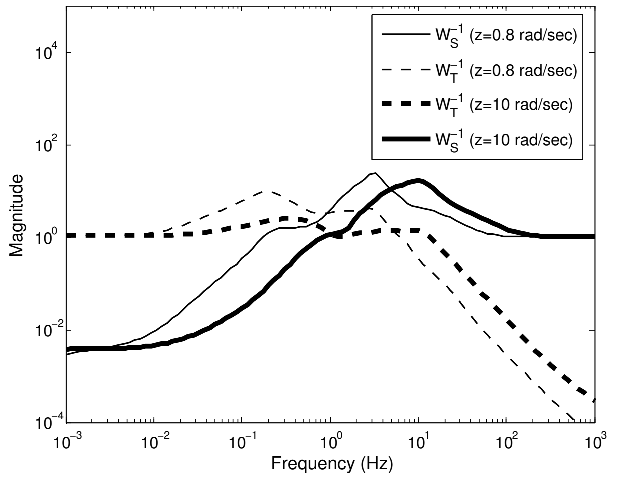

Fig. 5. Optimal inverse performance weights $|W_S(j\omega)|^{-1}$ and $|W_T(j\omega)|^{-1}$ for $z = 0.8$ rad/sec (light lines) and $z = 10$ rad/sec (bold lines).

Figure 5 the performance weights in output to the synthesis algorithm for both the two test cases are shown. In both cases at the end of the iterations the mixed μ computed for the closed loop system assumes a constant value of 1 across frequency, i.e. the sensitivity and complementary sensitivity of any plant in the uncertain plant set in (15) are below $|W_S(j\omega)|^{-1}$ and $|W_T(j\omega)|^{-1}$ respectively. With $z = 10$ rad/sec the open loop zero imposes no limitation upon the sensitivity properties of the system and, as expected, the desired bandwidth of 1 rad/sec is achieved (bold lines in Figure 5). On the other hand with $z = 0.8$ rad/sec the right half plane zero of the plant lies within the required closed loop bandwidth that, therefore, may not be achievable. Hence, a trade off must be performed between desired bandwidth and limitation due to the plant dynamic. This is automatically accounted for by the algorithm being the optimal closed loop bandwidth approximately 0.2 rad/sec (light lines in Figure 5). As a final remark note that the complex μ at the end of the iterations assumes the values of 1.43 and 1.67 for the two test cases. This shows that the algorithm takes advantage of the additional information about the real nature of the uncertainty to improve the robust performance of the system. In Figure 6 tracking and disturbance time responses to a step, for various $\delta \in [-1, 1]$, are presented. The system with $z = 0.8$ rad/sec presents a slower response and a degradation in terms of worst case overshoot with respect to the system with $z = 10$ rad/sec. However, this confirms the results displayed in Figure 5 that show how, for the system with $z = 10$ rad/sec, a higher closed loop bandwidth and lower peaks over frequency of $|W_S^{-1}|$ and $|W_T^{-1}|$ can be achieved.

7. CONCLUSION

In this paper we present a control synthesis technique that automatically performs an optimized trade off between achievable performance and limitations due to uncertainty

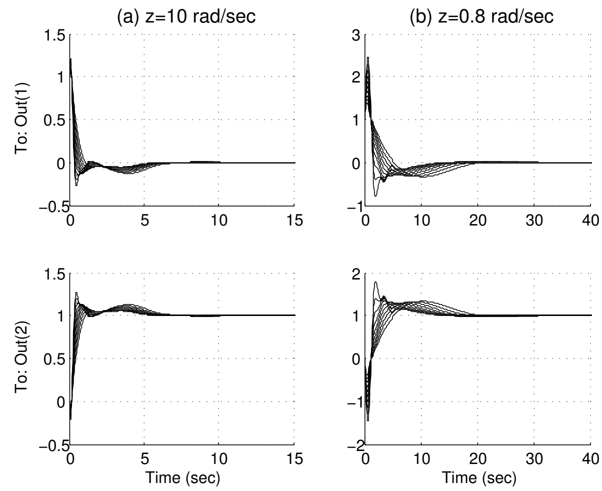

Fig. 6. Step responses for the closed loop uncertain systems. (First column system with $z = 10$ rad/sec, second column system with $z = 0.8$ rad/sec)

or plant dynamics for plants subject to mixed real and complex uncertainty. The proposed technique involves the optimization of the so-called performance weights that reflect the closed loop performance requirements of the system. This optimization is constrained by the fact that there must exist a controller that achieves robust performance with respect to the maximized weights in the presence of mixed complex and real uncertainties. The performance of the design algorithm has been tested through a numerical example that showed how the proposed synthesis technique simplifies the direct design of appropriate performance weights and provides an indication of the achievable performance for a given uncertain plant set.

REFERENCES

G.J. Balas, J.C. Doyle, K. Glover, A. Packard, and R. Smith. *$\mu-$Analysis and Synthesis Toolbox User's Guide.* The Mathworks Inc., 1998.

R. Braatz, P.M. Young, J.C. Doyle, and M. Morari. Computational complexity of μ calculation. *IEEE Transactions on Automatic Control*, 39(5):1000–1002, 1994.

J.C. Doyle. Structured uncertainty in control system design. In *IEEE Conference on Decision and Control*, Ft. Lauderdale, Florida, December 1985.

M.K.H. Fan and A.L. Tits. A measure of worst-case $\mathcal{H}_\infty$ performance and of largest acceptable uncertainty. *Systems and Control Letters*, 18(6):409–421, 1992.

A. Lanzon. Pointwise in frequency weight optimization in $\mu-$synthesis. *International Journal of Robust and Nonlinear Control*, 15:171–199, 2005a.

A. Lanzon. Weight optimization in $\mathcal{H}_\infty$ problems. *Automatica*, 41:1201–1208, 2005b.

A. Lanzon and M. Cantoni. On the formulation and solution of robust performance problems. *Automatica*, 39:1707–1720, 2003.

P.M. Young. Structured singular value approach for systems with parametric uncertainty. *International Journal of Robust and Nonlinear Control*, 11:653–680, 2001.

K Zhou and J.C. Doyle. *Essentials of Robust Control.* Prentice Hall, 1999.