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Robust multi-model predictive control using LMIs ⋆

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Abstract: This paper proposes a novel robust predictive control synthesis technique for constrained nonlinear systems based on linear matrix inequalities (LMIs) formalism. Local discrete-time polytopic models have been exploited for prediction of the system behavior. This design strategy can be applied to a wide class of nonlinear systems provided a suitable embedding is available. The devised procedure guarantees constraint satisfaction and asymptotic stability. The proposed result extends previous works by allowing different local descriptions of nonlinearity and uncertainty and by handling less conservative input constraints. The multi-model prediction shows significant improvements in terms of closed-loop performance and estimation of the feasibility domain.

1. INTRODUCTION

Model Predictive Control (MPC) is nowadays a well understood optimisation-based design technique for discrete-time systems allowing constraints handling from the design stage (Goodwin et al., 2004). Performance of the closed-loop system are directly related to the pertinence of the mathematical model used for predicting the system behavior (Mayne et al., 2000). The complexity of the underlying optimization problem is increased according to the model used for the prediction (linear, linear with uncertainties, nonlinear), the main practical limitations being related to the computational effort and the size of the feasibility region (Allgöwer and Zheng, 2000). The present contribution handles this trade-off between accurate prediction and complexity of optimization problems by proposing a new formulation which embeds the nonlinear model into linear models with uncertainty and casts the resulting optimal control problem into a convex optimization framework (known to provide polynomial-time solutions, (Ben-Tal and Nemirovski, 2001)). An elegant robust MPC formulation was proposed in (Kothare et al., 1996) by using linear matrix inequalities formulation for synthesizing the feedback control laws. Hereafter, this methodology is improved by considering local polytopic descriptions in order to decrease the conservatism in the approximation of nonlinearity and parametric uncertainty, while augmenting the feasible regions of the state space. Indeed, it is well known that efficient control solutions can be deviced exploiting piecewise linear model or other type of approximations (Özkan and Kothare, 2002; Lee et al., 2005; Lu and Arkun, 2002; Sontag, 1981; Rugh and Shamma, 2000; Chisci et al., 2003; Shamma and Xiong, 1999; Muñoz de la Peña et al., 2006). Moreover, the proposed formulation allows the device of an improved scheme handling input constraints. The novel input conditions are formulated in terms of bilinear matrix inequalities (BMI) and then, in order to be efficiently solved, a suitable iterative procedure based on LMIs problem is proposed. This procedure guarantees an improved solution in terms of performance and feasibility domain. The improved policy might turn to be unsuitable for on-line implementation within high sampling rate systems. For this reason, an off-line explicit solution is currently under investigation in order to improve on-line evaluation capabilities. The paper is organized as follows: in section 2 the problem formulation is introduced; section 3 presents the main conceptual results with respect to the optimal control problem formulation while in section 4 are detailed the constraints handling extensions of the optimisation problem. Section 5 details the practical adaptation for obtaining suboptimal solutions while section 6 presents an illustrative example.

2. PROBLEM FORMULATION

Consider a discrete-time nonlinear system

$$x(t+1) \;=\; f(x(t), u(t)) \tag{1}$$

where $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$ are respectively the state and control input at sample time t. Without loss of generality we assume $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Moreover $x(t)$ is available for feedback and the system is subject to the constraints

$$u(t) \in U(x), \quad x(t) \in X \tag{2}$$

where X is a compact set containing the origin in the interior and $U : X \to \mathcal{U}$ is a compact set valued map where $\mathcal{U} \triangleq \{V \subseteq \mathbb{R}^m \;:\; V \text{ is compact}, 0 \in \operatorname{int}(V)\}$. The function $f : \mathcal{D} \to \mathbb{R}^n$ is assumed to be defined in $\mathcal{D} \triangleq \bigcup_{x \in X} \{x\} \times U(x)$. The control objective is to regulate the system to the origin while satisfying the constraints (2). In order to solve the problem with computationally efficient tools, a convenient approach is to embed the nonlinear dynamics into an LPV representation following

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the lines discussed in (Rugh and Shamma, 2000). The LPV model will be subsequently adopted for designing MPC algorithms. Adopting an LPV paradigm and applying a suitable embedding technique, whose description is beyond the scope of this paper, the dynamics (1) can often (Boyd et al., 1994; Rugh and Shamma, 2000) be recast as

$$\begin{aligned} &x(t+1) = A\left(p(t)\right)x(t) + B\left(p(t)\right)u(t) \\ &p(t+1) \in \mathcal{G}(p(t)) \end{aligned} \tag{3}$$

where $p(t)$ is a measurable, time-varying parameter vector which depends on the state variables. The parameter $p(t)$ is confined to be in a compact set P and evolves according to a prescribed set-valued map, denoted as $\mathcal{G} : P \to P$, in order to take into account all possible dynamics of the original systems in the considered state space region. It is worth pointing out that there exist systems which can be stabilized by a parameter dependent feedback but not robustly stabilized if $p(t)$ takes value in a sufficiently large interval (Blanchini and Miani, 2003). Then, it is convenient to exploit the knowledge of parameter value during operation to reduce conservatism. However, in order to develop constructive algorithms it is necessary to consider suitable finite parameterizations of the LPV model (Shamma and Xiong, 1999; Chisci et al., 2003). In order to build a finite parametrization, a region $S \subseteq X$ of the state space is selected and further partitioned into compact sub-regions X_i, $i = 1, 2, \ldots, \ell$, such that

$$S \subseteq \bigcup_{i=1}^{\ell} X_i. \tag{4}$$

Noticed that it is not assumed that $X_i \cap X_j = \emptyset$ for $i \neq j$. This further degree of freedom can be exploited to improve numerical robustness of the algorithm. In each sub-region X_i, we embed the nonlinear dynamics into a polytopic model.

$$x(t+1) \;=\; \mathcal{A}^{(i)}(t)\; x(t) + \mathcal{B}^{(i)}(t)\; u(t), \tag{5}$$

where

$$\left[\mathcal{A}^{(i)}(t),\, \mathcal{B}^{(i)}(t)\right] = \sum_{h=1}^{p_i} \lambda_h^{(i)}(t)[A_h^{(i)},\, B_h^{(i)}], \tag{6}$$

$\lambda_h^{(i)}(t) \geq 0$, $\sum_{h=1}^{p_i} \lambda_h^{(i)}(t) = 1$, for $i \in \mathcal{I} \triangleq \{1, 2, \ldots, \ell\}$. In order to complete the embedding procedure it is customary to define appropriate transition equations for the dynamics of the index i on the basis of the set valued map $\mathcal{G}$. In this paper we are not going to define explicitly dynamics for $i(t)$ but we essentially allow for arbitrary variations of $i \in \mathcal{I}$. The following assumption is however fulfilled.

Assumption 1. - For any value $i \in \mathcal{I}$ of the discrete parameter, the sets

$$\Omega_i \triangleq Co\left\{[A_1^{(i)},\, B_1^{(i)}],\; [A_2^{(i)},\, B_2^{(i)}], \cdots, [A_{p_i}^{(i)},\, B_{p_i}^{(i)}]\right\} \tag{7}$$

are such that $f(x,u) \subseteq \Omega_i\,[x'\; u']'\,, \forall (x,u) \in \mathcal{D} \bigcap [X_i \times \mathbb{R}^m]$.

In words, this means that each set of linear models indexed by i provides a valid one-step ahead prediction of the nonlinear dynamics for any $x \in X_i$ and $u \in U(x)$.

For the moment, we consider the set of time varying models (5) subject to the subsequent constraints on the state variables and the control variables

$$(x(t), u(t)) \in \bigcup_{i \in \mathcal{I}} X_i \times U_i, \tag{8}$$

where X_i and U_i can be ellipsoidal and/or polyhedral sets. Notice that constraints on the input may differ in every region X_i due to the dependence of U on the variables x as in (2). In particular, then, it is possible to compute different norm bounds in every region. The choice of number ℓ of regions X_i is driven by a trade-off between conservatism and model complexity and making sure that:

$$U_i \subseteq U(x),\; \forall x \in X_i \tag{9}$$

3. MULTI-MODEL MIN-MAX ROBUST ALGORITHM

In this section the robust model predictive control scheme for the set of models (5) is introduced by extending ideas developed in (Kothare et al., 1996). The optimal control problem will be reformulated as an LMI synthesis procedure. The next section will extend the technique for handling input constraints.

At each time instant the measured state $x(t)$ is collected and a robust performance objective is minimized with respect to linearly parameterized state feedback

$$\begin{aligned} \min_{F^{(i)},\, i \in \mathcal{I}} \; & \max_{\left[\mathcal{A}^{(i)}(t+k),\, \mathcal{B}^{(i)}(t+k)\right] \in \Omega_i,\, k \geq 0} J_\infty(t) \\ & \text{subject to} \\ & x^{(i)}(t+k+1|t) = \mathcal{A}^{(i)}(t+k)x^{(i)}(t+k|t) \\ & \qquad\qquad + \mathcal{B}^{(i)}(t+k)u^{(i)}(t+k|t) \\ & u^{(i)}(t+k|t) = F^{(i)}x^{(i)}(t+k|t) \\ & x^{(i)}(t|t) = x(t) \\ & x^{(i)}(t+k+1|t) \in S \end{aligned} \tag{10}$$

with

$$\begin{aligned} J_\infty(t) \triangleq \sum_{k=0}^{\infty} & [x^{(i)}(t+k|t)' Q_1 x^{(i)}(t+k|t) \\ & + u^{(i)}(t+k|t)' R u^{(i)}(t+k|t)] \end{aligned} \tag{11}$$

where $Q_1 \succ 0$, $R \succ 0$ (positive definite matrices) are suitable weighting matrices and $x^{(i)}(t+k|t)$, $u^{(i)}(t+k|t)$ denote respectively the predicted state and input at time $t+k$, by means of the $i-$th model based on the measured x at time t. The considered quadratic performance objective (10) results in an infinite horizon MPC using worst-case closed-loop predictions. In order to devise an efficient algorithm we consider, along the same lines as in (Kothare et al., 1996), an upper bound on the performance objective by assuming, first of all, the following linear parameterization of the feedback control laws

$$u^{(i)}(t+k|t) = F^{(i)}x^{(i)}(t+k|t), \quad k \geq 0,\; i = 1, \ldots, \ell. \tag{12}$$

Notice that the control laws differ for every polytopic description. Moreover, an upper bound of the cost functional is achieved in the form of some quadratic function of the state $V(x) = x'Px$, with $P = P'$ and $P \succ 0$. At sampling time t we assume that $V(x)$ satisfies the following ℓ inequalities for all $x^{(i)}(t+k|t)$ and $u^{(i)}(t+k|t)$ satisfying (5), (6) and (12)

$$\begin{array}{r} V(x^{(i)}(t+k+1|t)) - V(x^{(i)}(t+k|t)) \leq \\ -[x^{(i)}(t+k|t)'Q_1x^{(i)}(t+k|t)+ \\ u^{(i)}(t+k|t)'Ru^{(i)}(t+k|t)], \\ \forall \left[\mathcal{A}^{(i)}(t+k),\, \mathcal{B}^{(i)}(t+k)\right] \in \Omega_i,\ \forall i \in \mathcal{I},\ k \geq 0 \end{array} \quad (13)$$

for some P to be determined possibly as a function of $F^{(i)}$. For any $i \in \mathcal{I}$ the robust performance objective (11) is finite if $\lim_{k\to\infty} x^{(i)}(t+k|t) = 0$ and hence $\lim_{k\to\infty} V(x^{(i)}(t+k|t)) = 0$. Thus, by summing (13) from $k = 0, \ldots, \infty$ the following upper bound for the performance index is obtained readily

$$\max_{\left[\mathcal{A}^{(i)}(t+k),\, \mathcal{B}^{(i)}(t+k)\right]\in\Omega_i,\, \forall i,\, k\geq 0} J_\infty \leq V(x(t|t)) \quad (14)$$

Then the robust MPC strategy can be recast as follows: synthesize a set of constant feedback laws $u^{(i)}(t+k|t) = F^{(i)}x^{(i)}(t+k|t)$ $i = 1, \ldots, \ell$, that minimize the function $V(x(t|t))$ while satisfying constraints in (10). As it is customary in MPC, at each time instant apply the first input action $u(t|t) = F^{(i^*)}x(t|t)$ with $i^* \in \mathcal{I}$ such that $x(t|t) \in X_{i^*}$. Notice that i^* does not need to be unique.

Conditions for the existence of P together with the set of feedback laws $F^{(i)}$ which guarantee fulfillment of inequalities (13) together with the following performance requirement

$$V(x(t|t)) = x(t|t)'Px(t|t) < \gamma \text{ for some } \gamma \geq 0 \quad (15)$$

are supplied by the following theorem

Theorem 1. Let $x(t|t)$ the measured state $x(t)$ at the time instant t. Assume that the predictions are carried out with the set of polytopic models (5) and the uncertainty in (6), without taking into account input and state constraints. Then the set of feedback control laws that, at the time instant t, minimize the upper bound γ while satisfying the inequalities (13) is given by

$$F^{(i)} = Y^{(i)}Q^{-1} \quad i = 1, \ldots, \ell \quad (16)$$

where $Q = \gamma P^{-1} \succ 0$ and $Y^{(i)}$ for $i = 1, \ldots, \ell$ are obtained from the following optimization problem whenever feasible:

$$\begin{array}{c} \gamma^* = \min_{Q,\, Y^{(i)}\, i=1,\cdots,\ell} \gamma \\ \begin{bmatrix} 1 & x(t|t)' \\ x(t|t) & Q \end{bmatrix} \succeq 0 \end{array} \quad (17)$$

$$\begin{array}{c} \begin{bmatrix} Q & * & * & * \\ A_h^{(i)}Q + B_h^{(i)}Y^{(i)} & Q & * & * \\ Q_1^{1/2}Q & 0 & \gamma I & * \\ R^{1/2}Y^{(i)} & 0 & 0 & \gamma I \end{bmatrix} \succ 0, \\ h = 1, \cdots, p_i, \qquad i = 1, \cdots, \ell \end{array} \quad (18)$$

where $\succeq 0$ denotes a semidefinite positive condition.

Proof - The proof of the theorem can be derived following the steps of theorem 1 in (Kothare et al., 1996). The only difference comes from the fact that there are ℓ distinct polytopic models (rather than a single one) and, for every such model, a different feedback law is designed that guarantees the satisfaction of the related LMIs with performance not worst than γ. ■

The solution of the optimization problem (10) requires the fulfillment of the state constraints $x^{(i)}(t+k+1|t) \in S$, $i = 1, \ldots, \ell$. Thus in order to consider state and/or output constraints, let us introduce the output variables:

$$y^{(i)}(t) = Cx^{(i)}(t) \quad i = 1, \ldots, \ell \quad (19)$$

where $y^{(i)}(t) \in \mathbb{R}^s$. Consider Euclidean norm and/or componentwise peak bounds on the output variables (19) for $t \geq 0$, $k \geq 1$ and $\forall i \in \mathcal{I}$

$$\begin{array}{l} \|y^{(i)}(t+k+1|t)\|_2 \leq y_{max} \\ |y_j^{(i)}(t+k+1|t)| \leq y_{j,max} \quad j = 1, \ldots, s \end{array} \quad (20)$$

where $y_j^{(i)}$ is the $j-th$ component of $y^{(i)}$. The constraints (20) are satisfied if the following set of LMIs hold (see (Kothare et al., 1996))

$$\begin{array}{c} \begin{bmatrix} Q & (A_h^{(i)}Q + B_h^{(i)}Y^{(i)})'C' \\ C(A_h^{(i)}Q + B_h^{(i)}Y^{(i)}) & y_{max}^2 \end{bmatrix} \succeq 0 \\ \begin{bmatrix} Q & (A_h^{(i)}Q + B_h^{(i)}Y^{(i)})'C_j' \\ C_j(A_h^{(i)}Q + B_h^{(i)}Y^{(i)}) & y_{j,max}^2 \end{bmatrix} \succeq 0 \\ j = 1, \cdots, s, \qquad h = 1, \cdots, p_i, \qquad i = 1, \cdots, \ell \end{array} \quad (21)$$

where C_j is the ith row of the matrix C. Then an LMI synthesis procedure for the solution of the robust performance problem (10) is provided by solving the optimization problem (17-18) together with the additional constraints (21).

Theorem 2. Let consider the system (1) and assume that assumption 1 is satisfied. Providing $x(0) \in S$, at each time instant t the state is measured and the optimization problem (17) together with the constraints (18) and (21) is solved. Thus at each time instant determine $i^*(t)$ such that $x(t) \in X_{i^*(t)}$ and apply

$$u(t) = F^{(i^*(t))}x(t). \quad (22)$$

Then, if the considered optimization problem is feasible at $t = 0$, it is feasible for all times $t > 0$. Moreover, the proposed control algorithm guarantees that: (i) the state constraints are satisfied and (ii) $\lim_{t\to\infty} x(t) = \mathbf{0}$ where $x(t)$ denotes the trajectory of the nonlinear system under the control law (22).

In order to prove the above theorem, the following Lemma is fundamental.

Lemma 1. Consider the following prediction model

$$x(t+k+1|t) = f(x(t+k|t), u(t+k|t)), \ \ x(t|t) = x(t) \quad (23)$$

and let assumption 1 be satisfied. At sampling time t, suppose there exist $Q \succ 0$, $\gamma > 0$ and $Y^{(i)} = F^{(i)}Q$, $i = 1, \ldots, \ell$ such that (18) and (21) hold. Moreover suppose that

$$u(t+k|t) = F^{(i^*(t+k|t))}x(t+k|t), \quad (24)$$

$$\forall i^*(t+k|t) \text{ s.t. } x(t+k|t) \in X_{i^*(t+k|t)}.$$

Define $\mathcal{E} \triangleq \{x \ : \ x'Q^{-1}x \leq 1\} = \{x \ : \ x'Px \leq \gamma\}$, then $\mathcal{E} \cap S$ is an invariant set for the predicted states of the nonlinear system (1)

Sketch of the proof - From theorem 1 we have that satisfaction of (18) implies fulfillment of (13) for all the set

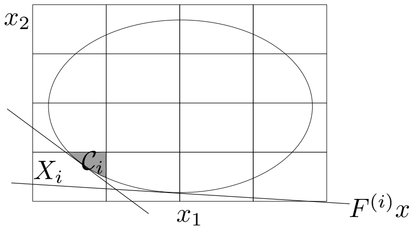

Fig. 1. Saturation of input constraints

of uncertain models (5) under the set of control laws (12) computed according to (16). Thus, since $Q_1 \succ 0$, we have that $V(x) = x'Px$ is a decreasing Lyapunov function for all the polytopic models. Moreover, thanks to constraints (21), the requirement $x^{(i)}(t+k|t) \in S \ \forall k > 0$ is satisfied for any $x(t) \in \mathcal{E}$. Therefore, it is possible to show that $\mathcal{E} \cap S$ is an invariant set for all the polytopic models under the set of control laws (12). Finally, exploiting assumption 1, it can be proved that the invariance of $\mathcal{E} \cap S$ holds also for the nonlinear system. ∎

Sketch of the proof of theorem 2 - First of all it is possible to show that any feasible solution of the optimization problem (17) together with the constraints (18) and (21) at time t is also feasible for all times $k > t$. Then the convergence of the algorithm can be carried out showing that the control algorithm supplies a strictly decreasing Lyapunov function for the set of polytopic models Then, by assumption 1, the contractivity property of control laws holds also for the nonlinear system. ∎.

4. INPUT CONSTRAINTS

Euclidean norm and/or component peak bounds on the input variables can be considered in the robust predictive control synthesis. Their formulation can be carried out in a similar way. In the following only the peak bounds case will be described.

$$|(F^{(i)}x(t+k|t))_j| \leq u^{(i)}_{j,\max} \quad j = 1, \cdots, m, \ \forall i \in \mathcal{I}, \ k \geq 0 \tag{25}$$

In order to recast (25) in terms of LMIs (see (Boyd et al., 1994; Kothare et al., 1996)) the following sufficient conditions are imposed for $j = 1, \cdots, m$

$$|(F^{(i)}z)_j| \leq u^{(i)}_{j,\max}, \ \forall i \in \mathcal{I}, \ k \geq 0, \ \forall z \ s.t. \ z'Q^{-1}z \leq 1 \tag{26}$$

Thus the final input constraints are given by the following LMIs

$$\begin{bmatrix} V^{(i)} & Y^{(i)} \\ Y^{(i)'} & Q \end{bmatrix} \succeq 0, \ V^{(i)}_{jj} \leq (u^{(i)}_{j,\max})^2, \ \begin{matrix} j = 1, \cdots, m \\ i = 1, \cdots, \ell \end{matrix} \tag{27}$$

where $V^{(i)} = V^{(i)'}$ are additional decisional variables. Notice that it is also possible to take into account different peak bounds for each region X_i, thus reducing the conservatism. The input constraints (27) can be incorporated in the optimization problem (17) together with the constraints (18) and (21). The devised MPC algorithm guarantees constraints satisfaction and convergence to zero of the controlled nonlinear system.

However conservatism is still substantial since condition (27) require satisfaction for any state in $\mathcal{E}$. Indeed it would be preferable to enforce constraints satisfaction for each $F^{(i)}$, $i = 1, \cdots, \ell$ considering the specific region X_i where $F^{(i)}$ is actually applied. The main idea is illustrated in figure 1. The control law $F^{(i)}x$ can reach its saturation level in a region that is far from X_i. Clearly it would be possible to improve performance and enlarge the stability domain imposing constraints on $F^{(i)}x$ only for the states belonging to the intersection, indicated as $\mathcal{C}_i$ in figure 1, between $\mathcal{E}$ and the region X_i. Formally, let us define $\mathcal{C}_i \triangleq \mathcal{E} \cap X_i$ The goal is to impose the following less conservative input limitations for $j = 1, \ldots, m$:

$$u^{(i)}_{j,\min} \leq (F^{(i)}z)_j \leq u^{(i)}_{j,\max}, \ \forall z \in \mathcal{C}_i, \ \forall i \in \mathcal{I} \tag{28}$$

A powerful tool, that allows to tackle constraints (28), is the so called $\mathcal{S}$-procedure (Boyd et al., 1994). Hereafter it is assume that the sets X_i are polytopes. For space reason only the case of upper bounds in (28) is detailed, the construction for the lower bounds will follow analogously. Let the following quadratic functions for $i = 1, \ldots, \ell$ and $j = 1, \ldots, m$: $H^{(i)}_0(z) \triangleq (u^{(i)}_{\max})^2 - z'(F^{(i)})'(F^{(i)})z$, $H^{(i)}_{j,U}(z) \triangleq 2u^{(i)}_{j,\max} - 2(F^{(i)}z)_j$, $E(z) \triangleq 1 - z'Q^{-1}z$, and $H^{(i)}_l(z) \triangleq 2h^{(i)'}_l z + 2c^{(i)}_l$, where $H^{(i)}_l(z)$ for $l = 1, \cdots, n^{(i)}_c$ are the linear equations describing the set X_i. If there exist $F^{(i)} = Y^{(i)}Q^{-1}$ and scalars $\lambda^{(i)}_{j,U} \geq 0$, $\tau^{(i)}_{j,l} \geq 0$ for $j = 1, \cdots, m$, $l = 1, \cdots, n^{(i)}_c$ and $i = 1, \cdots, \ell$ such that

$$H^{(i)}_{j,U}(z) - \lambda^{(i)}_{j,U}E(z) - \sum_{l=1}^{n^{(i)}_c} \tau^{(i)}_{j,l} H^{(i)}_l(z) \geq 0, \tag{29}$$

$\forall z, \ i \in \mathcal{I}, \ j = 1, \cdots, m$ then the constraints on the upper bound of the input in (28) hold. Conditions (29) can be written as

$$\begin{aligned} &\begin{bmatrix} \mathbf{0} & -F^{(i)'}_j \\ -F^{(i)}_j & 2u^{(i)}_{j,\max} \end{bmatrix} - \lambda^{(i)}_{j,U} \begin{bmatrix} -Q^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} - \\ &\qquad - \sum_{l=1}^{n^{(i)}_c} \tau^{(i)}_{j,l} \begin{bmatrix} \mathbf{0} & h^{(i)}_l \\ h^{(i)'}_l & 2c^{(i)}_l \end{bmatrix} \succeq 0 \end{aligned} \tag{30}$$

for $j = 1, \ldots, m$, and for all $i \in \mathcal{I}$. Summing up all the terms in (30) and multiplying the obtained matrices on the left and the right side by $[Q, \mathbf{0}; \mathbf{0}, 1]$ we get the following BMIs for $j = 1, \ldots, m$, and for all $i \in \mathcal{I}$

$$\begin{bmatrix} \lambda^{(i)}_{j,U}Q & -Y^{(i)'}_j - \sum_{l=1}^{n^{(i)}_c} \tau^{(i)}_{j,l} Q h^{(i)}_l \\ * & 2u^{(i)}_{j,\max} - \lambda^{(i)}_{j,U} - \sum_{l=1}^{n^{(i)}_c} \tau^{(i)}_{j,l} 2c^{(i)}_l \end{bmatrix} \succeq 0. \tag{31}$$

The fundamental difference with LMIs is that BMI problems are non-convex, and algorithms for their effective solution are not known to exist. Recently several optimization problems with BMI constraints have been formulated and solved with good results exploiting the path-following algorithm (Hassibi et al., 1999) and/or direct iterative algorithms (VanAntwerp and Braatz, 2000). Experience with low order systems has shown that these BMIs can be solved rather effectively. Numerical difficulties, instead, can arise for higher order ones. Having in mind these kind of problems in the next section we propose an ad hoc iteration method to solve the optimization problem (17) together with the constraints (18) and (21) and considering the devised less conservative input constraints analogous to (31). The proposed procedure guarantees a result at least as good as the result obtained exploiting the most conservative input LMI conditions (25). In most examples, significant improvements on system performances and enlargement of basin of attraction have been achieved, demonstrating the advantages of constraints like in (31).

5. SUBOPTIMAL SOLUTION

This section describes an iterative procedure that alternately solves the following two problems based on LMI techniques.

Multi-model PC problem (MmPCp) - Given $x \in S$, choose input bound values $\overline{u}^{(i)}_{j,\max} \geq \min\{|u^{(i)}_{j,\min}|, u^{(i)}_{j,\max}\}$, $j = 1, \ldots, m$ and $i = 1, \ldots, \ell$, the *"Multi-model PC problem"* consists in the solution of the optimization problem (17) together with the constraints (18), (21) and (27) considering $\overline{u}^{(i)}_{j,\max}$ $j = 1, \ldots, m$ and $i = 1, \ldots, \ell$ as input limitations.

Input feasibility problem (IFP) - Given Q and $Y^{(i)}$, $i = 1, \ldots, \ell$, computed solving the *Multi-model PC problem*, the *"Input feasibility problem"* consists in checking feasibility of constraints like in (31) considering the real input bounds $u^{(i)}_{j,\min}$, and $u^{(i)}_{j,\max}$, $i = 1, \ldots, \ell$ and $j = 1, \ldots, m$.

The starting point is the observation that the *Multi-model PC problem* with the physical input constraints can give rise to conservative conditions on the synthesized feedback laws. Roughly speaking, we may solve the *Multi-model PC problem* with increased input bounds and a posteriori verify the effective constraint satisfaction for the obtained set of feedback laws by solving the *Input feasibility problem* with the effective real bounds.
Thus, a suitable procedure, referred as *MmPC-IF* problem, can be formalized applying a bisection strategy on the input bounds $\overline{u}^{(i)}_{j,\max}$. The main idea of the method is summarized in figure 2 for a scalar input. The algorithm is initialized assigning to $\overline{u}^{(i)}_{\max}$ the effective input bounds in the *MmPCp* and a possible maximum bound $\overline{u}^{(i)}$ is imposed. Subsequently $\overline{u}^{(i)}_{\max}$ is updated according to the result given by the *Multi-model PC problem* and the *Input feasibility problem* following a bisection strategy between the minimum and the maximum obtained bounds. Whenever the solution of *Multi-model PC problem* is infeasible the minimum possible bounds for the input is updated, while if the solution of *Input feasibility problem* is infeasible the maximum possible upper bound is updated. Notice that solution of the *Input feasibility problem* requires a feasible solution of the *Multi-model PC problem.* The proposed *MmPC-IF* algorithm provides a nondecreasing sequence of feasible upper bounds corresponding to an admissible decreasing sequence of the optimal values γ^*.

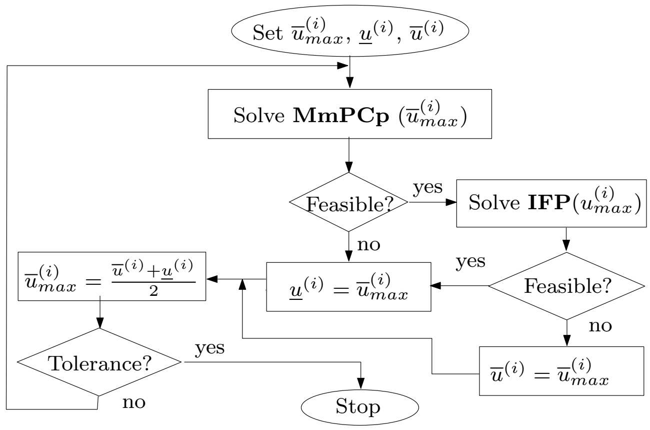

Fig. 2. - Principle idea of the *MmPC-IF* algorithm for scalar input described by means of a block scheme

6. SIMULATION EXAMPLE

The effectiveness of the proposed control procedure is now illustrated by means of simulation experiments. We consider the application of our approach to the strongly nonlinear model of a *continuous stirred tank reactor* (CSTR) (Magni et al., 2005). The state variables are C_A, the reagent concentration in the reactor, T the reactor temperature and manipulated variable is T_c the temperature of the coolant stream. It is possible to obtain a suitable LPV description as illustrated in (Chisci et al., 2003). The nominal operating conditions correspond to an unstable equilibrium point $C_A^{eq} = 0.5$mol/l, $T^{eq} = 350°$K, $T_c^{eq} = 300°$K. The system is subject to following input and state constraints: $240°\text{K} \leq T_c \leq 350°$K and 313°K $\leq T \leq 387°$K. The system model has been discretized by the Euler's technique with a sampling time $T_s = 0.03\ min$. The weight matrices $Q_1 = I_2$ and $R = 0.00002$ have been selected. For each possible region, a polytopic embedding with a minimum number of vertices $p_i = 4$ has been found, adding some suitable additional uncertainty in the polytopic description. The behaviour of the proposed procedure is illustrated starting from the initial condition $T = 317.0°$K, $C_A \cong 0.881$mol/l. Figures 3 and 4 show the state response of the proposed algorithms. It can be noticed that the relaxed condition on the input constraints allows to achieve a faster regulation to the desired set-point giving arise to a significant improvement in terms of performance. Moreover several simulation experiments have shown a considerable enlargement of the estimated domain of attraction. The considered initial condition is infeasible for the algorithm introduced in (Kothare et al., 1996), while the feasibility of the *MmPC-IF* is obtained subdividing the interval $313°\text{K} \leq T \leq 387°$K into 12 subintervals of equal size. The optimizations required by the algorithms were performed in Matlab using SeDuMi toolbox together with Yalmip (Löfberg, 2004). Finally fig. 5 shows several invariant regions obtained considering several initial conditions.

7. CONCLUSIONS

In this paper a novel MPC algorithm for nonlinear systems has been proposed. It is based on the embedding of the nonlinear dynamics into several local polytopic models.

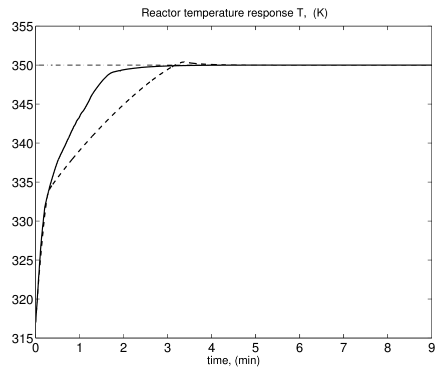

Fig. 3. Desired reference (dotted), temperature responses with input bounds (27) (dashed) and improved input constraints (31) (solid) for $\ell = 12$

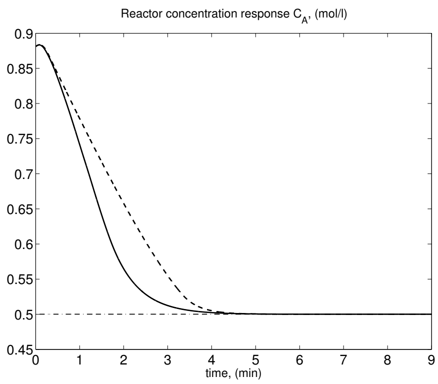

Fig. 4. C_A with input bounds (27) (dashed) and improved input constraints (31) (solid) for $\ell = 12$.

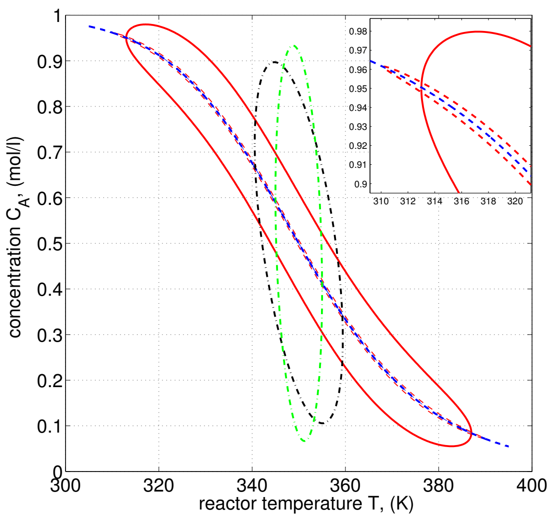

Fig. 5. Invariant regions in the coordinate space for different initial conditions and state partitions.

Convergence and stability properties are guaranteed by the invariance of sub-level sets under a control law composed by different state linear control laws defined for the different local descriptions. The computation of such control laws is performed by minimizing a worst case performance objective subject to input and state constraints. The advantages of the approach lie in a remarkable improvement in performance and size of guaranteed basin of attraction, even when nonlinear techniques are applicable with difficulty. It is worth to stress the fact that on-line optimization complexity can be critical from a practical point of view. For this reason, the future research prospective is focused on the development of an explicit formulation of the proposed algorithm, looking at a good compromise between complexity and optimality.

REFERENCES

F. Allgöwer and A.Z. Zheng, *Nonlinear Model Predictive Control: Assessment and Future Directions for Research*, Progress in Systems and Control Series, Birkhäuser Verlag, Basel, 2000.

A. Ben-Tal and A. Nemirovski, *Lectures on modern convex optimization*, MPS-SIAM Series on optimization, 2001.

F. Blanchini and S. Miani, Stabilization of LPV systems: state feedback, state estimation and duality, *SIAM J. Control Optim.*, volume 42(1), pages 76-97, 2003.

S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, USA, 1994.

L. Chisci, P. Falugi and G. Zappa, Gain-scheduling MPC control of nonlinear systems, *International Journal of Robust and Nonlinear Control*, volume 13, pages 295–308, 2003.

G. C. Goodwin, M.M. Seron and J.A. De Dona, *Constrained Control and Estimation*, Springer-Verlag, London, 2004.

A. Hassibi, J. How and S. Boyd, A path-following method for solving BMI problems in control, In *Proceedings of American control conference*, pages 1385-1389, 1999.

M.V. Kothare, V. Balakrishnan and M. Morari, Robust constrained model predictive control using linear matrix inequalities, *Automatica*, volume 32(10), pages 1361–1379, 1996.

Y.I. Lee, M. Cannon and B. Kouvaritakis, Extended invariance and its use in model predictive control, *Automatica*, volume 41, pages 2163–2169, 2005.

J. Löfberg, YALMIP : A Toolbox for Modeling and Optimization in MATLAB, In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. (URL: http://control.ee.ethz.ch/ joloef/yalmip.php)

Y. Lu and Y. Arkun, A scheduling quasimin-max model predictive control algorithm for nonlinear systems, *Journal of Process Control*, volume 12(5), pages 589-604, 2002.

L. Magni, G. De Nicolao, L. Magnani and R. Scattolini, A stabilizing model-based predictive control algorithm for nonlinear systems, *Automatica*, volume 37, pages 1351-1362, 2001.

D.Q. Mayne, J.B. Rawlings, C.V. Rao and P.O.M. Scokaert, Constrained model predictive control: Stability and optimality, *Automatica*, volume 36, pages 789–814, 2000.

D. Muñoz de la Peña, A. Bemporad and C. Filippi, Robust Explicit MPC Based on approximate Multi-parametric Convex Programming, *IEEE Trans. Automatic Control*, volume 51, pages 1399-1403, 2006.

L. Özkan, M.V. Kothare, Stability analysis of a multi-model predictive control algorithm with application to control of chemical reactors, *Journal of Process Control*, volume 16, pages 81-90, 2006.

W.J. Rugh and J.S. Shamma, Research on gain scheduling, *Automatica*, volume 36, pages 1401–1426, 2000.

J.S. Shamma and D. Xiong, Set valued methods for linear parameter varying systems, *Automatica*, volume 35, pages 1081–1089, 1999.

E.D. Sontag, Nonlinear regulation: The piecewise linear approach *IEEE Transactions on Automatic Control*, volume 26(2), pages 346–358, 1981.

J.G. VanAntwerp and R.D. Braatz, A tutorial on linear and bilinear matrix inequalities, *Journal of Process Control*, volume 10, pages 363–385, 2000.