Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A Formula for the Optimal Cost in the General Discrete-Time LEQG Problem ⋆

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Abstract: This paper is aimed at deriving an explicit formula for the optimal cost for the discrete-time linear exponential-of-quadratic Gaussian (LEQG) control problem. The paper considers the general case with cross terms in the cost and noise covariance matrices. The result is applicable to discrete time minimax LQG and state estimation problems.

Keywords: Linear optimal control, Discrete-time systems, Optimal estimation, Quadratic performance indices, Dynamic output feedback, Dynamic programming.

1. INTRODUCTION

The study of linear control problems with exponential-of-quadratic cost function was initiated in Jacobson (1973) where the author studies discrete-time and continuous-time state feedback problems. The discrete-time linear output feedback problems with exponential-of-quadratic cost function is studied in Whittle (1981). Bensousssan and Schuppen (1985) studied the continuous-time output feedback problem in this context, and, in Pan and Basar (1996), these results were given simpler proofs for the general case with cross terms in cost function and noise covariance matrices. Furthermore, an elegant expression for the optimal cost is derived in Pan and Basar (1996) which has been subsequently used to obtain explicit guaranteed cost bounds for continuous-time uncertain systems and to solve minimax LQG control problems (e.g., see Petersen et al. (2000)). This expression was also instrumental in the guaranteed cost approach to nonlinear control and state estimation presented in the papers Petersen (2006b,a, 2007); Ouyang and Petersen (2007).

The objective of this paper is to derive a formula for optimal cost for discrete-time finite horizon LEQG control problem with a cross term in the cost function and correlated system and observation noise. We take an information-state approach as in Collings et al. (1996), and present details of all of the main calculations in order to make the paper self contained. The motivation for undertaking this calculation is that it is required in order to develop discrete time algorithms for minimax LQG control and state estimation in the most general case of noise covariances and cost functional. In particular, this most general case is required in order to develop discrete time counter parts to the nonlinear guaranteed cost results of Petersen (2006b,a, 2007); Ouyang and Petersen (2007).

⋆ This work was supported by grants from Australian Research Council.

2. PRELIMINARIES

Consider the state space model given by

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + v_{k+1}, \\ y_{k+1} &= C_k x_k + w_{k+1}. \end{aligned} \tag{1}$$

This model is defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a complete filtration $\{\mathcal{F}_{k+1};\ 0 \le k \le T-1\}$. The time horizon is $k = 0, 1, \cdots, T-1$. The state x_k is an n-vector, the control input u_k is a m-vector and the observation y_k is a p-vector. The white noise $\begin{bmatrix} v_k \\ w_k \end{bmatrix}$ has joint normal density $\varphi_k \sim \mathsf{N}(0, \Delta_k)$; $\Delta_k = \begin{bmatrix} \Sigma_k & \Upsilon_k \\ \Upsilon'_k & \Gamma_k \end{bmatrix}$.

Hypothesis 1. The covariance matrices $\Delta_k; k = 1, \cdots, T$ are positive definite [1]. □

We note that $\Delta_k^{-1} =: \begin{bmatrix} \Sigma_{-k} & \Upsilon_{-k} \\ \Upsilon'_{-k} & \Gamma_{-k} \end{bmatrix}$, where $\Sigma_{-k} = (\Sigma_k - \Upsilon_k \Gamma_k^{-1} \Upsilon'_k)^{-1}$, $\Upsilon_{-k} = -\Sigma_{-k} \Upsilon_k \Gamma_k^{-1}$, $\Gamma_{-k} = \Gamma_k^{-1} + \Gamma_k^{-1} \Upsilon'_k \Sigma_{-k} \Upsilon_k \Gamma_k^{-1}$.
The complete filtration generated by $(y_1, \cdots, y_k)$ is denoted by $\mathcal{Y}_k$. The admissible controls $u = (u_0, u_1, \cdots, u_{T-1})$ are $\mathbf{R}^m$-valued $\{\mathcal{Y}_k\}$ adapted processes. The set of all admissible control processes on the time interval $k, \cdots, l$ is denoted by $\mathcal{U}_{k,l}$.

Fact 2. For $u \in \mathcal{U}_{0,T-1}$, define the random variables

$$\lambda_k^u := \frac{\varphi_k(x_k - A_{k-1}x_{k-1} - B_{k-1}u_{k-1}, y_k - C_{k-1}x_{k-1})}{\varphi(x_k, y_k)}.$$

$$\Lambda_{0,k}^u := \Pi_{\ell=1}^k \lambda_\ell^u.$$

Then $\{[\Lambda_{0,k}^u]^{-1}\}$ is an $\{\mathcal{F}_k\}$-martingale, and hence we can define a probability measure P^u on $(\Omega, \mathcal{F}_T)$ by setting

[1] This is equivalent to $\Gamma_k > 0$ and $\Sigma_k - \Upsilon_k \Gamma_k^{-1} \Upsilon'_k > 0$.

$$\frac{dP^u}{dP}|_{\mathcal{F}_k} = [\Lambda^u_{0,k}]^{-1}; \quad k = 1, 2, \cdots, T.$$

Note that, under P^u, (x_k, y_k) has Normal density φ. □

The cost function for the problem is

$$J(u) = \mathbf{E}\Big[e^{\theta\{\Psi^u_{0,T-1} + \frac{1}{2}x'_T M_T x_T\}}\Big], \quad u \in \mathcal{U}_{0,T-1}, \qquad (2)$$

where $\theta > 0$ is the *risk-sensitive* parameter,

$$\Psi^u_{j,k} = \frac{1}{2}\sum_{\ell=j}^{k}[x'_\ell M_\ell x_\ell + u'_\ell N_\ell u_\ell + 2x'_\ell S_\ell u_\ell], \qquad (3)$$

and $\mathbf{E}$ stands for expectation w.r.t. the reference probability measure P. In what follows, we use $\mathbf{E}^u$ to denote expectation w.r.t. the probability measure P^u.

3. INFORMATION-STATE

For $u \in \mathcal{U}_{0,T-1}$, define the measure valued process ν^u by

$$\nu^u_k(B) := \mathbf{E}^u[\Lambda^u_{0,k} e^{\theta\Psi^u_{0,k-1}} I_B(x_k) \mid \mathcal{Y}_k]; \quad k = 0, 1, \cdots, T.$$

Here $I_B(\cdot)$ is the indicator function of the Borel set B. Note that ν^u_0 is the distribution of x_0. Furthermore, let $\{\alpha^u_k\}$ be the density process associated with the measure valued process $\{\nu^u_k\}$ so that

$$\nu^u_k(B) = \int_B \alpha^u_k(x)\, dx.$$

This also yields

$$\langle \nu^u_k, g\rangle = \int \alpha^u_k(x) g(x)\, dx = \mathbf{E}^u[\Lambda^u_{0,k} e^{\theta\Psi^u_{0,k-1}} g(x_k) \mid \mathcal{Y}_k],$$

for all real-valued Borel measurable functions g.
Now, as in Collings et al. (1996), we can establish a recursion formula for the information-state density α^u_k:

Theorem 3. (Recursion for α^u_k)

$$\alpha^u_{k+1}(x) = \frac{1}{\phi_{k+1}(y_{k+1})} \int_{\mathbf{R}^n} \alpha^u_k(\xi) e^{\theta\Psi^u_{k,k}} \times \varphi_{k+1}(x - A_k\xi - B_k u_k, y_{k+1} - C_k\xi)\, d\xi, \qquad (4)$$

where $\Psi^u_{k,k} = \frac{1}{2}[\xi' M_k \xi + u'_k N_k u_k + 2\xi' S_k u_k]$ and ϕ_{k+1} is the marginal density of w_{k+1}. □

The linearity of the dynamics (1) implies that α^u_k are Normal densities. We denote

$$\alpha^u_k(x) =: Z_k\, e^{-\frac{1}{2}(x-\mu^u_k)' R_k^{-1}(x-\mu^u_k)}.$$

Note that

$$Z_k = (2\pi)^{-\frac{n}{2}}|R_k|^{-\frac{1}{2}}, \qquad (5)$$

where $|R_k|$ is the determinant of the covariance matrix R_k.
Notation: In subsequent sections, for notational convenience, we drop the superscript u from μ^u_k. For $G \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, we employ the simplifying notations

$$q[G, a, x] := x'Gx + 2x'a, \quad q[G, x] := q[G, 0, x]. \qquad \square\,(6)$$

4. RECURSION FORMULA FOR $\chi_K = (Z_K, R_K, \mu_K)$

By the recursion (4), we have

$$\alpha_{k+1}(x) = \frac{Z_{\varphi_{k+1}}}{Z_{\phi_{k+1}}} Z_k e^{-\frac{1}{2}\gamma_k} \int_{\mathbf{R}^n} e^{-\frac{1}{2}q[G_k, a_k, \xi]}\, d\xi, \qquad (7)$$

where

$$\begin{aligned}
\gamma_k &= q[R_k^{-1}, \mu_k] + q[\Sigma_{-(k+1)}, x - B_k u_k] - q[\theta N_k, u_k] \\
&\quad + q[\Gamma_{-(k+1)} - \Gamma^{-1}_{k+1}, y_{k+1}] + 2(x - B_k u_k)'\Upsilon_{-(k+1)} y_{k+1}, \\
G_k &= R_k^{-1} + D'_k \Delta^{-1}_{k+1} D_k - \theta M_k; \quad D_k = \begin{bmatrix} A_k \\ C_k \end{bmatrix}, \\
a_k &= -e_k - (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)})(x - B_k u_k); \\
e_k &:= \theta S_k u_k + R_k^{-1}\mu_k + (A'_k \Upsilon_{-(k+1)} + C'_k \Gamma_{-(k+1)}) y_{k+1}.
\end{aligned}$$

Hypothesis 4. $G_k > 0$ for every $k = 0, 1, \cdots T-1$. □

From (7), we now get

$$\begin{aligned}
\alpha_{k+1}(x) &= \frac{Z_{\varphi_{k+1}}}{Z_{\phi_{k+1}}} Z_k e^{-\frac{1}{2}\gamma_k} \times (2\pi)^{\frac{n}{2}} |G_k|^{-\frac{1}{2}} e^{\frac{1}{2}q[G_k^{-1}, a_k]} \\
&= Z_k \Big(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|}\Big)^{-\frac{1}{2}} |G_k|^{-\frac{1}{2}} e^{-\frac{1}{2}\gamma_k + \frac{1}{2}q[G_k^{-1}, a_k]} \\
&= Z_k \Big(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|}\Big)^{-\frac{1}{2}} |G_k|^{-\frac{1}{2}} e^{-\frac{1}{2}\{\delta_k + q[H_k, b_k, x - B_k u_k]\}},
\end{aligned}$$

where

$$\begin{aligned}
\delta_k &= q[R_k^{-1}, \mu_k] - q[\theta N_k, u_k] - q[G_k^{-1}, e_k] \\
&\quad + q[\Gamma_{-(k+1)} - \Gamma^{-1}_{k+1}, y_{k+1}], \\
H_k &= \Sigma_{-(k+1)} - (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)})' G_k^{-1} \times \\
&\quad (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)}), \\
b_k &= \Upsilon_{-(k+1)} y_{k+1} - \tilde{A}_k e_k; \\
\tilde{A}_k &:= (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)})' G_k^{-1}.
\end{aligned}$$

Hypothesis 5. For $k = 0, 1, \cdots, T-1$, the matrix H_k is positive definite. □

We now obtain the recursion formulae:

$$R_{k+1} = H_k^{-1} \qquad (8)$$

$$\mu_{k+1} = B_k u_k - R_{k+1} b_k, \qquad (9)$$

$$Z_{k+1} = Z_k \Big(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|}\Big)^{-\frac{1}{2}} |G_k|^{-\frac{1}{2}} e^{-\frac{1}{2}\{\delta_k - q[R_{k+1}, b_k]\}}. \qquad (10)$$

In the next three subsections, we will simplify the above recursions, and express them in standard form.

4.1 Simplification of recursion (8)

We first observe that

$$C'_k \Gamma^{-1}_{k+1} C_k = D'_k \Delta^{-1}_{k+1} D_k - (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)}) \times \Sigma^{-1}_{-(k+1)} (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)})'.$$

Hypothesis 6. For all k, the matrix $R_k^{-1} - \theta M_k + C'_k \Gamma^{-1}_{k+1} C_k =: \tilde{K}_k^{-1}$ is invertible. □

Now

$$\begin{aligned}
H_k^{-1} &= \Sigma^{-1}_{-(k+1)} \\
&\quad + \Sigma^{-1}_{-(k+1)} (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)})' \tilde{K}_k \times \\
&\quad (A'_k \Sigma_{-(k+1)} + C'_k \Upsilon'_{-(k+1)}) \Sigma^{-1}_{-(k+1)}.
\end{aligned}$$

For further simplification, we write

$$\Sigma_{-(k+1)}A_k + \Upsilon_{-(k+1)}C_k = \Sigma_{-(k+1)}\hat{A}_k,$$
$$\Upsilon'_{-(k+1)}A_k + \Gamma_{-(k+1)}C_k = \Gamma_{k+1}^{-1}\hat{C}_k,$$

where

$$\hat{A}_k = A_k - \Upsilon_{k+1}\Gamma_{k+1}^{-1}C_k,$$
$$\hat{C}_k = C_k - \Upsilon'_{k+1}\Sigma_{-(k+1)}\hat{A}_k.$$

Therefore, one may write the recursion for R_k as

$$\boxed{R_{k+1} = \Sigma_{k+1} - \Upsilon_{k+1}\Gamma_{k+1}^{-1}\Upsilon'_{k+1} + \hat{A}_k\tilde{K}_k\hat{A}'_k} \quad (11)$$

Remark:

$$\tilde{K}_k = (\tilde{R}_k^{-1} + C'_k\Gamma_{k+1}^{-1}C_k)^{-1}; \quad \tilde{R}_k^{-1} := R_k^{-1} - \theta M_k.$$

Also, define $\tilde{\mu}_k$ by the relation

$$R_k^{-1}\mu_k + \theta S_k u_k = \tilde{R}_k^{-1}\tilde{\mu}_k. \qquad \square$$

4.2 Simplification of (9)

We may write the recursion (9) for μ_k in the form

$$\mu_{k+1} = A_k^{\mu}\mu_k + B_k^{\mu}u_k + C_k^{\mu}y_{k+1}, \quad (12)$$

where

$$A_k^{\mu} := H_k^{-1}\tilde{A}_k R_k^{-1}, \quad B_k^{\mu} := B_k + \theta H_k^{-1}\tilde{A}_k S_k,$$
$$C_k^{\mu} := H_k^{-1}\Big(\tilde{A}_k(A'_k\Upsilon_{-(k+1)} + C'_k\Gamma_{-(k+1)}) - \Upsilon_{-(k+1)}\Big).$$

Note that we can write

$$A_k^{\mu} = \hat{A}_k\tilde{K}_k R_k^{-1}, \quad B_k^{\mu} = B_k + \theta\hat{A}_k\tilde{K}_k S_k.$$

We also have

$$\begin{aligned} C_k^{\mu} &= H_k^{-1}\tilde{A}_k\hat{C}'_k\Gamma_{k+1}^{-1} - H_k^{-1}\Upsilon_{-(k+1)} \\ &= \hat{A}_k\tilde{K}_k\hat{C}'_k\Gamma_{k+1}^{-1} - \Sigma_{-(k+1)}^{-1}\Upsilon_{-(k+1)} \\ &\quad -\hat{A}_k\tilde{K}_k\hat{A}'_k\Upsilon_{-(k+1)} \\ &= \hat{A}_k\tilde{K}_k C'_k\Gamma_{k+1}^{-1} + \Upsilon_{k+1}\Gamma_{k+1}^{-1}. \end{aligned}$$

Furthermore

$$\begin{aligned} & A_k^{\mu}\mu_k + (B_k^{\mu} - B_k)u_k \\ &= H_k^{-1}\tilde{A}_k(R_k^{-1}\mu_k + \theta S_k u_k) \\ &= \hat{A}_k\tilde{K}_k\tilde{R}_k^{-1}\tilde{\mu}_k \\ &= \hat{A}_k\tilde{K}_k(\tilde{K}_k^{-1} - C'_k\Gamma_{k+1}^{-1}C_k)\tilde{\mu}_k \\ &= \Big(\hat{A}_k - \hat{A}_k\tilde{K}_k C'_k\Gamma_{k+1}^{-1}C_k\Big)\tilde{\mu}_k \\ &= \Big(A_k - C_k^{\mu}C_k\Big)\tilde{\mu}_k. \end{aligned}$$

Therefore we can write (12) in the alternate form:

$$\boxed{\mu_{k+1} = A_k\tilde{\mu}_k + B_k u_k + C_k^{\mu}(y_{k+1} - C_k\tilde{\mu}_k)}$$

4.3 Simplification of recursion (10)

To derive a simpler recursion for Z_k, we use the relation

$$\tilde{K}_k = G_k^{-1} + \tilde{A}'_k H_k^{-1}\tilde{A}_k,$$

and obtain

$$\begin{aligned} \delta_k - q[H_k^{-1}, b_k] &= \mu'_k\tilde{\Pi}_k\mu_k + u'_k\tilde{\Phi}_k u_k + 2u'_k\tilde{\Xi}_k\mu_k \\ &\quad + y'_{k+1}\bar{\Psi}_k y_{k+1} + 2u'_k\tilde{\Theta}_k y_{k+1} + 2y'_{k+1}\tilde{\Omega}_k\mu_k; \end{aligned}$$

where

$$\begin{aligned} \tilde{\Pi}_k &:= R_k^{-1} - -R_k^{-1}\tilde{K}_k R_k^{-1}, \\ \tilde{\Phi}_k &:= -\theta N_k - \theta^2 S'_k\tilde{K}_k S_k \\ \tilde{\Xi}_k &:= -\theta S'_k\tilde{K}_k R_k^{-1} \\ \bar{\Psi}_k &:= \Gamma_{-(k+1)} - \Gamma_{k+1}^{-1} - \Gamma_{k+1}^{-1}\hat{C}_k\tilde{K}_k\hat{C}'_k\Gamma_{k+1}^{-1} \\ &\quad -\Upsilon'_{-(k+1)}H_k^{-1}\Upsilon_{-(k+1)} + 2\langle\Upsilon'_{-1}H_k^{-1}\tilde{A}_k\hat{C}'\Gamma^{-1}\rangle \\ \tilde{\Theta}_k &:= -\theta S'_k\tilde{K}_k\hat{C}'_k\Gamma_{k+1}^{-1} + \theta S'_k\tilde{A}'_k H_k^{-1}\Upsilon_{-(k+1)}, \\ \tilde{\Omega}_k &:= -\Gamma_{k+1}^{-1}\hat{C}_k\tilde{K}_k R_k^{-1} + \Upsilon'_{-(k+1)}H_k^{-1}\tilde{A}_k R_k^{-1}. \end{aligned}$$

Therefore,

$$\boxed{\begin{aligned} Z_{k+1} &= Z_k\Big(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|}\Big)^{-\frac{1}{2}}|G_k|^{-\frac{1}{2}}e^{-\frac{1}{2}\times} \\ &\Big(\mu'_k\tilde{\Pi}_k\mu_k + u'_k\tilde{\Phi}_k u_k + y'_{k+1}\bar{\Psi}_k y_{k+1} \\ &\quad +2u'_k\tilde{\Xi}_k\mu_k + 2u'_k\tilde{\Theta}_k y_{k+1} + 2y'_{k+1}\tilde{\Omega}_k\mu_k\Big) \end{aligned}}$$

Remark: The notation $\tilde{\Psi}_k$ denotes the quantity $\Gamma_{k+1}^{-1} + \bar{\Psi}_k$. That is, $\tilde{\Psi}_k := \Gamma_{k+1}^{-1} + \bar{\Psi}_k$. Such a notation will be convenient in subsequent calculations. $\square$

Remark: We observe that

$$\begin{aligned} \Gamma_{-(k+1)} - \Gamma_{k+1}^{-1} - \Upsilon'_{-(k+1)}H_k^{-1}\Upsilon_{-(k+1)} \\ = -\Upsilon'_{-(k+1)}\hat{A}_k\tilde{K}_k\hat{A}'_k\Upsilon_{-(k+1)}, \\ H_k^{-1}\tilde{A}_k = H_k^{-1}\Sigma_{-1}\hat{A}_k G_k^{-1} = \hat{A}_k\tilde{K}_k. \end{aligned}$$

Therefore, by using the fact

$$\Gamma_{k+1}^{-1}\hat{C}_k - \Upsilon'_{-(k+1)}\hat{A}_k = \Gamma_{k+1}^{-1}C_k,$$

we obtain

$$\tilde{\Psi}_k = \Gamma_{k+1}^{-1} - \Gamma_{k+1}^{-1}C_k\tilde{K}_k C'_k\Gamma_{k+1}^{-1} = (\Gamma_{k+1} + C_k\tilde{R}_k C'_k)^{-1},$$

$$\tilde{\Theta}_k = -\theta S'_k\tilde{K}_k C'_k\Gamma_{k+1}^{-1}, \quad \tilde{\Omega}_k = -\Gamma_{k+1}^{-1}C_k\tilde{K}_k R_k^{-1}. \qquad \square$$

5. VALUE FUNCTION AND DYNAMIC PROGRAMMING

Let $\beta_T(x) := e^{\frac{\theta}{2}q[M_T,x]}$. Note that

$$\begin{aligned} J(u) &= \mathbf{E}\Big[e^{\theta\{\Psi_{0,T-1}^u + \frac{1}{2}x'_T M_T x_T\}}\Big] \\ &= \mathbf{E}^u\Big[\Lambda_{0,T}^u e^{\theta\Psi_{0,T-1}^u}\beta_T(x_T)\Big] \\ &= \mathbf{E}^u\big[\langle\alpha_T^u(\cdot), \beta_T(\cdot)\rangle\big]. \end{aligned}$$

Now define the adjoint process

$$\beta_k^u(x) := \mathbf{E}^u\Big[\Lambda_{k+1,T}^u e^{\theta\Psi_{k,T-1}^u}\beta_T(x_T)/x_k = x, \mathcal{Y}_T\Big].$$

The value function for the control problem is

$$V_k(\chi) = \inf_{u \in \mathcal{U}_{k,T-1}} \mathbf{E}^u[\langle \alpha_k^u(\cdot), \beta_k^u(\cdot)\rangle \big/ \alpha_k^u = \alpha(\chi)].$$

Note that, if $\chi_T = (Z_T, R_T, \mu_T)$, then

$$\begin{aligned} V_T(\chi_T) &= \langle \alpha(\chi_T), \beta_T \rangle \\ &= Z_T \int e^{-\frac{1}{2}q[R_T^{-1}, x-\mu_T]}\ e^{\frac{1}{2}q[\theta M_T, x]}\ dx \\ &= Z_T e^{-\frac{1}{2}q[R_T^{-1},\mu_T]} \int e^{-\frac{1}{2}q[R_T^{-1}-\theta M_T, -R_T^{-1}\mu_T, x]}\ dx \\ &= Z_T e^{-\frac{1}{2}q[R_T^{-1},\mu_T]} \times (2\pi)^{n/2}|R_T^{-1} - \theta M_T|^{-1/2} \times \\ &\quad e^{\frac{1}{2}q[R_T^{-1}(R_T^{-1}-\theta M_T)^{-1}R_T^{-1},\mu_T]}. \end{aligned}$$

Hypothesis 7. The matrix $(R_T)^{-1} - \theta M_T > 0$. □

Under this assumption, we obtain

$$\begin{aligned} R_T^{-1}(R_T^{-1} &- \theta M_T)^{-1}R_T^{-1} - R_T^{-1} \\ &= R_T^{-1}((R_T^{-1} - \theta M_T)^{-1}R_T^{-1} - I) \\ &= R_T^{-1}((I - \theta R_T M_T)^{-1} - I) \\ &= R_T^{-1}\theta R_T M_T (I - \theta R_T M_T)^{-1} \\ &= \theta M_T (I - \theta R_T M_T)^{-1}. \end{aligned}$$

Therefore,

$$V_T(\chi_T) = Z_T (2\pi)^{n/2}|R_T^{-1} - \theta M_T|^{-1/2} \times e^{\frac{\theta}{2}q[M_T(I-\theta R_T M_T)^{-1},\mu_T]}. \tag{13}$$

If we use the notation:

$$\begin{aligned} S_T^a &:= M_T(I - \theta R_T M_T)^{-1} \\ e^{\frac{\theta}{2}s_T^c} &:= (2\pi)^{n/2}|R_T^{-1} - \theta M_T|^{-1/2}, \end{aligned}$$

then we can express (13) as

$$V_T(\chi_T) = Z_T e^{\frac{\theta}{2}\{\mu_T' S_T^a \mu_T + s_T^c\}}.$$

One can also prove the following *dynamic programming recursion.*

Theorem 8. (DYNAMIC PROGRAMMING RECURSION)

$$V_k(\chi) = \inf_{u_k \in \mathcal{U}_{k,k}} \mathbf{E}^u \Big[V_{k+1}(\chi_{k+1}(\chi_k, u_k, y_{k+1})) \big/ \chi_k = \chi \Big]. \tag{14}$$

□

Let $V_{k+1}(\chi_{k+1}) =: Z_{k+1}\ e^{\frac{\theta}{2}\{(\mu_{k+1})' S_{k+1}^a \mu_{k+1} + s_{k+1}^c\}}$. This together with the recursion formulae for Z_{k+1}, μ_{k+1} implies that

$$V_k(\chi) = \inf_{u_k} J_k(u_k), \tag{15}$$

where

$$\begin{aligned} J_k(u_k) &= \int_{\mathbf{R}^p} Z_k(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|})^{-\frac{1}{2}}|G_k|^{-\frac{1}{2}}\ e^{-\frac{1}{2}\{\delta_k - q[R_{k+1},b_k]\}} \times \\ &\quad e^{\frac{\theta}{2}q[S_{k+1}^a, B_k u_k - R_{k+1}b_k]}\ e^{\frac{\theta}{2}s_{k+1}^c}\phi_{k+1}(y_{k+1})\ dy_{k+1}. \end{aligned} \tag{16}$$

Notation:

$$\begin{aligned} r[\Pi, \Phi, \Psi, \Theta, \Xi, \Omega] &:= \mu_k'\Pi\mu_k + u_k'\Phi u_k + y_{k+1}'\Psi y_{k+1} \\ &+ 2u_k'\Theta y_{k+1} + 2u_k'\Xi\mu_k + 2y_{k+1}'\Omega_k\mu_k. \end{aligned}$$ □

The equation (16) can now be written as

$$\begin{aligned} J_k(u_k) &= Z_k(\frac{|\Delta_{k+1}|}{|\Gamma_{k+1}|})^{-\frac{1}{2}}|G_k|^{-\frac{1}{2}} Z_\phi\ e^{\frac{\theta}{2}s_{k+1}^c} \times \\ &\int_{\mathbf{R}^p} e^{-\frac{1}{2}r[\Pi_k,\Phi_k,\Psi_k,\Theta_k,\Xi_k,\Omega_k]}\ dy_{k+1}; \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Phi_k &:= \tilde{\Phi}_k - \theta B_k^{\mu'} S_{k+1}^a B_k^\mu, & \Psi_k &:= \tilde{\Psi}_k - \theta C_k^{\mu'} S_{k+1}^a C_k^\mu, \\ \Theta_k &:= \tilde{\Theta}_k - \theta B_k^{\mu'} S_{k+1}^a C_k^\mu, & \Xi_k &:= \tilde{\Xi}_k - \theta B_k^{\mu'} S_{k+1}^a A_k^\mu, \\ \Omega_k &:= \tilde{\Omega}_k - \theta C_k^{\mu'} S_{k+1}^a A_k^\mu, & \Pi_k &:= \tilde{\Pi}_k - \theta A_k^{\mu'} S_{k+1}^a A_k^\mu. \end{aligned}$$

Note that $\Phi_k < 0$.

Hypothesis 9. $\Psi_k > 0$. □

Now, using Lemma 13 in the appendix, we obtain the optimal control in the form:

$$u_k^* = -(\Phi_k - \Theta_k\Psi_k^{-1}\Theta_k')^{-1}(\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k)\mu_k, \tag{18}$$

and the optimal cost in the form

$$V_k(\chi) = Z_k|\Delta_{k+1}|^{-\frac{1}{2}}|G_k|^{-\frac{1}{2}}\ e^{\frac{\theta}{2}s_{k+1}^c}|\Psi_k|^{-\frac{1}{2}}\ e^{-\frac{1}{2}q[Q_k,\mu_k]}, \tag{19}$$

where

$$\begin{aligned} Q_k &= \Pi_k - \Omega_k'\Psi_k^{-1}\Omega_k - (\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k)' \times \\ &\quad (\Phi_k - \Theta_k\Psi_k^{-1}\Theta_k')^{-1}(\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k) \end{aligned}$$

Thus

$$\begin{aligned} \theta S_k^a &= (\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k)'(\Phi_k - \Theta_k\Psi_k^{-1}\Theta_k')^{-1} \times \\ &\quad (\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k) - (\Pi_k - \Omega_k'\Psi_k^{-1}\Omega_k), \end{aligned} \tag{20}$$

and

$$e^{\frac{\theta}{2}s_k^c} = |\Delta_{k+1}|^{-\frac{1}{2}}|G_k|^{-\frac{1}{2}}|\Psi_k|^{-\frac{1}{2}}\ e^{\frac{\theta}{2}s_{k+1}^c}. \tag{21}$$

From (21), it follows that

$$s_k^c = s_{k+1}^c - \frac{1}{\theta}\log(|\Delta_{k+1}|.|G_k|.|\Psi_k|).$$

Therefore

$$\theta s_k^c = -\log\frac{|R_T^{-1} - \theta M_T|}{(2\pi)^n} - \sum_{j=k}^{T-1}\log(|\Delta_{j+1}|.|G_j|.|\Psi_j|) \tag{22}$$

5.1 An explicit expression for s_k^c

Note that

$$\begin{aligned} G_k &= R_k^{-1} - \theta M_k + C_k'\Gamma_{k+1}^{-1}C_k \\ &\quad + \hat{A}_k'(\Sigma_{k+1} - \Upsilon_{k+1}\Gamma_{k+1}^{-1}\Upsilon_{k+1}')^{-1}\hat{A}_k, \\ &= \tilde{K}_k^{-1} + \hat{A}_k'(\Sigma_{k+1} - \Upsilon_{k+1}\Gamma_{k+1}^{-1}\Upsilon_{k+1}')^{-1}\hat{A}_k. \end{aligned}$$

Now

$$\Psi_k = (\Gamma_{k+1} + C_k\tilde{R}_kC_k')^{-1} - \theta C_k^{\mu'} S_{k+1}^a C_k^\mu.$$

Recall that

$$(\Gamma_{k+1} + C_k\tilde{R}_kC_k')C_k^{\mu'} = (\Upsilon_{k+1} + A_k\tilde{R}_kC_k')'.$$

Therefore

$$\begin{aligned}\Psi_k^{-1} &= (\Gamma_{k+1} + C_k\tilde{R}_kC_k') - (\Upsilon_{k+1} + A_k\tilde{R}_kC_k')' \times \\ &\quad (-\frac{\tilde{P}_{k+1}^{-1}}{\theta} + A_k\tilde{R}_kA_k')^{-1}(\Upsilon_{k+1} + A_k\tilde{R}_kC_k') \\ &= \Gamma_{k+1} + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}\Upsilon_{k+1} \\ &\quad +(C_k + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}A + k) \times \\ &\quad (\tilde{R}_k^{-1} - \theta A_k'\tilde{P}_{k+1}A_k)^{-1}(C_k + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}A_k)'.\end{aligned}$$

Alternatively

$$\begin{aligned}\Psi_k &= \Gamma_{k+1}^{-1} - \Gamma_{k+1}^{-1}C_k\tilde{K}_kC_k'\Gamma_{k+1}^{-1} \\ &\quad -\theta\Gamma_{k+1}^{-1}(\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k')'S_{k+1}^a \\ &\quad \times(\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k')\Gamma_{k+1}^{-1} \\ &= \Gamma_{k+1}^{-1} \\ &\quad -\Gamma_{k+1}^{-1}\Big[C_k\tilde{K}_kC_k' + (\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k')'(\theta S_{k+1}^a) \\ &\quad \times(\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k')\Big]\Gamma_{k+1}^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}\Gamma_{k+1}\Psi_k\Gamma_{k+1} &= \Gamma_{k+1} - C_k\tilde{K}_kC_k' \\ &\quad -(\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k')'(\theta S_{k+1}^a)(\Upsilon_{k+1} + \hat{A}_k\tilde{K}_kC_k').\end{aligned}$$

Applying Lemma 14, we obtain

$$\begin{aligned}\Gamma_{k+1}\Psi_k\Gamma_{k+1} &= Z_k^c + 2\Gamma_{k+1} \\ &\quad -(Z_k^c + \Gamma_{k+1})(Z_k^c)^{-1}(Z_k^c + \Gamma_{k+1}) \\ &= -\Gamma_{k+1}(Z_k^c)^{-1}\Gamma_{k+1},\end{aligned}$$

where

$$\begin{aligned}-Z_k^c &= \Gamma_{k+1} + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}\Upsilon_{k+1} \\ &\quad +(C_k + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}A_k) \times \\ &\quad (\tilde{R}_k^{-1} - \theta A_k'\tilde{P}_{k+1}A_k)^{-1}(C_k + \theta\Upsilon_{k+1}'\tilde{P}_{k+1}A_k)' \\ &= \Gamma_{k+1} + C_k\tilde{R}_kC_k' + (\Upsilon_{k+1} + A_k\tilde{R}_kC_k')' \times \\ &\quad \Big(\frac{\tilde{P}_{k+1}^{-1}}{\theta} - A_k\tilde{R}_kA_k'\Big)^{-1}(\Upsilon_{k+1} + A_k\tilde{R}_kC_k').\end{aligned}$$

Therefore we have

$$\begin{aligned}&(\Gamma_{k+1} + C_k\tilde{R}_kC_k')\Psi_k(\Gamma_{k+1} + C_k\tilde{R}_kC_k') \\ &= \Gamma_{k+1} + C_k\tilde{R}_kC_k' \\ &\quad - (\Upsilon_{k+1} + A_k\tilde{R}_kC_k')'(\frac{P_{k+1}^{-1}}{\theta} - R_{k+1})^{-1} \times \\ &\quad (\Upsilon_{k+1} + A_k\tilde{R}_kC_k').\end{aligned}$$

Thus

$$\begin{aligned}|\Psi_k| &= \frac{1}{\left|\Gamma_{k+1} + C_k\tilde{R}_kC_k'\right|^2}\times \\ &\Big|\Gamma_{k+1} + C_k\tilde{R}_kC_k' - (\Upsilon_{k+1} + A_k\tilde{R}_kC_k')'\times \\ &\quad (\frac{P_{k+1}^{-1}}{\theta} - R_{k+1})^{-1}(\Upsilon_{k+1} + A_k\tilde{R}_kC_k')\Big|.\end{aligned}$$

Hence

$$\begin{aligned}\theta s_k^c &= \log\Big((2\pi)^n|R_T|\Big) - \log\big(|I - \theta R_TM_T|\big) \\ &\quad -(T-k)\log\Big(|\Delta_{k+1}|\Big) \\ &\quad -\sum_{j=k}^{T-1}\log\Big(|R_j^{-1} - \theta M_j + C_j'\Gamma_{j+1}^{-1}C_j \\ &\quad +\hat{A}_j'(\Sigma_{j+1} - \Upsilon_{j+1}\Gamma_{j+1}^{-1}\Upsilon_{j+1}')^{-1}\hat{A}_j|\Big) \\ &\quad -\sum_{j=k}^{T-1}\log\Big(\Big|\Gamma_{j+1} + C_j\tilde{R}_jC_j' - (\Upsilon_{j+1} + A_j\tilde{R}_jC_j')' \times \\ &\quad (\frac{P_{j+1}^{-1}}{\theta} - R_{j+1})^{-1}(\Upsilon_{j+1} + A_j\tilde{R}_jC_j')\Big|\Big) \\ &\quad +\sum_{j=k}^{T-1}\log\Big(|\Gamma_{j+1} + C_j\tilde{R}_jC_j'|^2\Big).\end{aligned} \tag{23}$$

5.2 Further simplification of recursion (20):

Recall that

$$\begin{aligned}\tilde{\Pi}_k &= R_k^{-1} - R_k^{-1}\tilde{K}_kR_k^{-1}, \\ \tilde{\Phi}_k &= -\theta N_k - \theta^2S_k'\tilde{K}_kS_k, \\ \tilde{\Psi}_k &= \Gamma_{k+1}^{-1} - \Gamma_{k+1}^{-1}C_k\tilde{K}_kC_k'\Gamma_{k+1}^{-1}, \\ \tilde{\Theta}_k &= -\theta S_k'\tilde{K}_kC_k'\Gamma_{k+1}^{-1}, \\ \tilde{\Xi}_k &= -\theta S_k'\tilde{K}_kR_k^{-1}, \\ \tilde{\Omega}_k &= -\Gamma_{k+1}^{-1}C_k\tilde{K}_kR_k^{-1},\end{aligned}$$

and

$$\begin{aligned}\Pi_k &= \tilde{\Pi}_k - \theta A_k^{\mu'}S_{k+1}^aA_k^\mu, & \Phi_k &= \tilde{\Phi}_k - \theta B_k^{\mu'}S_{k+1}^aB_k^\mu, \\ \Psi_k &= \tilde{\Psi}_k - \theta C_k^{\mu'}S_{k+1}^aC_k^\mu, & \Theta_k &= \tilde{\Theta}_k - \theta B_k^{\mu'}S_{k+1}^aC_k^\mu, \\ \Xi_k &= \tilde{\Xi}_k - \theta B_k^{\mu'}S_{k+1}^aA_k^\mu, & \Omega_k &= \tilde{\Omega}_k - \theta C_k^{\mu'}S_{k+1}^aA_k^\mu.\end{aligned}$$

Furthermore

$$\begin{aligned}A_k^\mu &= \hat{A}_k\tilde{K}_kR_k^{-1}, \quad B_k^\mu = B_k + \theta\hat{A}_k\tilde{K}_kS_k, \\ C_k^\mu &= \hat{A}_k\tilde{K}_kC_k'\Gamma_{k+1}^{-1} + \Upsilon_{k+1}\Gamma_{k+1}^{-1}.\end{aligned}$$

Note that the recursion (20) can also be expressed as

$$\begin{aligned}\theta S_k^a &= (\Omega_k - \Theta_k'\Phi_k^{-1}\Xi_k)'(\Psi_k - \Theta_k'\Phi_k^{-1}\Theta_k)^{-1}\times \\ &\quad (\Omega_k - \Theta_k'\Phi_k^{-1}\Xi_k) - (\Pi_k - \Xi_k'\Phi_k^{-1}\Xi_k).\end{aligned} \tag{24}$$

Now apply Lemma 14 to get

$$\theta S_k^a = -\tilde{X}_k + \tilde{Y}_k'\tilde{Z}_k^{-1}\tilde{Y}_k, \tag{25}$$

where

$$\tilde{X}_k = \Big(\tilde{\Pi}_k - \tilde{\Omega}_k'\tilde{\Psi}_k^{-1}\tilde{\Omega}_k\Big) - (\tilde{\Xi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Omega}_k)' \times (\tilde{\Phi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Theta}_k')^{-1}(\tilde{\Xi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Omega}_k),$$
$$\tilde{Y}_k = \sqrt{\theta}\Big(A_k^{\mu} - C_k^{\mu}\tilde{\Psi}_k^{-1}\tilde{\Omega}_k\Big) - \sqrt{\theta}(B_k^{\mu'} - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}C_k^{\mu'})' \times (\tilde{\Phi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Theta}_k')^{-1}(\tilde{\Xi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Omega}_k),$$
$$\tilde{Z}_k = \Big((S_{k+1}^a)^{-1} - \theta C_k^{\mu}\tilde{\Psi}_k^{-1}C_k^{\mu'}\Big) - \theta(B_k^{\mu'} - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}C_k^{\mu'})' \times(\tilde{\Phi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Theta}_k')^{-1}(B_k^{\mu'} - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}C_k^{\mu'}).$$

The next lemma will be useful in simplifying the above expressions for $\tilde{X}_k, \tilde{Y}_k, \tilde{Z}_k$.

Lemma 10. We have
$$\tilde{R}_k = \tilde{K}_k + \tilde{K}_k C_k'\Gamma_{k+1}^{-1}\tilde{\Psi}_k^{-1}\Gamma_{k+1}^{-1}C_k\tilde{K}_k. \tag{26}$$

PROOF: Let $\tilde{S}_k$ denote the R.H.S. of the above equality:
$$\tilde{S}_k^{-1} = (\tilde{K}_k + \tilde{K}_k C_k'(\Gamma_{k+1} - C_k\tilde{K}_k\Gamma_{k+1}C_k')^{-1}C_k\tilde{K}_k)^{-1} = \tilde{K}_k^{-1} - C_k'\Gamma_{k+1}^{-1}C_k = \tilde{R}_k^{-1}. \qquad \square$$

With the help of this lemma, we can show the following:
$$\begin{aligned}
\tilde{\Pi}_k - \tilde{\Omega}_k'\tilde{\Psi}_k^{-1}\tilde{\Omega}_k &= R_k^{-1} - R_k^{-1}\tilde{R}_k R_k^{-1},\\
\tilde{\Xi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Omega}_k &= -\theta S_k'\tilde{R}_k R_k^{-1},\\
\tilde{\Phi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Theta}_k' &= -\theta(N_k + \theta S_k'\tilde{R}_k S_k),\\
A_k^{\mu} - C_k^{\mu}\tilde{\Psi}_k^{-1}\tilde{\Omega}_k &= A_k\tilde{R}_k R_k^{-1},\\
B_k^{\mu} - C_k^{\mu}\tilde{\Psi}_k^{-1}\tilde{\Theta}_k' &= B_k + \theta A_k\tilde{R}_k S_k,\\
C_k^{\mu}\tilde{\Psi}_k^{-1}C_k^{\mu'} &= \Upsilon_{k+1}\Gamma_{k+1}^{-1}\Upsilon_{k+1}' + A_k\tilde{R}_k A_k' - \hat{A}_k\tilde{K}_k\hat{A}_k'\\
&= \Sigma_{k+1} - R_{k+1} + A_k\tilde{R}_k A_k'.
\end{aligned}$$

Therefore
$$\begin{aligned}
\tilde{X}_k &= R_k^{-1} - R_k^{-1}\Big(\tilde{R}_k - \theta\tilde{R}_k S_k(N_k + \theta S_k'\tilde{R}_k S_k)^{-1}S_k'\tilde{R}_k\Big) \times R_k^{-1}\\
&= R_k^{-1} - R_k^{-1}\Big(\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k'\Big)^{-1}R_k^{-1}\\
&= \Big(R_k - (\theta\breve{M}_k)^{-1}\Big)^{-1}\\
&= -\theta\breve{M}_k(I - \theta R_k\breve{M}_k)^{-1}\\
&= -\theta\bar{M}_k;
\end{aligned}$$
$$\begin{aligned}
\frac{1}{\sqrt{\theta}}\tilde{Y}_k &= \Big[A_k\Big\{\tilde{R}_k - \theta\tilde{R}_k S_k(N_k + \theta S_k'\tilde{R}_k S_k)S_k'\tilde{R}_k\Big\}\\
&\quad -B_k\left\{\begin{matrix} N_k^{-1} - N_k^{-1}S_k'\times \\ (\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k')^{-1}S_k N_k^{-1}\end{matrix}\right\}S_k'\tilde{R}_k\Big]R_k^{-1}\\
&= \Big[A_k(\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k')^{-1}\\
&\quad -B_k N_k^{-1}S_k'\left\{\begin{matrix}\tilde{R}_k - \theta(\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k')^{-1}\times \\ S_k N_k^{-1}S_k'\tilde{R}_k\end{matrix}\right\}\Big]\\
&\quad \times R_k^{-1}\\
&= \breve{A}_k(\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k')^{-1}R_k^{-1}\\
&= \breve{A}_k(I - \theta R_k\breve{M}_k)^{-1}\\
&=: \bar{A}_k;
\end{aligned}$$
$$\begin{aligned}
\tilde{Z}_k &= (S_{k+1}^a)^{-1} - \theta(\Sigma_{k+1} - R_{k+1} + A_k\tilde{R}_k A_k')\\
&\quad +(B_k + \theta A_k\tilde{R}_k S_k)(N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\\
&\quad \times(B_k + \theta A_k\tilde{R}_k S_k)\\
&= (S_{k+1}^a)^{-1} - \theta(\Sigma_{k+1} - R_{k+1} + A_k\tilde{R}_k A_k')\\
&\quad -\left\{\begin{matrix} -\theta A_k\tilde{R}_k A_k' - B_k N_k^{-1}B_k' \\ +\theta\breve{A}_k(\tilde{R}_k^{-1} + \theta S_k N_k^{-1}S_k')^{-1}\breve{A}_k'\end{matrix}\right\}\\
&= (S_{k+1}^a)^{-1} - \theta\Sigma_{k+1} + \theta R_{k+1} + B_k N_k^{-1}B_k' \times\\
&\qquad -\theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'
\end{aligned}$$
where
$$\begin{aligned}
\breve{M}_k &:= M_k - S_k N_k^{-1}S_k';\\
\bar{M}_k &:= \breve{M}_k(I - \theta R_k\breve{M}_k)^{-1};\\
\breve{A}_k &:= A_k - B_k N_k^{-1}S_k'.
\end{aligned}$$

Hence
$$\boxed{S_k^a = \bar{M}_k + \bar{A}_k'\Big((S_{k+1}^a)^{-1} - \theta\Sigma_{k+1} + \theta R_{k+1} + B_k N_k^{-1}B_k' - \theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\Big)^{-1}\bar{A}_k} \tag{27}$$

Now using the Matrix Inversion Lemma, we have
$$\begin{aligned}
(S_k^a)^{-1} = \bar{M}_k^{-1} - \bar{M}_k^{-1}\bar{A}_k'\Big(&(S_{k+1}^a)^{-1} - \theta\Sigma_{k+1} + \theta R_{k+1}\\
&+B_k N_k^{-1}B_k' - \theta\breve{A}_k\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\\
&+\bar{A}_k\bar{M}_k^{-1}\bar{A}_k'\Big)^{-1}\bar{A}_k\bar{M}_k^{-1}.
\end{aligned}$$

Now note that
$$\begin{aligned}
\bar{M}_k^{-1} = \breve{M}_k^{-1} - \theta R_k, &\quad \bar{A}_k\bar{M}_k^{-1} = \breve{A}_k\breve{M}_k^{-1},\\
\bar{A}_k\bar{M}_k^{-1}\bar{A}_k' &= \breve{A}_k\breve{M}_k^{-1}(I - \theta\breve{M}_k R_k)^{-1}\breve{A}_k'\\
&= \breve{A}_k\breve{M}_k^{-1}R_k^{-1}(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\\
&= \theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k' + \breve{A}_k\breve{M}_k^{-1}\breve{A}_k'.
\end{aligned}$$

Therefore, we obtain
$$\begin{aligned}
(S_k^a)^{-1} = \bar{M}_k^{-1} - \bar{M}_k^{-1}\bar{A}_k'\Big(&(S_{k+1}^a)^{-1} - \theta\Sigma_{k+1} + \theta R_{k+1}\\
&+B_k N_k^{-1}B' + \breve{A}_k\breve{M}_k^{-1}\breve{A}_k'\Big)^{-1}\bar{A}_k\bar{M}_k^{-1}.
\end{aligned}$$

By letting $P_k = ((S_k^a)^{-1} + \theta R_k)^{-1}$, we obtain
$$P_k^{-1} = \breve{M}_k^{-1} - \breve{M}_k^{-1}\breve{A}_k'\left(\begin{matrix}P_{k+1}^{-1} - \theta\Sigma_{k+1} + B_k N_k^{-1}B_k' \\ +\breve{A}_k\breve{M}_k^{-1}\breve{A}_k'\end{matrix}\right)^{-1}\breve{A}_k\breve{M}_k^{-1}.$$

Thus
$$\boxed{P_k = \breve{M}_k + \breve{A}_k'\Big(P_{k+1}^{-1} - \theta\Sigma_{k+1} + B_k N_k^{-1}B_k'\Big)^{-1}\breve{A}_k} \tag{28}$$

We can express S_k^a in terms of P_k as
$$\boxed{S_k^a = P_k(I - \theta R_k P_k)^{-1}} \tag{29}$$

5.3 Simplification of the expression (18) for the optimal policy

Theorem 11. The optimal control policy is given by

$$\boxed{u_k^* = K_k(I-\theta R_k P_k)^{-1}\mu_k;} \quad k=0,1,\cdots,T-1. \quad (30)$$

where

$$K_k := -N_k^{-1}\Big[S_k' + B_k'(P_{k+1}^{-1} - \theta\Sigma_{k+1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\Big]. \quad (31)$$

PROOF: In view of (18) and Lemma 15, it suffices to prove that

$$N_k^{-1}\Big[S_k' + B_k'(P_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\Big](I-\theta R_k P_k)^{-1} = \bar{Z}_k^{-1}\bar{Y}_k \quad (32)$$

where

$$\bar{Y}_k = \tilde{\Xi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Omega}_k - \theta(B_k^\mu - C_k^\mu\tilde{\Psi}_k^{-1}\tilde{\Theta}_k')' \times ((S_{k+1}^a)^{-1} - \theta C_k^\mu\tilde{\Psi}_k^{-1}C_k^{\mu'})^{-1}(A_k^\mu - C_k^\mu\tilde{\Psi}_k^{-1}\tilde{\Omega}_k),$$
$$\bar{Z}_k = \tilde{\Phi}_k - \tilde{\Theta}_k\tilde{\Psi}_k^{-1}\tilde{\Theta}_k' - \theta(B_k^\mu - C_k^\mu\tilde{\Psi}_k^{-1}\tilde{\Theta}_k')' \times ((S_{k+1}^a)^{-1} - \theta C_k^\mu\tilde{\Psi}_k^{-1}C_k^{\mu'})^{-1}(B_k^\mu - C_k^\mu\tilde{\Psi}_k^{-1}\tilde{\Theta}_k').$$

Therefore

$$-\frac{1}{\theta}\bar{Y}_k = \Big[S_k' + (B_k+\theta A_k\tilde{R}_k S_k)' \times \Big((S_{k+1}^a)^{-1} - \theta\begin{pmatrix}\Sigma_{k+1} - R_{k+1}\\ +A_k\tilde{R}_k A_k'\end{pmatrix}\Big)^{-1}A_k\Big]\tilde{R}_k R_k^{-1} = \begin{bmatrix} S_k' + (B_k+\theta A_k\tilde{R}_k S_k)' \\ \times\Big(\tilde{P}_{k+1}^{-1} - \theta A_k\tilde{R}_k A_k'\Big)^{-1}A_k\end{bmatrix}\tilde{R}_k R_k^{-1},$$

and

$$-\frac{1}{\theta}\bar{Z}_k = (N_k + \theta S_k'\tilde{R}_k S_k) + (B_k+\theta A_k\tilde{R}_k S_k)' \times \Big(\tilde{P}_{k+1}^{-1} - \theta A_k\tilde{R}_k A_k'\Big)^{-1}(B_k+\theta A_k\tilde{R}_k S_k).$$

This yields

$$\bar{Z}_k^{-1}\bar{Y}_k = W_k\tilde{R}_k R_k^{-1}, \quad (33)$$

where

$$W_k = \Big[(N_k + \theta S_k'\tilde{R}_k S_k) + (B_k+\theta A_k\tilde{R}_k S_k)' \times \Big(\tilde{P}_{k+1}^{-1} - \theta A_k\tilde{R}_k A_k'\Big)^{-1}(B_k+\theta A_k\tilde{R}_k S_k)\Big]^{-1} \times \Big[S_k' + (B_k+\theta A_k\tilde{R}_k S_k)'\Big(\tilde{P}_{k+1}^{-1} - \theta A_k\tilde{R}_k A_k'\Big)^{-1}A_k\Big]$$
$$= (N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\Big[S_k' + (B_k+\theta A_k\tilde{R}_k S_k)' \times \Big(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k' - \theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\Big)^{-1} \times \Big(A_k - (B_k+\theta A_k\tilde{R}_k S_k)(N_k+\theta S_k'\tilde{R}_k S_k)^{-1}S_k'\Big)\Big]$$
$$= (N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\Big[S_k' + (B_k+\theta A_k\tilde{R}_k S_k)' \times \Big(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k' - \theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\Big)^{-1} \times \breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\tilde{R}_k^{-1}\Big].$$

Now note that

$$\Big(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k' - \theta\breve{A}_k(R_k^{-1} - \theta\breve{M}_k)^{-1}\breve{A}_k'\Big)^{-1}\breve{A}_k$$
$$= (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1} \times \Big[I + \theta\breve{A}_k(R_k^{-1} - \theta P_k)^{-1}\breve{A}_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\Big]\breve{A}_k$$
$$= (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k \times \Big[I + \theta(R_k^{-1} - \theta P_k)^{-1}\breve{A}_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\Big]$$
$$= (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k(R_k^{-1} - \theta P_k)^{-1} \times \Big[R_k^{-1} - \theta P_k + \theta\breve{A}_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\Big]$$
$$= (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k(R_k^{-1} - \theta P_k)^{-1}(R_k^{-1} - \theta\hat{M}_k)$$
$$= (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k(I - \theta R_k P_k)^{-1}R_k \times(R_k^{-1} - \theta\hat{M}_k).$$

This gives

$$W_k = (N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\Big[S_k' + (B_k+\theta A_k\tilde{R}_k S_k)' \times (\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k(I-\theta R_k P_k)^{-1}R_k\tilde{R}_k^{-1}\Big]$$
$$= (N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\tilde{W}_k(I-\theta R_k P_k)^{-1}R_k\tilde{R}_k^{-1},$$

where

$$\tilde{W}_k = S_k'\tilde{R}_k R_k^{-1}(I - \theta R_k P_k) + (B_k+\theta A_k\tilde{R}_k S_k)'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k$$
$$= S_k'(\theta\tilde{R}_k)\begin{bmatrix}(\theta R_k)^{-1} - P_k + \\ A_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\end{bmatrix} + B_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k$$
$$= S_k'(\theta\tilde{R}_k)\Big[(\theta R_k)^{-1} - \breve{M}_k + S_k N_k^{-1}B_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k\Big] + B_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k.$$

Therefore

$$(N_k + \theta S_k'\tilde{R}_k S_k)^{-1}\tilde{W}_k$$
$$= N^{-1}\Big[B_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k + S_k'\Big((\theta\tilde{R}_k)^{-1} + S_k N_k^{-1}S_k'\Big)^{-1}\Big((\theta R_k)^{-1} - \breve{M}_k\Big)\Big]$$
$$= N_k^{-1}\Big[B_k'(\tilde{P}_{k+1}^{-1} + B_k N_k^{-1}B_k')^{-1}\breve{A}_k + S_k'\Big].$$

This completes the proof. □

6. CONCLUSIONS

For the discrete-time LEQG problem (1),(2), we have obtained the explicit expression for the optimal cost as

$V_k(\chi_k) = Z_k e^{\frac{\theta}{2}\left(\mu_k' S_k^a \mu_k + s_k^c\right)}$ through the formulas (5), (11), (23), (28) and (29). This is useful in constructing controllers that lead to explicit guaranteed cost bounds for discrete-time uncertain systems (e.g., see Shaiju and Petersen (2007)).

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Appendix A. SOME BASIC RESULTS

Lemma 12. Let $q[G,a,x]$, $q[G,x]$ be defined as in (6). If $G > 0$, then

$$q[G,a,x] = q[G, x + G^{-1}a] - q[G^{-1}, a],$$

and

$$\int_{\mathbf{R}^n} e^{-\frac{1}{2}q[G,a,x]}\, dx = (2\pi)^{\frac{n}{2}} |G|^{-\frac{1}{2}} e^{\frac{1}{2}q[G^{-1},a]}. \qquad \square$$

Lemma 13. Consider the quadratic functional

$$\begin{aligned} &r[x,y,u] \\ &\quad = x'\Pi x + y'\Psi y + u'\Phi u + 2u'\Theta y + 2u'\Xi x + 2y'\Omega x, \end{aligned}$$

in the variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$.

Assume that $\Phi < 0$ and $\Psi > 0$. Then:
(i).

$$\begin{aligned} &\min_u \int e^{-\frac{1}{2}r[x,y,u]}\, dy \\ &= (2\pi)^{p/2} |\Psi|^{-1/2} \times e^{-\frac{1}{2}\max_u \min_y r[x,y,u]}; \end{aligned}$$

(ii).

$$\begin{aligned} x'(\breve{\Pi} - \breve{\Xi}'\breve{\Phi}^{-1}\breve{\Xi})x &= \max_u \min_y r[x,y,u] \\ &= \min_y \max_u r[x,y,u] \\ &= x'(\hat{\Pi} - \hat{\Omega}'\hat{\Psi}^{-1}\hat{\Omega})x, \end{aligned}$$

where

$$\begin{aligned} \breve{\Pi} &= \Pi - \Omega'\Psi^{-1}\Omega, & \breve{\Xi} &= \Xi - \Theta\Psi^{-1}\Omega, \\ \breve{\Phi} &= \Phi - \Theta\Psi^{-1}\Theta', & \hat{\Pi} &= \Pi - \Xi'\Phi^{-1}\Xi, \\ \hat{\Omega} &= \Omega - \Theta'\Phi^{-1}\Xi, & \hat{\Psi} &= \Psi - \Theta'\Phi^{-1}\Theta. \end{aligned}$$

(iii). The optimizing u is given by

$$u^* = -\breve{\Phi}^{-1}\breve{\Xi}x.$$

Lemma 14. Let

$$\begin{aligned} \Pi &= \tilde{\Pi} - A'SA, & \Phi &= \tilde{\Phi} - B'SB, \\ \Psi &= \tilde{\Psi} - C'SC, & \Theta &= \tilde{\Theta} - B'SC, \\ \Xi &= \tilde{\Xi} - B'SA, & \Omega &= \tilde{\Omega} - C'SA. \end{aligned}$$

Then

$$\begin{aligned} \Pi - \Xi'\Phi^{-1}\Xi - (\Omega - \Theta'\Phi^{-1}\Xi)'(\Psi - \Theta'\Phi^{-1}\Theta)^{-1} \times \\ (\Omega - \Theta'\Phi^{-1}\Xi) = X - Y'Z^{-1}Y, \end{aligned}$$

where

$$\begin{aligned} X &= \tilde{\Pi} - \tilde{\Omega}'\tilde{\Psi}^{-1}\tilde{\Omega} - (\tilde{\Xi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Omega})' \times \\ &\quad (\tilde{\Phi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Theta}')^{-1}(\tilde{\Xi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Omega}), \\ Y &= A - C\tilde{\Psi}^{-1}\tilde{\Omega} - (B' - \tilde{\Theta}\tilde{\Psi}^{-1}C')' \times \\ &\quad (\tilde{\Phi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Theta}')^{-1}(\tilde{\Xi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Omega}), \\ Z &= S^{-1} - C\tilde{\Psi}^{-1}C' - (B' - \tilde{\Theta}\tilde{\Psi}^{-1}C')' \times \\ &\quad (\tilde{\Phi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Theta}')^{-1}(B' - \tilde{\Theta}\tilde{\Psi}^{-1}C'). \qquad \square \end{aligned}$$

Lemma 15. With the same notation as in Lemma 14,

$$(\Phi_k - \Theta_k\Psi_k^{-1}\Theta_k')^{-1}(\Xi_k - \Theta_k\Psi_k^{-1}\Omega_k) = \bar{Z}^{-1}\bar{Y},$$

where

$$\begin{aligned} \bar{Y} &= \tilde{\Xi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Omega} - (B - C\tilde{\Psi}^{-1}\tilde{\Theta}')' \times \\ &\quad (S^{-1} - C\tilde{\Psi}^{-1}C')^{-1}(A - C\tilde{\Psi}^{-1}\tilde{\Omega}), \\ \bar{Z} &= \tilde{\Phi} - \tilde{\Theta}\tilde{\Psi}^{-1}\tilde{\Theta}' - (B - C\tilde{\Psi}^{-1}\tilde{\Theta}')' \times \\ &\quad (S^{-1} - C\tilde{\Psi}^{-1}C')^{-1}(B - C\tilde{\Psi}^{-1}\tilde{\Theta}'). \qquad \square \end{aligned}$$