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Stability Analysis for an Euler-Bernoulli Beam under Local Internal Control and Boundary Observation ⋆

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Abstract: In this paper, an Euler-Bernoulli beam system under the local internal distributed control and a boundary point observation is studied. We design an infinite-dimensional observer for the open-loop system. The closed loop-system that is non-dissipative is obtained by estimated state feedback. By a detailed spectral analysis, it is shown that there is a set of generalized eigenfunctions, which forms a Riesz basis for the state space. As consequences, both the spectrum-determined growth condition and exponential stability are concluded.

1. INTRODUCTION

In the output stabilization for control systems described by partial differential equations (PDEs), collocated control design, that is, the actuators and sensors are in the same positions and designed in a "collocated" fashion, is the main method. This is natural in the sense that the proportional output feedback for an collocated system produces a dissipative closed-loop system, by which the methods like Lyapunov function methods and energy multiplier methods could be used to get the stability of the system.

On the other hand, in engineering practice, it has been found long time that the performance of the collocated control design is not always good enough ([3]). Although the non-collocated control has been widely used in engineering systems control ([1, 16, 17]), the theoretical studies from mathematical control point of view for these systems are quite few. The first difficult problem is that the open-loop forms of non-collocated systems are usually not minimum-phase. This leads to the closed-loop systems unstable for large feedback controller gains. Secondly, the closed-loop forms of non-collocate systems are usually non-dissipative, which gives rise to the difficulty in applying the traditional Lyapunov function methods or the energy multiplier methods to the analysis of the stability. Recently, the estimated state feedbacks are designed through backstepping observers in [14] to stabilize a class of one-dimensional parabolic PDEs. The abstract observers design for a class of well-posed regular infinite-dimensional systems can be found in [5] but the stabilization is not addressed.

Recently, some efforts have been made for non-collocated system control. In [11] and [10], the stabilization of wave and beam equation under boundary control with non-collocated observation has been handled respectively.

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The objective of this paper is to study the stabilization of an Euler-Bernoulli beam system under local distributed internal control with point boundary observation. This design is a typical non-collocated system of PDEs. Such a distributed control is feasible in engineering practice due to the application of the smart materials. The pointwise measurement is the common observation for distributed parameter systems.

The system we are concerned with is the following Euler-Bernoulli beam with local internal distributed control and boundary point observation on the domain $Q = \{(x,t) : 0 \le x \le 1, t > 0\}$:

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) + \sigma(x)u(x,t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = w_{xxx}(1,t) = 0, \\ y(t) = w_t(1,t), \end{cases} \tag{1}$$

where $w(x,t)$ represents the transverse displacement of the beam at position $x \in [0,1]$ and time $t \ge 0$, $u(x,t)$ is the locally distributed control (input), $y(t)$ is the output (observation), and

$$\sigma(x) = \begin{cases} 1, \ x \in (a,b) \in (0, \frac{1}{2}), b > a, \\ 0, \ \ \text{otherwise}. \end{cases} \tag{2}$$

We choose the energy state space as $\mathbb{H} = H_E^2(0,1) \times L^2(0,1)$, $H_E^2(0,1) = \{f |\ f \in H^2(0,1),\ f(0) = f'(0) = 0\}$. $\mathbb{H}$ is equipped with the obvious inner product induced norm $\|(f,g)\|_{\mathbb{H}}^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2]dx$ for any $(f,g) \in \mathbb{H}$. The input and output spaces are $U = L^2(0,1)$ and $Y = \mathbb{C}^1$, respectively.

Define the operator $\mathbb{A} : D(\mathbb{A})(\subset \mathbb{H}) \to \mathbb{H}$ as following:

$$\begin{cases} \mathbb{A}(f,g) = (g, -f^{(4)}), \\ D(\mathbb{A}) = \{(f,g) \in \mathbb{H} |\ \mathbb{A}(f,g) \in \mathbb{H}, \\ \qquad\qquad f''(1) = f'''(1) = 0\}. \end{cases} \tag{3}$$

Then system (1) can be written as

$$\sum: \begin{cases} \dfrac{d}{dt}\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} + \mathbb{B}u(x,t), \\ y(t) = \mathbb{C}\begin{pmatrix} w(x,t) \\ w_t(x,t) \end{pmatrix} = w_t(1,t), \end{cases} \tag{4}$$

where

$$\mathbb{B} = \begin{pmatrix} 0 \\ -\sigma(x) \end{pmatrix}, \quad \mathbb{C} = (0, \ \langle \delta(x-1), \ \cdot \rangle),$$

and $\delta(\cdot)$ denotes the Dirac delta distribution. Obviously, $\mathbb{B}$ is a bounded control operator while $\mathbb{C}$ is an unbounded observation operator.

Theorem 1. For each $u \in L^2_{loc}(0,\infty)$ and initial datum $(w(\cdot,0), w_t(\cdot,0)) \in \mathbb{H}$, there exists a unique solution $(w, w_t) \in C(0,\infty;\mathbb{H})$ to equation (1), and for each $T > 0$, there exists a $C_T > 0$ independent of u and $(w(\cdot,0), w_t(\cdot,0))$ such that

$$\|(w(\cdot,T), w_t(\cdot,T))\|^2_{\mathbb{H}} + \int_0^T |y(\tau)|^2 d\tau$$
$$\leq C_T \left[\|(w(\cdot,0), w_t(\cdot,0))\|^2_{\mathbb{H}} + \int_0^T \|u(\cdot,\tau)\|^2_{L^2} d\tau \right].$$

Proof. Since $\mathbb{B}$ is bounded, by the well-posed linear infinite-dimensional system theory ([4, 12]), it is equivalent to showing that $\mathbb{C}$ is admissible for $e^{\mathbb{A}t}$. This is a well-known fact (see [8]).

Remark 2. The significance of Theorem 1 is that it not only gives the well-posedness of the open-loop system (1) but also shows that for any L^2 control, the output y makes sense and is also in L^2. This fact is very important to the design of the observer because for the observer, y becomes input. The L^2 property of y plays a key role in the solvability of the observer. This is attributed to the well-posed infinite-dimensional systems theory ([4]).

The remaining part of this paper are organized as follows. In Section 2, we construct an observer for the system (1) and show that this observer is exponentially convergent. Section 3 is devoted to the estimated state feedback design. In section 4, we analyze the asymptotic behavior of the eigenpairs. The Riesz basis property and exponential stability are developed in Section 5.

2. OBSERVER DESIGN

We design an observer to system (1) as following:

$$\begin{cases} \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) + \sigma(x)u(x,t) = 0, \\ \widehat{w}(0,t) = \widehat{w}_x(0,t) = \widehat{w}_{xx}(1,t) = 0, \\ \widehat{w}_{xxx}(1,t) = \alpha\widehat{w}_t(1,t) - \alpha y(t), \end{cases} \tag{5}$$

where $\alpha \in \mathbb{R}^+$ is a positive constant. The system (5) can be written as ([8])

$$\begin{cases} \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) = -\sigma(x)u(x,t) + \alpha\delta(x-1)y(t), \\ \widehat{w}(0,t) = \widehat{w}_x(0,t) = \widehat{w}_{xx}(1,t) = 0, \\ \widehat{w}_{xxx}(1,t) = \alpha\widehat{w}_t(1,t), \end{cases} \tag{6}$$

or

$$\widehat{w}_{tt} + \mathbb{A}\widehat{w} + \alpha\mathbf{b}\mathbf{b}^*\widehat{w}_t = -\sigma(x)u(x,t) + \alpha\delta(x-1)y(t), \tag{7}$$

where $\mathbb{A}$ is given by (3) and $\mathbf{b} = -\delta(x-1)$.

The following result comes directly from [8].

Theorem 3. The system (7) is well-posed. That is, for any $(\widehat{w}(\cdot,0), \widehat{w}_t(\cdot,0)) \in \mathbb{H}$ and $u, y \in L^2_{loc}(0,\infty)$, there exists a unique solution to (7) such that $(\widehat{w}(\cdot,t), \widehat{w}_t(\cdot,t)) \in C(0,\infty;\mathbb{H})$, and for any $T > 0$, there exists a constant C_T such that

$$\|(\widehat{w}(\cdot,T), \widehat{w}_t(\cdot,T))\|^2_{\mathbb{H}} \leq C_T \left[\|(\widehat{w}(\cdot,0), \widehat{w}_t(\cdot,0))\|^2_{\mathbb{H}} \right. \tag{8}$$
$$\left. + \int_0^T \|u(\cdot,t)\|^2_{L^2} dt + \int_0^T y^2(t) dt \right].$$

Let $e(x,t)$ be the error of solutions of (1) and (5):

$$e(x,t) = \widehat{w}(x,t) - w(x,t). \tag{9}$$

Then $e(x,t)$ satisfies

$$\begin{cases} e_{tt}(x,t) + e_{xxxx}(x,t) = 0, \\ e(0,t) = e_x(0,t) = e_{xx}(1,t) = 0, \\ e_{xxx}(1,t) = \alpha e_t(1,t). \end{cases} \tag{10}$$

It is well-known that the above system is exponentially stable in $\mathbb{H}$ ([2]).

3. ESTIMATED STATE FEEDBACK CONTROL DESIGN

Having obtained the estimated state through observer, we can now naturally design the following output feedback based on estimated state as what we have done for collocated system:

$$u(t) = \gamma\widehat{w}_t(x,t), \ \gamma > 0.$$

The closed-loop system now becomes on Q:

$$\begin{cases} \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) + \gamma\sigma(x)\widehat{w}_t(x,t) = 0, \\ \widehat{w}(0,t) = \widehat{w}_x(0,t) = \widehat{w}_{xx}(1,t) = 0, \\ \widehat{w}_{xxx}(1,t) = \alpha\widehat{w}_t(1,t) - \alpha w_t(1,t), \\ w_{tt}(x,t) + w_{xxxx}(x,t) + \gamma\sigma(x)\widehat{w}_t(x,t) = 0, \\ w(0,t) = w_x(0,t) = w_{xx}(1,t) = w_{xxx}(1,t) = 0. \end{cases} \tag{11}$$

By $e(x,t) = \widehat{w}(x,t) - w(x,t)$ defined in (9), we get the equivalent system of (11):

$$\begin{cases} e_{tt}(x,t) + e_{xxxx}(x,t) = 0, \\ e(0,t) = e_x(0,t) = e_{xx}(1,t) = 0, \\ e_{xxx}(1,t) = \alpha e_t(1,t), \\ \widehat{w}_{tt}(x,t) + \widehat{w}_{xxxx}(x,t) + \gamma\sigma(x)\widehat{w}_t(x,t) = 0, \\ \widehat{w}(0,t) = \widehat{w}_x(0,t) = \widehat{w}_{xx}(1,t) = 0, \\ \widehat{w}_{xxx}(1,t) = e_{xxx}(1,t). \end{cases} \tag{12}$$

We consider the system (12) in the state space $X = \mathbb{H} \times \mathbb{H}$ with the obvious inner product induced norm: $\forall\, (f, g, \phi, \psi) \in X$,

$$\|(f, g, \phi, \psi)\|^2 = \int_0^1 [|f''|^2 + |g|^2 + |\phi''|^2 + |\psi|^2] dx.$$

The system operator $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ for (12) is defined by

$$\begin{cases} \mathcal{A}(f, g, \phi, \psi) = (g, -f^{(4)}, \psi, -\phi^{(4)} - \gamma\sigma\psi), \\ D(\mathcal{A}) = \{(f, g, \phi, \psi) \in X \mid \mathcal{A}(f, g, \phi, \psi) \in X, \\ f''(1) = \phi''(1) = 0, \\ f'''(1) = \phi'''(1), \ f'''(1) = \alpha g(1)\}. \end{cases} \tag{13}$$

With the operator $\mathcal{A}$ at hand, we can write (12) as an evolution equation in X:

$$\begin{aligned}&\frac{d}{dt}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t),\widehat{w}(\cdot,t),\widehat{w}_t(\cdot,t))\\&=\mathcal{A}(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t),\widehat{w}(\cdot,t),\widehat{w}_t(\cdot,t)).\end{aligned}\tag{14}$$

We observe that $\mathcal{A}$ is not dissipative. Actually, let $(f,g,\phi,\psi)\in D(\mathcal{A})$. We have

$$\begin{aligned}&\operatorname{Re}\langle\mathcal{A}(f,g,\phi,\psi),\ (f,g,\phi,\psi)\rangle_X\\&=-\alpha|g(1)|^2-\alpha\operatorname{Re}\left(g(1)\overline{\psi(1)}\right)-\gamma\int_0^1\sigma|\psi|^2dx,\end{aligned}\tag{15}$$

which shows that $\mathcal{A}$ is not dissipative.

Lemma 4. $\mathcal{A}^{-1}$ is compact on X and hence $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues only.

Proof. For any $(p_1,q_1,p_2,q_2)\in X$, solve $\mathcal{A}(f,g,\phi,\psi)=(p_1,q_1,p_2,q_2)$ to obtain

$$\begin{cases}g(x)=p_1(x),\ \psi(x)=p_2(x),\\-f^{(4)}(x)=q_1(x),\ -\phi^{(4)}(x)-\gamma\sigma(x)\psi(x)=q_2(x),\\f(0)=f'(0)=f''(1)=\phi(0)=\phi'(0)=\phi''(1)=0,\\f'''(1)=\alpha g(1),\ \phi'''(1)=f'''(1).\end{cases}$$

This gives

$$\begin{cases}g(x)=p_1(x),\ \psi(x)=p_2(x),\\f(x)=\alpha p_1(1)\left[\frac{1}{6}x^3-\frac{1}{2}x^2\right]\\\qquad+\int_0^x q_1(\xi)\left(\frac{1}{6}\xi^3-\frac{1}{2}\xi^2x\right)d\xi\\\qquad+\frac{1}{2}\int_x^1 q_1(\xi)\left(\frac{1}{3}x^3-x^2\xi\right)d\xi,\\\phi(x)=\alpha p_1(1)\left[\frac{1}{6}x^3-\frac{1}{2}x^2\right]\\\qquad+\int_0^x q_3(\xi)\left(\frac{1}{6}\xi^3-\frac{1}{2}\xi^2x\right)d\xi\\\qquad+\frac{1}{2}\int_x^1 q_3(\xi)\left(\frac{1}{3}x^3-x^2\xi\right)d\xi,\\q_3(x)=\gamma\sigma(x)p_2(x)+q_2(x).\end{cases}$$

Hence $\mathcal{A}^{-1}$ is defined everywhere on X and $\mathcal{A}^{-1}$ maps X into a subset of the space $(H^4(0,1)\times H^2(0,1))^2$, which is compact in X. By the Sobolev embedding theorem ([13]), $\mathcal{A}^{-1}$ is compact on X, proving the required result.

Lemma 5. $\operatorname{Re}(\lambda)<0$ for any $\lambda\in\sigma(\mathcal{A})$.

Proof. This is a direct verification. We omit the details here.

4. SPECTRAL ANALYSIS

Let $\lambda\in\sigma(\mathcal{A})$ and $(f,g,\phi,\psi)\neq 0$ be a corresponding eigenfunction. Then $\mathcal{A}(f,g,\phi,\psi)=\lambda(f,g,\phi,\psi)$ means that

$$g=\lambda f,\ \psi=\lambda\phi,$$

and (f,ϕ) satisfies the following eigenvalue problem:

$$\begin{cases}\lambda^2 f+f^{(4)}=0,\\\lambda^2\phi+\phi^{(4)}+\gamma\lambda\sigma\phi=0,\\f(0)=f'(0)=f''(1)=\phi(0)=0,\\\phi'(0)=\phi''(1)=0,\\f'''(1)=\alpha\lambda f(1),\ \ f'''(1)=\phi'''(1).\end{cases}\tag{16}$$

By Lemma 5 and the fact that the eigenvalues are symmetric about the real axis, we consider only those λ that are located in the second quadrant of the complex plane:

$$\lambda=i\rho^2,\ \rho\in\mathcal{S}=\left\{\rho\in\mathbb{C}\mid 0\le\arg\rho\le\frac{\pi}{4}\right\}.\tag{17}$$

Note that for any $\rho\in\mathcal{S}$, it has

$$\operatorname{Re}(-\rho)\le\operatorname{Re}(i\rho)\le\operatorname{Re}(-i\rho)\le\operatorname{Re}(\rho),\tag{18}$$

and

$$\begin{cases}\operatorname{Re}(-\rho)=-|\rho|\cos(\arg\rho)\le-\frac{\sqrt{2}}{2}|\rho|<0,\\\operatorname{Re}(i\rho)=-|\rho|\sin(\arg\rho)\le 0.\end{cases}\tag{19}$$

Lemma 6. For $\rho\in\mathcal{S}$ and $x\in[0,1]$,

$$e^{i\rho x},\ e^{-\rho x},\ e^{-i\rho x},\ e^{\rho x}\tag{20}$$

are linearly independent fundamental solutions of $f^{(4)}(x)-\rho^4f(x)=0$, and as $|\rho|$ is large enough, $\phi^{(4)}(x)-\rho^4\phi(x)+i\gamma\rho^2\sigma(x)\phi(x)=0$, where $\sigma(x)$ is given by (2), has the following asymptotic fundamental solutions:

$$\begin{cases}\phi_1(x)=e^{i\rho x}\left[1+\widetilde{\sigma}(x)\rho^{-1}\left(1+\rho^{-1}\right)\right],\\\phi_2(x)=e^{-\rho x}\left[1-i\widetilde{\sigma}(x)\rho^{-1}\left(1+\rho^{-1}\right)\right],\\\phi_3(x)=e^{-i\rho x}\left[1-\widetilde{\sigma}(x)\rho^{-1}\left(1+\rho^{-1}\right)\right],\\\phi_4(x)=e^{\rho x}\left[1+i\widetilde{\sigma}(x)\rho^{-1}\left(1+\rho^{-1}\right)\right],\end{cases}\tag{21}$$

where

$$\widetilde{\sigma}(x)=\frac{1}{4}\gamma\int_0^x\sigma(\xi)d\xi=\begin{cases}0, & x\in[0,a),\\\frac{\gamma(x-a)}{4}, & x\in[a,b),\\\frac{\gamma(b-a)}{4}, & x\in[b,1].\end{cases}\tag{22}$$

Proof. This is a direct result of [15].

From Lemma 6, we get that $f(x)$ and $\phi(x)$ have the following asymptotic forms in $\mathcal{S}$ respectively:

$$\begin{cases}f(x)=c_1e^{i\rho x}+c_2e^{-\rho x}+c_3e^{-i\rho x}+c_4e^{\rho x},\\\phi(x)=d_1\phi_1(x)+d_2\phi_2(x)+d_3\phi_3(x)+d_4\phi_4(x),\end{cases}$$

where c_i and d_i $(i=1,2,3,4)$ are constants. Substitute the above into the boundary conditions of (16), to obtain

$$\Delta(\rho)[c_1,c_2,c_3,c_4,d_2,d_2,d_3,d_4]^\top=0,\tag{23}$$

where

$$\Delta(\rho)=\begin{bmatrix}\Delta_{11}(\rho) & \Delta_{12}(\rho) & 0_{4\times2} & 0_{4\times2}\\\Delta_{21}(\rho) & \Delta_{22}(\rho) & \Delta_{23}(\rho) & \Delta_{24}(\rho)\end{bmatrix},\tag{24}$$

$$\Delta_{11}(\rho)=\begin{bmatrix}1 & 1\\i & -1\\-e^{i\rho} & e^{-\rho}\\-ie^{i\rho}(1+\alpha\rho^{-1}) & -e^{-\rho}(1+i\alpha\rho^{-1})\end{bmatrix},$$

$$\Delta_{12}(\rho)=\begin{bmatrix}1 & 1\\-i & 1\\-e^{-i\rho} & e^{\rho}\\ie^{-i\rho}(1-\alpha\rho^{-1}) & e^{\rho}(1-i\alpha\rho^{-1})\end{bmatrix},$$

$$\Delta_{21}(\rho)=\begin{bmatrix}0&0\\0&0\\0&0\\-ie^{i\rho}&-e^{-\rho}\end{bmatrix},\Delta_{22}(\rho)=\begin{bmatrix}0&0\\0&0\\0&0\\ie^{-i\rho}&e^{\rho}\end{bmatrix},$$

$$\Delta_{23}(\rho)=\begin{bmatrix}1&1\\i&-1\\-e^{i\rho}\left[1+\frac{\widetilde{\sigma}(1)}{\rho}\right]_2&e^{-\rho}\left[1-i\frac{\widetilde{\sigma}(1)}{\rho}\right]_2\\ie^{i\rho}\left[1+\frac{\widetilde{\sigma}(1)}{\rho}\right]_2&e^{-\rho}\left[1-i\frac{\widetilde{\sigma}(1)}{\rho}\right]_2\end{bmatrix},$$

$$\Delta_{24}(\rho)=\begin{bmatrix}1&1\\-i&1\\-e^{-i\rho}\left[1-\frac{\widetilde{\sigma}(1)}{\rho}\right]_2&e^{\rho}\left[1+i\frac{\widetilde{\sigma}(1)}{\rho}\right]_2\\-ie^{-i\rho}\left[1-\frac{\widetilde{\sigma}(1)}{\rho}\right]_2&-e^{\rho}\left[1+i\frac{\widetilde{\sigma}(1)}{\rho}\right]_2\end{bmatrix},$$

where $[a]_2 = a + \mathcal{O}(\rho^{-2})$. Hence, (23) has a nontrivial solution if and only if

$$\det(\Delta(\rho)) = 0, \tag{25}$$

which equals

$$\begin{aligned}&\det(\Delta(\rho))\\&= \det[\Delta_{11}(\rho),\ \Delta_{12}(\rho)]\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)] = 0.\end{aligned} \tag{26}$$

Theorem 7. Let $\lambda = i\rho^2$ and let $\Delta(\rho)$ be given by (24). Then $\det\Delta(\rho)$ is the characteristic determinant of the eigenvalue problem (16) and it has the asymptotic expression in $\mathcal{S}$:

$$\begin{aligned}&\det(\Delta(\rho))\\&= \det[\Delta_{11}(\rho),\ \Delta_{12}(\rho)]\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)] = 0,\end{aligned}$$

where

$$\begin{aligned}\det[\Delta_{11}(\rho),\ \Delta_{12}(\rho)] = 2e^{\rho}\big(&e^{-i\rho}(i+(1-i)\alpha\rho^{-1})\\&+e^{i\rho}(i+(1+i)\alpha\rho^{-1})+\mathcal{O}(\rho^{-2})\big),\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)]\\&= -2ie^{\rho}\big(e^{i\rho}\left(1+(1+i)\widetilde{\sigma}(1)\rho^{-1}\right)\\&\quad +e^{-i\rho}\left(1-(1-i)\widetilde{\sigma}(1)\rho^{-1}\right)+\mathcal{O}(\rho^{-2})\big).\end{aligned} \tag{28}$$

Moreover, the eigenvalue of (16) has the following asymptotic forms: as $n\to\infty$,

$$\begin{cases}\lambda_{1n} = -2\alpha + i\left(\dfrac{1}{2}+n\right)^2\pi^2+\mathcal{O}(n^{-1}),\\[2ex] \lambda_{2n} = -2\widetilde{\sigma}(1) + i\left(\dfrac{1}{2}+n\right)^2\pi^2+\mathcal{O}(n^{-1}),\end{cases} \tag{29}$$

where $n's$ are positive integers and $\widetilde{\sigma}(1) = \dfrac{\gamma(b-a)}{4}$ that is given by (22). Therefore, as $n\to\infty$,

$$\begin{cases}\mathrm{Re}\{\lambda_{1n},\overline{\lambda}_{1n}\}\to -2\alpha,\\[1ex] \mathrm{Re}\{\lambda_{2n},\overline{\lambda}_{2n}\}\to -\dfrac{\gamma(b-a)}{2}.\end{cases} \tag{30}$$

Proof. Since the expansion of $\det(\Delta(\rho))$ is a direct computation, we just need to show the asymptotic expression of eigenvalues. Let $\rho\in\mathcal{S}$ and let $\det(\Delta(\rho)) = 0$. Then we have $\det[\Delta_{11}(\rho),\ \Delta_{12}(\rho)] = 0$ or $\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)] = 0$. Thus, from (27), we get

$$e^{-i\rho}(i+(1-i)\alpha\rho^{-1})+e^{i\rho}(i+(1+i)\alpha\rho^{-1})+\mathcal{O}(\rho^{-2}) = 0. \tag{31}$$

This leads to

$$e^{-i\rho}+e^{i\rho}+\mathcal{O}(\rho^{-1}) = 0. \tag{32}$$

Notice that in the first quadrant of the complex plane, the solutions of the equation $e^{i\rho}+e^{-i\rho} = 0$ are given by

$$\tilde{\rho}_{1n} = \left(\frac{1}{2}+n\right)\pi,\ n = 0,1,2,\ldots.$$

Apply the Rouché's theorem to (32) to give the solutions of (32): as $n\to\infty$,

$$\rho_{1n} = \tilde{\rho}_{1n}+\alpha_{1n} = \left(\frac{1}{2}+n\right)\pi+\alpha_{1n},\ \alpha_{1n} = \mathcal{O}(n^{-1}). \tag{33}$$

Substitute ρ_{1n} into (31) and use the fact $e^{i\tilde{\rho}_{1n}} = -e^{-i\tilde{\rho}_{1n}}$, to obtain

$$\alpha_{1n} = -\frac{\alpha}{i(\frac{1}{2}+n)\pi}+\mathcal{O}(n^{-2}).$$

Substitute above into (33) to produce

$$\rho_{1n} = \left(\frac{1}{2}+n\right)\pi-\frac{\alpha}{i(\frac{1}{2}+n)\pi}+\mathcal{O}(n^{-2})\ \text{ as } n\to\infty. \tag{34}$$

Since $\lambda_{1n} = i\rho_{1n}^2$, we get eventually that

$$\lambda_{1n} = -2\alpha+i\left(\frac{1}{2}+n\right)^2\pi^2+\mathcal{O}(n^{-1})\ \text{ as } n\to\infty.$$

Now we investigate the second branch eigenvalues. Let $\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)] = 0$. From (28), it has

$$e^{i\rho}\big(1+(1+i)\frac{\widetilde{\sigma}(1)}{\rho}\big)+e^{-i\rho}\big(1-(1-i)\frac{\widetilde{\sigma}(1)}{\rho}\big)+\mathcal{O}(\rho^{-2}) = 0. \tag{35}$$

Similar analysis gives the asymptotic solutions for above equation: as $n\to\infty$,

$$\rho_{2n} = \tilde{\rho}_{1n}+\alpha_{2n} = \left(\frac{1}{2}+n\right)\pi+\alpha_{2n},\ \alpha_{2n} = \mathcal{O}(n^{-1}).$$

Substitute above into (35) and use the fact $e^{i\tilde{\rho}_{1n}} = -e^{-i\tilde{\rho}_{1n}}$ again, to obtain

$$\alpha_{2n} = -\frac{\widetilde{\sigma}(1)}{i(\frac{1}{2}+n)\pi}+\mathcal{O}(n^{-2}),$$

and

$$\rho_{2n} = \left(\frac{1}{2}+n\right)\pi-\frac{\widetilde{\sigma}(1)}{i(\frac{1}{2}+n)\pi}+\mathcal{O}(n^{-2})\ \text{ as } n\to\infty. \tag{36}$$

Since $\lambda_{2n} = i\rho_{2n}^2$, we get eventually that

$$\lambda_{2n} = -2\widetilde{\sigma}(1)+i\left(\frac{1}{2}+n\right)^2\pi^2+\mathcal{O}(n^{-1})\ \text{ as } n\to\infty.$$

The proof is complete.

Theorem 8. Let $\{\lambda_{1n},\overline{\lambda}_{1n},\lambda_{2n},\overline{\lambda}_{2n},\ n\in\mathbb{N}\}$ be the eigenvalues of $\mathcal{A}$ with λ_{in} being given in (29). Then the corresponding eigenfunctions

$$\{[f''_{in}(x),\lambda_{in}f_{in}(x),\phi''_{in}(x),\lambda_{in}\phi_{in}(x)], i = 1,2,n\in\mathbb{N}\} \tag{37}$$

have the following asymptotic expressions:

$$\begin{cases} f''_{1n}(x) = 2e^{-i\rho_{1n}}e^{-\rho_{1n}(1-x)} + (1-i)e^{i\rho_{1n}x} \\ \qquad +2ie^{-\rho_{1n}x} - (1+i)e^{-i\rho_{1n}x} + \mathcal{O}(n^{-1}), \\ \lambda_{1n}f_{1n}(x) = 2ie^{-i\rho_{1n}}e^{-\rho_{1n}(1-x)} - (1+i)e^{i\rho_{1n}x} \\ \qquad -2e^{-\rho_{1n}x} - (1-i)e^{-i\rho_{1n}x} + \mathcal{O}(n^{-1}), \\ \phi''_{1n}(x) = f''_{1n}(x) + \mathcal{O}(n^{-1}), \\ \lambda_{1n}\phi_{1n}(x) = \lambda_{1n}f_{1n}(x) + \mathcal{O}(n^{-1}), \\ f''_{2n}(x) = \lambda_{2n}f_{2n}(x) = 0, \\ \phi''_{2n}(x) = 2e^{-i\rho_{2n}}e^{-\rho_{2n}(1-x)} + (1-i)e^{i\rho_{2n}x} \\ \qquad +2ie^{-\rho_{2n}x} - (1+i)e^{-i\rho_{2n}x} + \mathcal{O}(n^{-1}), \\ \lambda_{2n}\phi_{2n}(x) = 2ie^{-i\rho_{2n}}e^{-\rho_{2n}(1-x)} - (1+i)e^{i\rho_{2n}x} \\ \qquad -2ie^{-\rho_{2n}x} - (1-i)e^{-i\rho_{2n}x} + \mathcal{O}(n^{-1}), \end{cases} \tag{38}$$

where ρ_{1n} and ρ_{2n} are given by (34) and (36) respectively. Moreover,

$$\Big\{\big[f_{in}, \lambda_{in}f_{in}, \phi_{in}, \lambda_{in}\phi_{in}\big],\ i = 1, 2, n \in \mathbb{N}\Big\}$$

are approximately normalized in X in the sense that there exist positive constants c_1 and c_2, independent of n, such that for $n \in \mathbb{N}$,

$$c_1 \le \|f''_{in}\|_{L^2}, \|\lambda_{in}f_{in}\|_{L^2}, \|\phi''_{in}\|_{L^2}, \|\lambda_{in}\phi_{in}\|_{L^2} \le c_2. \tag{39}$$

Proof. Due to the tremendous computations, we omit the details here.

5. RIESZ BASIS GENERATION AND EXPONENTIAL STABILITY

This section is devoted to the Riesz basis property for system (14). The main result is the following Theorem 9.

Theorem 9. Let $\mathcal{A}$ be define by (13). Then each eigenvalue with large modulus is algebraically simple. Moreover, there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X.

Proof. Define a linear operator $\mathbf{A} : D(\mathbf{A})(\subset \mathbb{H}) \to \mathbb{H}$:

$$\begin{cases} \mathbf{A}(f, g) = (g, -f^{(4)}),\ \forall\,(f, g) \in D(\mathbf{A}), \\ D(\mathbf{A}) = \{(f, g) \in \mathbb{H} |\ \mathbf{A}(f, g) \in \mathbb{H}, \\ \qquad\qquad f''(1) = 0,\ f'''(1) = \alpha g'(1)\}. \end{cases} \tag{40}$$

Let $\lambda = i\rho^2$. It is easily to check that $\det[\Delta_{11}(\rho),\ \Delta_{12}(\rho)]$ is the characteristic determinant of $\mathbf{A}$, which has the asymptotic expression (27). The eigenvalues of $\mathbf{A}$, which is denoted by $\{\lambda_{1n},\ \overline{\lambda}_{1n},\ n \in \mathbb{N}\}$ and corresponding generalized eigenfunctions $\{(f_{1n}, \lambda_{1n}f_{1n}),\ (\overline{f_{1n}}, \overline{\lambda}_{1n}\overline{f_{1n}}),\ n \in \mathbb{N}\}$ have the asymptotic expressions (29) and (38), respectively. In terms of regular theory of the second order partial differential equations (see e.g., [9]), we know that each eigenvalue with sufficiently large module is algebraically simple and there is a set of generalized eigenfunctions of $\mathbf{A}$, which forms a Riesz basis for $\mathbb{H}$.

Now define another operator $\widetilde{\mathbb{A}} : D(\widetilde{\mathbb{A}})(\subset \mathbb{H}) \to \mathbb{H}$ as following:

$$\begin{cases} \widetilde{\mathbb{A}}(f, g) = (g, -f^{(4)} - \gamma\sigma g), \\ D(\widetilde{\mathbb{A}}) = \{(f, g) \in \mathbb{H} |\ \widetilde{\mathbb{A}}(f, g) \in \mathbb{H}, \\ \qquad\qquad f''(1) = f'''(1) = 0\}. \end{cases} \tag{41}$$

Then $\widetilde{\mathbb{A}}$ is a bounded perturbation of $\mathbb{A}$ and we have the following results on it (see [15]): for $\lambda = i\rho^2$,

a) $\det[\Delta_{23}(\rho),\ \Delta_{24}(\rho)]$ is the characteristic determinant of eigenvalues of $\widetilde{\mathbb{A}}$, which has the asymptotic expression (28);

b) the eigenvalues $\{\lambda_{2n},\ \overline{\lambda_{2n}},\ n \in \mathbb{N}\}$ have the asymptotic expansions (29) and each eigenvalue is algebraically simple when its modulus is large enough;

c) the corresponding eigenfunctions

$$\{(\phi_{2n}, \lambda_{2n}\phi_{2n}),\ (\overline{\phi_{2n}}, \overline{\lambda_{2n}\phi_{2n}}),\ n \in \mathbb{N}\}$$

have the asymptotic expressions (38);

d) there is a set of generalized eigenfunctions of $\widetilde{\mathbb{A}}$, which forms a Riesz basis for $\mathbb{H}$.

To sum up, we have obtained that

$$\{((f_{1n}, \lambda_{1n}f_{1n}), 0, 0),\ (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n},\ n \in \mathbb{N}\}$$

and their conjugates

$$\{(\overline{f_{1n}}, \overline{\lambda}_{1n}\overline{f_{1n}}, 0, 0),\ (0, 0, \overline{\phi_{2n}}, \overline{\lambda_{2n}\phi_{2n}},\ n \in \mathbb{N}\}$$

form a Riesz basis for $\mathbb{H} \times \mathbb{H}$, that is X.

Furthermore, due to the fact that

$$\begin{pmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{I}_{2\times 2} \end{pmatrix} (f_{1n}, \lambda_{1n}f_{1n}, 0, 0)^T = (f_{1n}, \lambda_{1n}f_{1n}, f_{1n}, \lambda_{1n}f_{1n})^T$$

and

$$\begin{pmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{I}_{2\times 2} \end{pmatrix} (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n})^T = (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n})^T,$$

we conclude that

$$\{(f_{1n}, \lambda_{1n}f_{1n}, f_{1n}, \lambda_{1n}f_{1n}),\ (0, 0, \phi_{2n}, \lambda_{2n}\phi_{2n}),\ n \in \mathbb{N}\}$$

together with their conjugates

$$\{(\overline{f_{1n}}, \overline{\lambda_{1n}f_{1n}}, \overline{f_{1n}}, \overline{\lambda_{1n}f_{1n}}),\ (0, 0, \overline{\phi_{2n}}, \overline{\lambda_{2n}\phi_{2n}}),\ n \in \mathbb{N}\}$$

form a Riesz basis for X. Now, by virtue of the Bari's theorem (see [6, 7]) and the expressions (38), we know that each eigenvalue with sufficiently large module is algebraically simple and there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X. The proof is complete.

Theorem 10. Let $\mathcal{A}$ be defined by (13). Then

(i). $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on X.

(ii). The spectrum-determined growth condition holds true for $e^{\mathcal{A}t}$, that is to say, $S(\mathcal{A}) = \omega(\mathcal{A})$, where

$$S(\mathcal{A}) := \sup_{\lambda \in \sigma(\mathcal{A})} \mathrm{Re}\lambda$$

is the spectral bound of $\mathcal{A}$, and

$$\omega(\mathcal{A}) := \inf\left\{\omega \mid \exists M > 0 \text{ such that } \left\|e^{\mathcal{A}t}\right\| \le Me^{\omega t}\right\}$$

is the growth order of $e^{\mathcal{A}t}$.

(iii). System (14) is exponentially stable.

Proof. Since there is a set of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for X, (i) and (ii) then follow from the asymptotic expansion (29) for eigenvalues. By (ii), the stability of (14) can be determined by the maximal value of the real parts of eigenvalues of $\mathcal{A}$. Now, by Lemma

5, $\mathrm{Re}(\lambda) < 0$ for any $\lambda \in \sigma(\mathcal{A})$, and from Theorem 7, the imaginary axis is not the asymptote of the eigenvalues. Hence, system (14) is exponentially stable. The proof is complete.

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