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Robust Fault-Tolerant Control for Systems with Extended Bounded-Sensor-Faults ⋆

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Abstract: In this paper an observer-based novel design of robust control system with an estimate scheme of sensor states to accommodate extended bounded-sensor-faults is proposed. The sensor faults taken into consideration are, in general, modeled as polytopic bounds in robust control framework and are usually given as *a priori* assumption. But, in practice, the sensors that are subject to fault are especially vulnerable to various conditions, such as temperature, humidity, etc. and, thus, their faults may fall outside the presumed polytopic bounds easily. An estimate scheme of sensor state is integrated into the observer-based control system where the sensor fault outside the presumed region is captured and, then, the notion of the well-known quadratic stability is used to stabilize the system, while, in the mean time, a robust performance measure of an output error signal is guaranteed in the presence of a set of extended admissible sensor faults. The effectiveness of the proposed approaches is shown by a numerical example.

1. INTRODUCTION

The problem of designing fault-tolerant control systems has attracted considerable attention, and a number of theoretical results and application examples have now been described in the literature; see (Feng, 2007; Yang, 2001; Stoustrup, 2004; Blanke et al., 2003; Zhou, 2001; Alwi, 2006; Liao, 2002; Veillette, 1995; Zhao, 1998; Kim, 2003; Tao, 2001; Jiang, 1994; Gao, 1991; Ye, 2006), for example. The approaches to fault-tolerant control are, in general, divided into two broad spectrums: *active* approach (Tao, 2001; Jiang, 1994; Gao, 1991; Ye, 2006) and *passive* approach (Feng, 2007; Yang, 2001; Stoustrup, 2004; Blanke et al., 2003; Zhou, 2001; Alwi, 2006; Liao, 2002; Veillette, 1995; Zhao, 1998; Kim, 2003). In the active approach, the reconfigurable mechanism of systems has been design in the event of bounded faults. Due to flexible capacity, the controller in such systems may not be in a fixed form. Thus increasing the complexity of controller is inevitable. In contrast, the passive fault-tolerant control is to exploit the inherent redundancy of the system components or to use the remaining functions of the component to design a fixed compensator so as to achieve a tolerable system performance in the presence of component faults. The designed fixed controller guarantees satisfactory system performance not merely during normal operations, but under various fault conditions.

In this paper we deal with extended bounded-sensor-faults in a passive form, where an observer-based controller with an integrated estimate scheme of sensor state is designed such that the system not merely is quadratically stable, but a robust performance measure of output error signals is guaranteed in the presence of a set of extended admissible sensor faults.

A basic idea to handle bounded sensor faults in an observer-based control system relies on the *plant states* being correctly estimated under the corrupted measurement signals. To reach this idea, Feng (2007) designed a novel observer that integrates an estimate scheme of sensor state into it. The preliminary results of asymptotic stability against sensor faults are stated in detail in Feng (2007), in which the sensor faults are restricted to be bounded and piece-wise constant function and which are somewhat unrealistic. We will, however, in this paper, relax the piece-wise constant restriction to allow the sensor faults to be time-varying and/or nonlinear with bounds, which in most cases are bounded in a polytopic fashion. But for some cases these presumed faulty sensors are vulnerable due to various conditions, such as temperature, humidity, etc., which their faults may fall outside the designed bounds easily. Thus, an estimate scheme of sensor state is integrated into the observer-based control system where the sensor fault outside the presumed region is captured. The difference of true sensor function, representing the remaining function that sensor has, and the estimate sensor function is assumed to be within a norm-bound, which results in an extended ball that contains the newly happened faults. The notion of quadratic stability will be used to ensure the stability and $\mathscr{L}_2$-gain measures the robust performance of an error signal in response to the signals produced by the difference of estimated sensor function and true sensor function.

It is worth noting that our goal as stated is to observe the plant states. Once we have the correct states we can design the control law accordingly. The results of estimate scheme are used to monitor the condition of sensors and to assist in observing the correct plant states. In this paper the sensor fault will not be identified and thus although we do not intend to find the true sensor functions, we, however, assume their differences can be bounded.

This paper is organized as follows. In Section 2, the system including observer structure and fault models of bounded piece-wise constant function are formulated. To proceed with Section 2, Section 3 gives the preliminary results developed in Feng (2007). Section 4 presents the main results, which deal with the robust performance against extended bounded-sensor-faults, which is divided into three subsections including system reformulation, quadratic stability and LMI (Linear matrix Inequality) characterizations. Section 5 includes the synthesis of observer and control gains in terms of LMIs. Section 6 demonstrates effectiveness of proposed method by a numerical example. Lastly, in the Section 7 we conclude the overall results shown in this paper.

2. PROBLEM FORMULATION

Consider a linear time-invariant dynamical system with sensor faults

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_1 d(t), \;\; x(0) = x_0 \\ y(t) = Cx(t) \\ y_s(t) = diag[y(t)]\phi(t) = diag[\phi(t)]y(t), \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state vectors and $u(t) \in \Re^m$ is the control signal of actuator. $y(t) \in \Re^l$ is the output of the system and $y_s(t) \in \Re^l$ is the true measured output of sensor. $d(t) \in \Re^d$ is the disturbance. The representation of $diag[y(t)]$, when the vector $y(t)$ is a vector with l components, is a square matrix of dimension l with the elements of $y(t)$ on the diagonal. The sensor function, $\phi(t) \in \Re^l$, is to represent the remaining function of the associated sensor. For example, if a sensor $\phi_k(t) = 0.8$, in which $\phi_k(t)$ represents the remaining function of k^{th} sensor in the vector $\phi(t)$, then we say the sensor is 80% functioning.

Now, consider a state observer with control law of the following form:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\big(y_s(t) - diag[\hat{\phi}(t)]C\hat{x}(t)\big) \\ u(t) = K\hat{x}(t) \end{cases} \tag{2}$$

where the vectors $\hat{x} \in \Re^n$ is the state of observer, which is an estimate of $x(t)$. Similarly, $\hat{\phi}(t)$, an estimate of $\phi(t)$, which will be shown later in the sequel, is useful to observe the state $x(t)$ in the presence of sensor faults. L and K

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are the observer gain and control gain to be designed such that the control objectives are achieved. Notice that the following expressions are interchangeable when later deriving the formulas, $diag[Cx(t)]\phi(t) = diag[\phi(t)](Cx(t))$ and $diag[C\hat{x}(t)]\phi(t) = diag[\phi(t)](C\hat{x}(t))$.

3. PRELIMINARIES

The following assumptions are used for demonstrating the asymptotic stability based on Lyapunov method shown in the Theorem 1, which will be then relaxed while $\mathscr{L}_2$-gain robust performance is pursued in the next section.

(1) $\phi_k(t) \in [0,\ 1]$.
(2) $\lim_{\Delta t \to 0} \frac{\Delta\phi_k(t_i)}{\Delta t} = 0$, where $\Delta\phi_k(t_i) = \Delta\phi_k(t_i + \Delta t) - \Delta\phi_k(t_i)$, except at some time instants that $\phi_k(t)$ jump toward zero.

The assumptions addressed above have the following interpretations: $\phi_k(t) = 0$ means the sensor fails. $\phi_k(t) = 1$ means the sensor works properly. A fault sensor will be such that $0 < \phi_k(t) < 1$. Thus it is a bounded sensor fault. The $\lim_{\Delta t \to 0} \frac{\Delta\phi_k(t_i)}{\Delta t} = 0$ and $\phi_k(t)$ jumping toward zero mean that the sensor fault not only is a piecewise constant process but also indicate that $\phi_k(t)$ is a bounded above function. The analysis of asymptotic stability of the closed-loop system against bounded and piece-wise constant sensor faults is first studied in Feng (2007), which is revealed in the following theorem and is stated for the completeness. We define the following sets,

$$\mathbf{\Phi} = \left\{ diag[\phi] \,\middle|\, \phi = (\phi_1^T \ \ \phi_2^T \ \ \cdots \ \ \phi_l^T)^T, \ \phi_k \in \Re,\ \phi_k \in [\,\underline{\phi}_k,\ \overline{\phi}_k\,] \right\}$$

and the vertex set of $\mathbf{\Phi}$ is defined as

$$\mathscr{B}\mathbf{\Phi} = \left\{ diag[\phi] \,\middle|\, \phi = (\phi_1^T \ \cdots \ \phi_l^T)^T, \ \phi_k = \underline{\phi}_k \text{ or } \phi_k = \overline{\phi}_k \right\} \tag{3}$$

Notice that it is easy to see that there are 2^l vertices in $\mathscr{B}\mathbf{\Phi}$ to represent the possible faults in a known convex set.

Theorem 1. Assumptions (1) and (2) hold. Consider the system (1) and (2) for the case $B_1 = 0$ and $diag[\phi] \in \mathscr{B}\mathbf{\Phi}$.

If there exist
(1) the matrices Q and L satisfying

$$\begin{cases} Q = Q^T > 0 \\ \Xi_1(Q) < 0, \end{cases} \tag{4}$$

where

$$\Xi_1(Q) = (A - L(diag[\phi])C)^T Q + Q(A - L(diag[\phi])C),$$

(2) the matrices P and K satisfying

$$\begin{cases} P = P^T > 0 \\ \Xi_2(P) < 0, \end{cases} \tag{5}$$

where

$$\Xi_2(P) = (A + BK)^T P + P(A + BK),$$

(3) for a given matrix $S > 0$, the matrices $W > 0$ and Γ satisfying

$$\begin{cases} \Gamma = \Gamma^T > 0 \\ W + W^T - S > 0, \end{cases} \tag{6}$$

then the closed-loop system:

$$\begin{cases} \dot{x} = Ax + BK\hat{x} \\ \dot{\hat{x}} = (A + BK - L(diag[\hat{\phi}])C)\hat{x} + Ly_s \\ \dot{\hat{\phi}} = \begin{cases} \Gamma\big(diag[C\hat{x}]\big)L^T Q\tilde{x} \ \text{ for } \ \hat{\phi} \in D \\ -\Gamma W\hat{\phi} + \Gamma\big(diag[C\hat{x}]\big)L^T Q\tilde{x} \ \text{ for } \ \hat{\phi} \in \overline{D} \end{cases} \end{cases} \tag{7}$$

is asymptotically stable for $\hat{\phi} \in D$, where D is as follows:

$$D = \left\{ \hat{\phi} \,\middle|\, \|\hat{\phi}\|^2 \le \frac{\overline{\lambda} l}{\underline{\rho}} \right\} \text{ and } \overline{D} = \left\{ \hat{\phi} \,\middle|\, \|\hat{\phi}\|^2 > \frac{\overline{\lambda} l}{\underline{\rho}} \right\}.$$

The parameters, $\overline{\lambda}$ and $\underline{\rho}$, are the maximum and minimum eigenvalues of $W^T S^{-1} W$ and $W + W^T - S$, respectively, for some positive definite symmetric matrices W and S and $W + W^T - S > 0$.

Proof. Refer to Feng (2007) for the proof.

4. ROBUST PERFORMANCE

In the last section the asymptotic stability, based on Lyapunov method, has been shown for the system with bounded sensor faults under piecewise constant assumption. Now, we will turn our focus on the system (7) where the robust performance is studied for a set of extended admissible sensor faults. Notice that, from now on, we will relax the piecewise constant assumption and not only admit bounded time varying and/or nonlinear sensor function in the vertex set, $\mathscr{B}\mathbf{\Phi}$, but also let the true sensor faults, which may fall outside the presumed bound, be norm-bound. A closed-loop autonomous system will be reformulated in the following subsection.

4.1 System Reformulation

Consider closed-loop system (7), which can be rewritten as

$$\begin{cases} \dot{x}_c = A_c x_c + B_c w, \ \ x_c(0) \in \mathscr{B} \\ q = C_q x_c \\ e = C_e x_c \\ w = diag[\tilde{\phi}]q, \end{cases} \tag{8}$$

where

$$x_c = \begin{pmatrix} \tilde{x} \\ x \\ \hat{\phi} \end{pmatrix}, \ B_c = \begin{pmatrix} -L \\ 0 \\ 0 \end{pmatrix},$$

$$C_q = \begin{pmatrix} C & C & 0 \end{pmatrix}, \qquad C_e = \begin{pmatrix} C & 0 & 0 \end{pmatrix},$$

$$A_c = \begin{pmatrix} A - L(diag[\phi])C & 0 & 0 \\ BK & A + BK & 0 \\ \Gamma(diag[C\hat{x}])L^T X_1 & 0 & -\Gamma W \end{pmatrix},$$

for $\tilde{x} = \hat{x} - x$, $diag[\phi] \in \mathscr{B}\mathbf{\Phi}$, and $diag[\tilde{\phi}] = diag(\tilde{\phi}_1,\ \cdots,\ \tilde{\phi}_l)$. We have used the case where $\hat{\phi} \in \overline{D}$ for the representation purpose in (8). The matrices K, L, Γ, and X_1 are to be determined in the sequel. The signal, w, is a bounded exogenous signal that comprises $\tilde{\phi}$ and $C\hat{x}$, i.e. $w = diag[\tilde{\phi}]C\hat{x}$, where $\tilde{\phi}$ is the difference of estimated sensor function, $\hat{\phi}$, and true sensor function, ϕ, i.e. $\tilde{\phi} = \hat{\phi} - \phi$. The estimation error $\tilde{\phi}(t)$ is assumed to belong to the following diagonal norm-bound set:

$$\mathbf{\Delta} \triangleq \left\{ \tilde{\phi}_i(t) \,\middle|\, |\tilde{\phi}_i(t)| \le 1,\ i = 1, \cdots, l \right\},$$

which defines the extended sensor faults.

Remark 2. We have noticed that if the true sensor function falls within the presumed polytopic bound, $\mathscr{B}\mathbf{\Phi}$, then the diagonal norm-bound set, which is confined within $\mathbf{\Delta}$, are partially (or maybe completely) overlaid with the $\mathscr{B}\mathbf{\Phi}$. However, if the true sensor function does not fall into the set $\mathscr{B}\mathbf{\Phi}$, then it is possible that $\mathscr{B}\mathbf{\Phi}$ intersects $\mathbf{\Delta}$ partially or their intersection is an empty set.

Remark 3. It is worth noting that, under the above assumption, the signal w is such that

$$w^T w = \hat{x}^T C^T (diag[\tilde{\phi}])^T (diag[\tilde{\phi}])C\hat{x} \le \hat{x}^T C^T C\hat{x},$$

which for asymptotically stable systems can be bounded above and will be demonstrated in the sequel.

4.2 Robust Performance Measure

In this subsection, we shall define a robust performance measure, which is used in the rest of the paper and also state an important theorem on which the robust performance is established. We assume that not all state information is available and we are concerned with designing a fixed structure of observer-based controller to stabilize the system (8) with a given $\mathscr{L}_2$-gain constraint of an error signal in respond to an exogenous signal for all admissible sensor faults.

We will be concerned with the following notion of stabilizability for the system (8).

Definition 4. Let the constant $\gamma > 0$ be given. The system (8) is said to be stabilizable with a robust $\mathscr{L}_2$-gain measure γ if there exists a fixed control law $u = K\hat{x}$ and observer gain L such that for any admissible sensor faults the following conditions are satisfied.
(1) The system is uniformly asymptotically stable.
(2) Subject to the assumption of zero initial condition, the controlled output e satisfies

$$\frac{\|e\|_2}{\|w\|_2} \le \gamma.$$

Here, we will also use the notion of quadratic stability with an $\mathscr{L}_2$-gain measure which was introduced in Boyd (1994). This concept is a generalization of that of quadratic stabilization to handle $\mathscr{L}_2$-gain measure constraint on exogenous attenuation. To this end, the characterizations of robust performance based on quadratic stability will be given in terms of matrix inequalities, where if LMIs can be found then the computations by finite dimensional convex programming are efficient.

Theorem 5. Consider the closed-loop system (8), the following statement hold: the closed-loop system is said to be quadratically stable with a robust $\mathscr{L}_2$-gain measure γ from input w to output e if there exists $X > 0$ and $\lambda \geq 0$ such that

$$\Pi < 0, \tag{9}$$

where

$$\Pi = \begin{pmatrix} A_c^T X + XA_c + C_e^T C_e + \lambda C_q^T C_q & XB_c \\ B_c^T X & -\gamma^2 I \end{pmatrix}. \tag{10}$$

Proof. Let quadratic Lyapunov function be $V(x_c) = x_c^T X x_c^T$, with $X > 0$ such that

$$\forall\, x_c, \text{ and } w \text{ satisfying } (8),$$

$$\frac{d}{dt}V(x_c) + e^T e - \gamma_1^2 w^T w < 0, \tag{11}$$

and constraint

$$w^T w \leq x_c^T C_q^T C_q x_c.$$

Then, it follows from the S-procedure that the equivalent condition of (11) is the existence of $\lambda \geq 0$ satisfying

$$\frac{d}{dt}V(x_c) + e^T e - \gamma_1^2 w^T w + \lambda(x_c^T C_q^T C_q x_c - w^T w) \leq 0,$$

which can be equivalently written as

$$\frac{d}{dt}V(x_c) + \lambda x_c^T C_q^T C_q x_c + e^T e - \gamma^2 w^T w \leq 0 \tag{12}$$

for $\gamma^2 = \gamma_1^2 + \lambda$. Then the $\mathscr{L}_2$-gain of the (8) is less than γ. To show this, we integrate (12) from 0 to T, with the initial condition $x_c(0) = 0$, to get

$$V(x_c(T)) + \lambda \int_0^T x_c^T C_q^T C_q x_c dt + \int_0^T (e^T e - \gamma^2 w^T w) dt \leq 0. \tag{13}$$

Since $V(x_c(T)) + \lambda \int_0^T x_c^T C_q^T C_q x_c dt \geq 0$, this implies

$$\frac{\|e\|_2}{\|w\|_2} \leq \gamma. \tag{14}$$

The inequality (9) and definition (10) are obtained by substituting (8) into (12). Without loss of generality, we will adopt only strict inequality in this paper. This completes the proof.

4.3 Matrix Inequality Characterizations

The following lemma is to show that the energy of the estimated output signals by observer can be limited by some matrix inequalities, which can actually provide an upper bound of the exogenous signal, w.

Lemma 6. Given

$$\begin{cases} \dot{x}_c = A_c x_c, \ x_c(0) \in \mathscr{B}, \\ \hat{y}_i = C_{q,i} x_c \\ \mathscr{B} = \left\{ x_c \middle| x_c^T X x_c \leq \nu \right\}, \ X > 0, \ \nu > 0 \end{cases} \tag{15}$$

If there exist $X > 0$ and $\theta_i > 0$ satisfying

$$A_c^T X + XA_c < 0$$

$$\Upsilon_i > 0, \ i = 1, \cdots, l, \tag{16}$$

then the following statements are equivalent,
(1) $\hat{y}_i^2 < \frac{\theta_i}{\nu}$,
(2) $\hat{\Theta} > 0$, where

$$\Upsilon_i = \begin{pmatrix} X & C_{q,i}^T \\ C_{q,i} & \frac{\theta_i}{\nu} I \end{pmatrix}, \tag{17}$$

$$\hat{\Theta} = (diag[C\hat{x}])(-\nu I)(diag[C\hat{x}]) + \Theta, \tag{18}$$

$$\Theta = diag(\theta_1, \cdots, \theta_l).$$

Proof. $\mathscr{B}$ is an invariant ellipsoid: Let $V(x_c) = x_c^T X x_c$. Since

$$\dot{V}(x_c) = x_c^T \left(A_c^T X + XA_c \right) x_c < 0$$

then we have

$$V(x_c(t)) \leq V(x_c(0)) \leq \nu.$$

Hence, $x_c(t) \in \mathscr{B}, \ \forall t \geq 0$ and

$$\hat{y}_i^2 \leq \max_{x_c \in \mathscr{B}} x_c^T C_{q,i}^T C_{q,i} x_c. \tag{19}$$

Let $\theta_i = \max_{x_c \in \mathscr{B}} x_c^T C_{q,i}^T C_{q,i} x_c$. We have

$$\begin{pmatrix} X & C_{q,i}^T \\ C_{q,i} & \frac{\theta_i}{\nu} I \end{pmatrix} > 0, \ \Leftrightarrow \ C_{q,i}^T C_{q,i} < \frac{\theta_i}{\nu} X, \ \forall i \tag{20}$$

From (19) and (20), we have

$$\hat{y}_i^2 \leq \max_{x_c \in \mathscr{B}} x_c^T C_{q,i}^T C_{q,i} x_c < \frac{\theta_i}{\nu}, \ \forall i. \tag{21}$$

This completes the proof of (1). The equivalence of (1) and (2) is straightforward.

Before stating the main theorem for the robust $\mathscr{L}_2$-gain measure γ of the closed-loop system (8), which ensures the robust performance of the original system (1) and (2) against sensor faults, the following matrices are defined:

$$\Pi_1 = \begin{pmatrix} \Xi_2(X_2) + \lambda C^T C & X_2 BK + \lambda C^T C & 0 \\ (BK)^T X_2 + \lambda C^T C & (\star) & -X_1 L \\ 0 & -L^T X_1 & -\gamma^2 I \end{pmatrix}, \tag{22}$$

$$\Pi_2 = \begin{pmatrix} -(\Gamma W)^T X_3 - X_3(\Gamma W) + 2\kappa X_3 & X_3 \Gamma \\ \Gamma^T X_3 & -\nu I \end{pmatrix}, \tag{23}$$

where

$$(\star) = \Xi_1(X_1) + (\lambda + 1) C^T C + (X_1 L)\Theta(L^T X_1).$$

Theorem 7. Let the $\gamma > 0$, $\kappa > 0$, and $\nu > 0$ be given. The closed-loop system (8) with the admissible bounded sensor faults is said to be quadratically stable with a robust $\mathscr{L}_2$-gain measure γ, if, let the matrix X be in the set χ

$$\chi = \left\{ block \ diag(X_1, X_2, X_3) | X_i = X_i^T > 0 \right\},$$

there exist K, L, $\Gamma > 0$, $\Theta > 0$, and $X \in \chi$ such that $\Pi_1 < 0$, $\Pi_2 < 0$, and $\Upsilon_i > 0, \ i = 1, \cdots, l$.

Proof. We consider the signals $e(\cdot)$ in response to the signals $w(\cdot)$ with the zero initial states, and a (candidate) quadratic Lyapunov function $V(x_c) \triangleq x_c^T X x_c$. For any nonzero vectors, ς_1 and ς_2, are defined as

$$\varsigma_1 = \begin{pmatrix} x \\ \tilde{x} \\ w \end{pmatrix} \text{ and } \varsigma_2 = \begin{pmatrix} \hat{\phi} \\ diag[C\hat{x}] L^T X_1 \tilde{x} \end{pmatrix}.$$

We have

$$\begin{aligned} \varpi &= \varsigma_1^T \Pi_1 \varsigma_1 + \varsigma_2^T \Pi_2 \varsigma_2 \\ &= \begin{pmatrix} \dot{V}(x_c) + \lambda q^T q + e^T e - \gamma^2 w^T w \\ \qquad + \tilde{x}^T (X_1 L) \hat{\Theta} (L^T X_1) \tilde{x} + \hat{\phi}^T (2\kappa X_3) \hat{\phi} \end{pmatrix}, \end{aligned} \tag{24}$$

where $\hat{\Theta}$ is defined in (18). To prove statement (1), let $\zeta = \begin{pmatrix} x_c \\ w \end{pmatrix}$, thus

$$\varpi = \zeta^T \Pi \zeta + \tilde{x}^T (X_1 L) \hat{\Theta} (L^T X_1) \tilde{x} + \hat{\phi}^T (2\kappa X_3) \hat{\phi},$$

where Π is defined by (10). If the matrix inequalities, $\Pi_1 < 0$, $\Pi_2 < 0$, and $\Upsilon_i > 0$, are satisfied, we have $\varpi < 0$, or equivalently,

$$\zeta^T \Pi \zeta < -\tilde{x}^T (X_1 L) \hat{\Theta} (L^T X_1) \tilde{x} - \hat{\phi}^T (2\kappa X_3) \hat{\phi} < 0, \tag{25}$$

for nonzero $\tilde{x}$ and $\hat{\phi}$, which implies $\Pi < 0$. Hence, by Theorem 5, the closed-loop system is quadratically stable with a robust $\mathscr{L}_2$-gain measure γ is ensured. This completes the proof.

Remark 8. It is highlighted that the κ is defined to be the decay rate of the estimated sensor function $\hat{\phi}$ shown in (7) such that $\lim_{t \to \infty} e^{\kappa t} \|\hat{\phi}\| = 0$, when $\hat{\phi} \in \overline{D}$.

Before presenting the synthesis results in the next section, a useful and important lemma will be stated for clarity:

Lemma 9. (Elimination Lemma). (Boyd, 1994) Given $\mathscr{H} = \mathscr{H}^T \in \Re^{n\times n}$, $\mathscr{V} \in \Re^{n\times m}$, and $\mathscr{U} \in \Re^{n\times p}$ with Rank$(\mathscr{V}) < n$ and Rank$(\mathscr{U}^T) < n$. There exists a matrix K such that

$$\mathscr{H} + \mathscr{V}K\mathscr{U}^T + \mathscr{U}K^T\mathscr{V}^T < 0$$

if and only if

$$\mathscr{V}_\perp^T \mathscr{H} \mathscr{V}_\perp < 0 \text{ and } \mathscr{U}_\perp^T \mathscr{H} \mathscr{U}_\perp < 0,$$

where $\mathscr{V}_\perp$ and $\mathscr{U}_\perp$ are orthogonal complement of $\mathscr{V}$ and $\mathscr{U}$, respectively, i.e. $\mathscr{V}_\perp^T\mathscr{V} = 0$ and $(\mathscr{V}_\perp \;\; \mathscr{V})$ is of maximum rank.

5. CONTROL AND OBSERVER GAIN SYNTHESIS

In this section according to the analyzed results shown in the last section, the observer gain, L, and control gain, K, will be synthesized. The general LMI synthesis problem involves sets of the form $X \in \chi$ and a list of matrices A, B, C, L, K, Γ, and W and scalars ν, κ, and θ_i. We will conclude the quadratic stability with a robust $\mathscr{L}_2$-gain measure γ control problem in an convex optimization fashion. We will also specify the details in the following and the results will be concluded in an algorithm of computation.

Assume $diag[\phi] \in \mathscr{B}\mathbf{\Phi}$ and $\tilde{\phi}_i \in \mathbf{\Delta}$, $\forall i$. Given pre-specified matrices A, B, C, $W > 0$ and scalars $\nu > 0$ and $\kappa > 0$. If, according to Theorem 7, there exist matrices $X \in \chi$, L, K, Γ, $\Theta > 0$, and γ^2 such that $\Pi_1 < 0$, $\Pi_2 < 0$, and $\Upsilon_i > 0$, $\forall\, i$, are satisfied, then the closed-loop system is quadratically stable with a robust $\mathscr{L}_2$-gain measure γ. Let the matrix Π_1 be decomposed into

$$\Pi_1 = \Pi_{1\_1} + \Pi_{1\_2} < 0, \tag{26}$$

where

$$\Pi_{1\_1} = \begin{pmatrix} \Xi_2(X_2) & X_2BK & 0 \\ (BK)^TX_2 & (\star\star) & -X_1L \\ 0 & -L^TX_1 & -\gamma^2 I \end{pmatrix},$$

$$\Pi_{1\_2} = \lambda \begin{pmatrix} C^T \\ C^T \\ 0 \end{pmatrix} \begin{pmatrix} C & C & 0 \end{pmatrix},$$

$$(\star\star) = \Xi_1(X_1) + C^TC + (X_1L)\Theta(L^TX_1).$$

Thus,

$$\varsigma_1^T\Pi_1\varsigma_1 = \varsigma_1^T\Pi_{1\_1}\varsigma_1 + \varsigma_1^T\Pi_{1\_2}\varsigma_1.$$

It is noted that

$$\varsigma_1^T\Pi_{1\_2}\varsigma_1 = \lambda q^Tq \ge 0 \tag{27}$$

for $\lambda \ge 0$. Next, by S-procedure, the requirement of $\varsigma_1^T\Pi_1\varsigma_1 < 0$ is equivalent to

$$\varsigma_1^T\Pi_{1\_1}\varsigma_1 < 0,$$

which implies $\Pi_{1\_1} < 0$. Hence, by the well known Schur Complement, the matrix $\Pi_{1\_1} < 0$ can be rewritten as

$$\begin{pmatrix} \Xi_2(X_2) & X_2BK & 0 & 0 \\ (BK)^TX_2 & \Xi_1(X_1)+C^TC & -X_1L & X_1L \\ 0 & -L^TX_1 & -\gamma^2I & 0 \\ 0 & L^TX_1 & 0 & -\Theta^{-1} \end{pmatrix} < 0. \tag{28}$$

The inequality (28) can be again rewritten as

$$\mathscr{H} + \mathscr{V}K\mathscr{U}^T + \mathscr{U}K\mathscr{V}^T < 0, \tag{29}$$

where

$$\mathscr{H} = \begin{pmatrix} A^TX_2+X_2A & 0 & 0 & 0 \\ 0 & \Xi_1(X_1)+C^TC & -X_1L & X_1L \\ 0 & -L^TX_1 & -\gamma^2I & 0 \\ 0 & L^TX_1 & 0 & -\Theta^{-1} \end{pmatrix}, \tag{30}$$

$$\mathscr{V} = \begin{pmatrix} X_2B \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \mathscr{U} = \begin{pmatrix} I \\ I \\ 0 \\ 0 \end{pmatrix}. \tag{31}$$

Next, the orthogonal complement of $\mathscr{V}$ and $\mathscr{U}$ is given by

$$\mathscr{V}_\perp = \begin{pmatrix} X_2^{-1}B_\perp & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \text{ and } \mathscr{U}_\perp = \begin{pmatrix} I & 0 & 0 \\ -I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \tag{32}$$

which $B_\perp$ is defined as the orthogonal complement of B and is such that $B_\perp^TB = 0$ and $[B \;\; B_\perp]$ is of maximum rank. Similarly, $\mathscr{V}$ and $\mathscr{U}$ are defined as orthogonal complement of V and U, respectively.

It is followed by using the well known Elimination Lemma stated in Lemma 9 that the matrix variable, K, can be eliminated from the inequality (29), which is equivalent to the following two inequalities,

$$\mathscr{V}_\perp^T\mathscr{H}\mathscr{V}_\perp < 0 \text{ and } \mathscr{U}_\perp^T\mathscr{H}\mathscr{U}_\perp < 0. \tag{33}$$

Substituting (30) and (32) into (33), we have

$$\mathscr{V}_\perp^T\mathscr{H}\mathscr{V}_\perp = \begin{pmatrix} B_\perp^T(X_2^{-1}A^T+AX_2^{-1})B_\perp & 0 & 0 & 0 \\ 0 & \Xi_1(X_1)+C^TC & -X_1L & X_1L \\ 0 & -L^TX_1 & -\gamma^2I & 0 \\ 0 & L^TX_1 & 0 & -\Theta^{-1} \end{pmatrix} < 0, \tag{34}$$

which is equivalent to

$$B_\perp^T(X_2^{-1}A^T+AX_2^{-1})B_\perp < 0, \tag{35}$$

$$\begin{pmatrix} \Xi_1(X_1)+C^TC & -X_1L & X_1L \\ -L^TX_1 & -\gamma^2I & 0 \\ L^TX_1 & 0 & -\Theta^{-1} \end{pmatrix} < 0, \tag{36}$$

and

$$\mathscr{U}_\perp^T\mathscr{H}\mathscr{U}_\perp = \begin{pmatrix} A^TX_2+X_2A+\Xi_1(X_1)+C^TC & X_1L & -X_1L \\ L^TX_1 & -\gamma^2I & 0 \\ -L^TX_1 & 0 & -\Theta^{-1} \end{pmatrix} < 0. \tag{37}$$

It is noted that (35), (36), and (37) can not be solved simultaneously using LMI Toolbox of Matlab due to its non-convexity in matrix variable, X_2. We therefore propose that X_2^{-1} of (35) be solved first. It is easy to find X_2 using X_2^{-1}. If we let $\hat{L} = X_1L$, then (36) and (37) are LMIs in variables, X_1, $\hat{L}$, γ^2, and Θ, which can be solved simultaneously by the LMI Toolbox of Matlab. The rest LMIs considered are $\Pi_2 < 0$ for a pre-specified decaying rate, $\kappa > 0$, $\nu > 0$, and matrix $W > 0$, we have

$$\begin{pmatrix} -\hat{\Gamma}W - W^T\hat{\Gamma}^T + 2\kappa X_3 & \hat{\Gamma} \\ \hat{\Gamma}^T & -\nu I \end{pmatrix} < 0,\; X_3 > 0, \tag{38}$$

where $\hat{\Gamma} = X_3\Gamma$. And lastly for $\Upsilon_i > 0$ using Schur Complement, we equivalently have

$$X - \left(\frac{\nu}{\theta_i}\right) C_{q,i}^TC_{q,i} > 0,\;\; i = 1, \cdots, l, \tag{39}$$

where $\Theta = diag(\theta_1, \cdots, \theta_l)$.

Remark 10. (Step of computation). Now, we can summarize the step of computation.
(1) Find feasible solutions of $X_2^{-1} > 0$ and thus X_2 where LMI (35) is satisfied.
(2) Use the computed matrices X_2 found in step 1), the robust $\mathscr{L}_2$-gain performance problem is placed as the following optimization problem:

$$\begin{aligned} &\text{minimize } \gamma^2 \\ &\text{subject to } (36),\ (37),\ (38), \text{ and } (39) \\ &\qquad X_1 > 0, X_3 > 0, \Theta > 0. \end{aligned} \tag{40}$$

(3) Reconstruct control gain, K, the matrices found in step (1) and (2) are substituted into the inequality (29), which is LMI on one matrix variable, K and can be solved by LMI Toolbox of Mtalab.

6. NUMERICAL EXAMPLE

This example adopted from Yang (2001) will be used to illustrate the proposed design with the following parameters:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix},\; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We will test three cases to verify the superiority of the designed observer-based controller. First, case #1: one sensor fault where the conditions for simulation are similar to those in Yang (2001), in which the sensor y_1 is perfectly normal and sensor y_2 is subject to faults with 50% reduction in signal strength. That is, by Theorem 1, ϕ_1 is equal to 1 and ϕ_2 is allowed to be time varying or nonlinear function and varies between 0 and 0.5, i.e. $0 \le \phi_2 \le 0.5$. In the simulation we let $\phi_2 = 0.25 + 0.1\sin(10t)$ to examine the design. Using Matlab LMI Control toolbox and following the step of computation in Remark 10, the matrix, X_2, is computed:

$$X_2 = \begin{pmatrix} 1.5829 & 1.6959 & 0.9610 \\ 1.6959 & 3.8158 & 1.6959 \\ 0.9610 & 1.6959 & 1.5829 \end{pmatrix}.$$

Then we solve optimization problem proposed by (40). We find

$$X_1 = 10^7 \begin{pmatrix} 9.3574 & -7.9408 & -1.4166 \\ -7.9408 & 8.8721 & -0.9313 \\ -1.4166 & -0.9313 & 2.3478 \end{pmatrix}, L = \begin{pmatrix} 20.0010 & 0.0263 \\ 20.0010 & 0.0263 \\ 20.0010 & 0.0263 \end{pmatrix}$$

$$K = \begin{pmatrix} -7.0000 & -1.7000 & 1.3000 \end{pmatrix}$$

and the optimal value of γ^2 is 56.0603. The control gain K places the eigenvalues of matrix $A+BK$ at {-1.00, -0.85$\pm j$0.5268}. It is of interest that the eigenvalues of $A - L(diag[\phi])C$ are placed around $\{-19, -2, -3\}$ for all $\phi_2 = 0.25 + 0.1\sin(10t)$. It is highlighted that $L(diag[\phi])C$ intend to closely map the fault sensor signal into the its null space. This claim can be verified by viewing,

$$L(diag[\phi])C = \begin{pmatrix} 20.0010\phi_1 & 0.0263\phi_2 & 0 \\ 20.0010\phi_1 & 0.0263\phi_2 & 0 \\ 20.0010\phi_1 & 0.0263\phi_2 & 0 \end{pmatrix},$$

where the effectiveness of ϕ_2 on the eigenvalues of $A - L(diag[\phi])C$ has been greatly reduced. Fig.1 and Fig.2 show the complete simulation results. In Fig.1, the observed states, $\hat{x}$, converge to the plant states, x, eventually. In Fig.2, it is noted that the estimate sensor function $\hat{\phi}_1$ approaches very closely to its true value. However, for fast variation of sensor function ϕ_2 the estimated sensor function, $\hat{\phi}_2$ does not follow it but stay at certain value as $t \to \infty$. We must highlight that the estimate scheme of sensor function may not be able to approach the true sensor function, see Astrom (1994) for detail, if lacking of persistent excitation, but to keep the the difference of estimated signal and true sensor function within a bound. It is seen from Fig7(a) that the $\tilde{\phi}_i \in \mathbf{\Delta}$, $i = 1,2$. In Fig.2, the control input, u, is also shown.

The second simulation, case #2, uses the condition, which is similar to the previous case, where true sensor function $\phi_1 = 1$ and the true sensor function ϕ_2 is not in the designed interval, i.e. $\phi_2 \notin \mathscr{B}\mathbf{\Phi}$, which makes $\phi_2 > 0.5$ or $\phi_2 < 0$. In the simulation, we let true sensor function, $\phi_2 = 1.5 + 0.1\sin(10t)$. The simulation results can actually be predicted and are similar to the previous case since the computed observer gain L closely maps the second sensor signal into its null space. It is easy to predict that the states, x and $\hat{x}$, will be similar to Fig.1 and is shown in Fig.3. In Fig.4, the true sensor functions, ϕ_1 and ϕ_2, estimated sensor functions, $\hat{\phi}_1$ and $\hat{\phi}_2$, and control signals, u are depicted. It is shown that although the estimated signal, $\tilde{\phi}_2$ is within the diagonal norm-bound, $\mathbf{\Delta}$, which is shown in Fig7(b), its results are deteriorated by the poor guess of its initial states.

The third simulation, i.e. case #3, is to show two sensor faults. We allow the true sensor function ϕ_1, shown in Fig.6, to be varying in the pulse form between 0.4 and 0.7 for the first 0.8 second and then $\phi_1 = 0.6 + 0.03\sin(10t)$, $t \geq 0.8$, while keeping $\phi_2 = 0.25 + 0.1\sin(10t)$ for $t \geq 0$. To compute K and L, we assume the polytopic bound with $0.4 \leq \phi_1 \leq 1$ and $0 \leq \phi_2 \leq 0.5$. We surprisingly found $K = \begin{pmatrix} -6.9980 & -1.6965 & 1.3033 \end{pmatrix}$, which is close to the one sensor fault case, and $L^T = \begin{pmatrix} 199.9852 & 199.9852 & 199.9852 \\ 0.0333 & 0.0333 & 0.0333 \end{pmatrix}$. Under two sensor fault case we have raised the $\gamma^2 = 222.5654$. It is not surprising to have the result since the signals for feedback is extremely week. We have shown the comparison of the true states and observed states in Fig.5, where the large ripples of the observed states are produced due to the pulses of the true sensor function in the first 0.8 second but soon died out. In Fig.6, the control input, the true sensor signals and estimate sensor signals are depicted. We recognize again from Fig.6 and Fig.7 that although we assume that $\tilde{\phi}_i \in \mathbf{\Delta}$, $i = 1,2$, it really depends on the good guess of initial states on the estimate scheme due to viewing the fact that $\hat{\phi}_2$ maintains at its initial states. This, however, can be understood that the associated sensor function is not subject to persistent excitation (Astrom, 1994) since it is mapped closely to its null space.

7. CONCLUSION

This paper has developed an observer-based robust control system with an estimate scheme of sensor states against extended bounded-sensor-faults. In this design the control system not only can deal with the sensor fault in a prescribed polytopic bound but also can endure the faults outside the bound. Based on the notion of quadratic stability with a robust $\mathscr{L}_2$-gain measure γ, sufficient conditions for the solvability of the robust control problem have been obtained and a complete solution was given in terms of LMIs. The numerical example shows the effectiveness of the designed method.

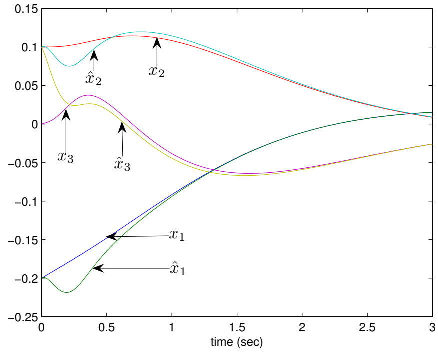

Fig. 1. Case #1: the comparison of the plant states, x, and observed states, $\hat{x}$, of one sensor fault for the true sensor function $\phi_2 \in \mathscr{B}\mathbf{\Phi}$.

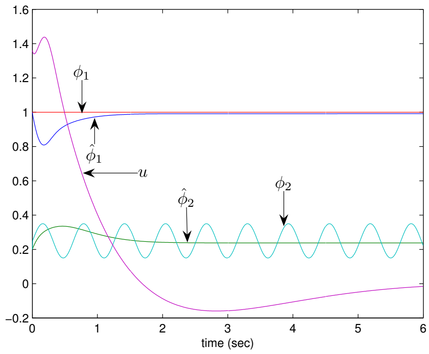

Fig. 2. Case #1: true sensor functions, $\phi_1 = 1$ and $\phi_2 = 0.25 + 0.1\sin(10t)$, and estimated sensor functions, $\hat{\phi}_1$ and $\hat{\phi}_2$, for $\phi_2 \in \mathscr{B}\mathbf{\Phi}$. The control input, u, is also shown.

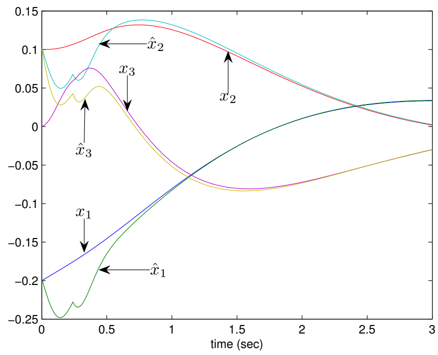

Fig. 3. Case #2: the comparison of the plant states, x, and observed states, $\hat{x}$, of one sensor fault for the true sensor function $\phi_2 \notin \mathscr{B}\mathbf{\Phi}$.

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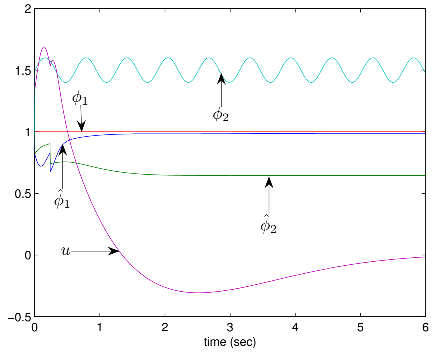

Fig. 4. Case #2: true sensor functions, $\phi_1 = 1$ and $\phi_2 = 1.5 + 0.1\sin(10t)$, and estimated sensor functions, $\hat{\phi}_1$ and $\hat{\phi}_2$, for $\phi_2 \notin \mathscr{B}\mathbf{\Phi}$. The control input, u, is also shown.

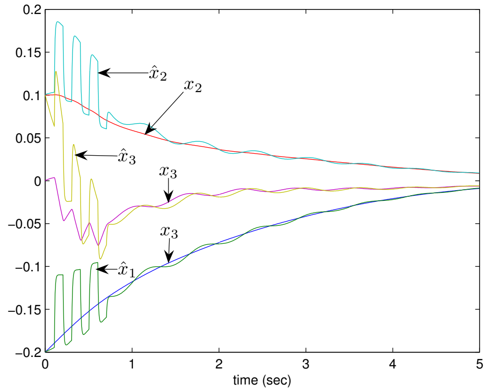

Fig. 5. Case #3: the comparison of the plant states, x, and observed states, $\hat{x}$, of two sensor fault for the true sensor function $\phi_i \in \mathscr{B}\mathbf{\Phi}$, $i = 1, 2$.

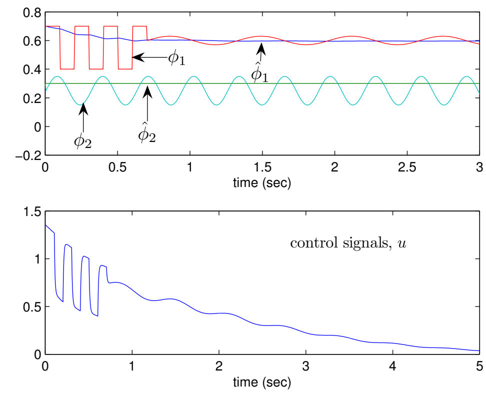

Fig. 6. Case #3: the true sensor functions, ϕ_1=pulses shown in the upper figure for $t \le 0.8$ and $\phi_1 = 0.6 + 0.03\sin(10t)$, $t \ge 0.8$ and $\phi_2 = 0.25 + 0.1\sin(10t)$, and the estimated sensor functions, $\hat{\phi}_1$ and $\hat{\phi}_2$. The control input, u, is also shown.

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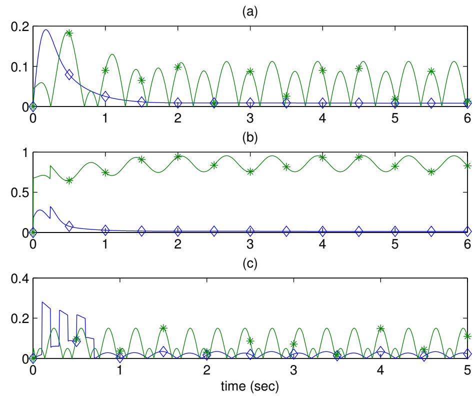

Fig. 7. Solid-diamond line represents $|\tilde{\phi}_1|$ and solid-star line represents $|\tilde{\phi}_2|$. (a), (b), and (c) are associated with simulation case 1, 2, and 3, respectively.

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