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A Parametric Lyapunov Equation Approach to Low Gain Feedback Design for Discrete-time Systems ⋆

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Abstract: Low gain feedback, a parameterized family of stabilizing state feedback gains whose magnitudes approach zero as the parameter decreases to zero, has found several applications in constrained control systems, robust control and nonlinear control. In the continuous-time setting, there are currently three ways of constructing low gain feedback laws the eigenstructure assignment approach, the parametric ARE based approach and the parametric Lyapunov equation based approach. The eigenstructure assignment approach leads to feedback gains explicitly parameterized in the low gain parameter. The parametric ARE based approach results in a Lyapunov function along with the feedback gain, but requires the solution of an ARE for each value of the parameter. The parametric Lyapunov equation based approach possesses the advantages of the first two approaches and results both an explicitly parameterized feedback gains and a Lyapunov function. The first two approaches have been extended to discrete-time setting. This paper develops the parametric Lyapunov equation based approach to low gain feedback design for discrete-time systems.

Keywords: Pole shift, circle-symmetry, parametric Lyapunov matrix equation, low gain feedback, actuator saturation, semi-global stabilization.

1. INTRODUCTION

Low gain feedback was first proposed in continuous-time setting in Lin & Saberi (1993) to achieve semi-global stabilization for linear systems under actuator saturation. Low gain feedback refers to a family of stabilizing state feedback gains, parameterized in a scalar, that tend to zeros as the parameter approaches zero. A key feature of the low gain feedback as constructed in Lin & Saberi (1993) is that, for a given stabilizable linear system with all its open loop poles in the closed left-half plane and its initial state in an arbitrarily large bounded set, the peak magnitude of the low gain feedback control goes to zero as the low gain parameters approaches zero. As a result, for such a linear system, actuator saturation can be avoided by decreasing the value of the low gain parameter as long as the initial state lies in a bounded, but arbitrarily large, set of the state space. In other words, a linear system subject to actuator saturation is semi-globally stabilizable by linear low gain feedback. Low gain feedback has also found applications in solving several other problems in robust control and nonlinear control (Lin (1999)).

The low gain feedback constructed in Lin & Saberi (1993) is based on an eigenstructure assignment algorithm and the resulting feedback gains are explicitly parameterized in the low gain parameter. Alternative approaches to the low gain feedback design were later developed based on the solution of parametric $\mathcal{H}_2$ algebraic Riccati equation (ARE) and $\mathcal{H}_\infty$ ARE in Lin et al. (1996) and Teel (1995a), respectively.

Both the eigenstructure assignment approach and the ARE-based approach have their own advantages. The biggest advantage of the eigenstructure assignment approach is that it results in feedback gains that are matrix polynomial matrix in the low gain parameter. Thus the design is non-repetitive in the sense that if the value of the low gain parameter required to change, the design process need not be repeated. The ARE-based approach is however conceptually appealing and directly results in a Lyapunov function along with the feedback gain. However, the resulting feedback gain is indirectly dependent on the low gain parameter. For every different values of the low gain parameter, the solution of a new ARE is required. The solution of these AREs may become numerically ill-conditioned as the value of the low gain parameter becomes small. This is the case, for example, when the value of the low gain parameter is adjusted on line to achieve

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global results, instead of semi-global ones (Teel (1995b); Suarez et al. (1997); Lin (1998)).

Recently, an alternative approach to low gain feedback design was proposed based on the solution of a parametric Lyapunov equation (Zhou et al. (2007)). This approach possesses the advantages of both the eigenstructure assignment approach and the ARE-based approach. On the one hand, it is conceptually appealing and directly results in a quadratic Lyapunov function for the closed-loop system. Furthermore, the low gain parameter also directly represents the convergence rate of the closed-loop system. On the other hand, it avoids the numerical stiffness encountered in the solution of an ARE with a small value of the low gain parameter.

Among the three approaches now available for the design of low gain feedback for continuous-time systems, both the eigenstructure assignment approach and the parametric ARE based approach have been extended to the discrete-time setting in Lin & Saberi (1995) and Lin et al. (1996), respectively. The objective of this paper is to develop the parametric Lyapunov equation based approach for discrete-time systems. Even though the development is parallel to that in Zhou et al. (2007), the result is not as obvious as expected. Indeed, in deriving some of the results needed for our design, we have discovered that some related results in the literature are incorrect. Also, in the continuous-time setting, the resulting feedback gain was shown to be a polynomial matrix in the low gain parameter if the system has a single input. Here in the discrete-time setting, the resulting feedback gain is in general only a rational matrix in the low gain parameter even for single input systems.

The remainder of this paper is organized as follows. In Section 2, a parametric discrete-time Lyapunov equation and the low gain feedback that results from it are introduced and some of their key properties are established. In Section 3, as an application of the proposed low gain feedback design, the problem of semi-global stabilization of discrete-time linear systems subject to actuator saturation by state feedback and output feedback is solved. A numerical example is given in Section 4 to illustrate the proposed results. Section 5 concludes the paper.

2. A PARAMETRIC DISCRETE-TIME LYAPUNOV EQUATION AND LOW GAIN FEEDBACK

We start by stating some preliminaries that are needed in deriving the results in this section. We first recall the following fact in matrix theory from Kailath (1980).

Lemma 1. Let A, B, C and D be some matrices of appropriate dimensions. Assume that $A, C, A + BCD$ and $C^{-1} + DA^{-1}B$ are all nonsingular. Then,

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1},$$

or equivalently,

$$A\left(A+BCD\right)^{-1}A = A - B\left(C^{-1}+DA^{-1}B\right)^{-1}D. \quad (1)$$

A set $\mathbb{F}$ on the complex plane is said to be symmetric with respect to the real axis if $\alpha \in \mathbb{F}$ implies $\bar{\alpha} \in \mathbb{F}$, where $\bar{\alpha}$ is the complex conjugate of α.

Definition 1. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two sets that are each symmetric with respect to the real axis. These two sets are said to be a mirror image of each other with respect to the circle $|z|^2 = r$ if for any $\alpha \in \mathbb{F}_1$, there exists a $\beta \in \mathbb{F}_2$ such that $\alpha\bar{\beta} = r$, and for any $\beta \in \mathbb{F}_2$, there exists an $\alpha \in \mathbb{F}_1$ such that $\alpha\bar{\beta} = r$ (see Fig. 1).

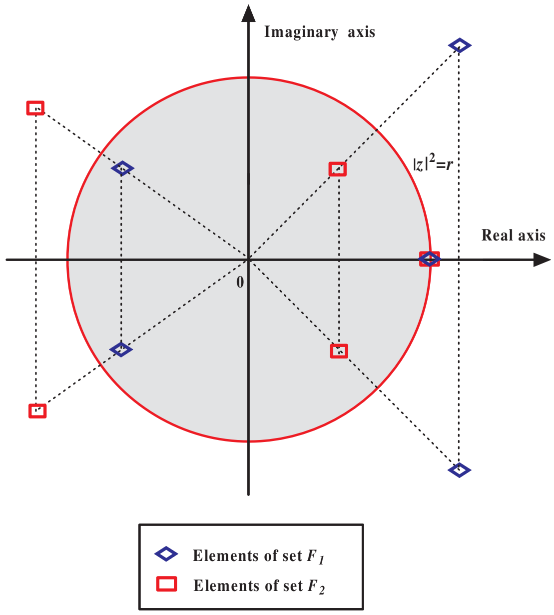

Fig. 1. Two sets $\mathbb{F}_1$ and $\mathbb{F}_2$ are mirror image of each other with respect to the circle $|z|^2 = r$.

We now consider a discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k), \quad (2)$$

where $A \in \mathrm{R}^{n\times n}$ and $B \in \mathrm{R}^{n\times m}$ are the system matrix and input matrix, respectively. An optimal control problem can be defined as finding the control sequence such that the cost function

$$J(u) = \sum_{k=0}^{\infty}(1-\gamma)^{-k}\left(x^{\mathrm{T}}(k)Qx(k) + u^{\mathrm{T}}(k)Ru(k)\right), \quad \gamma < 1, Q = C^{\mathrm{T}}C \geq 0, R > 0, \quad (3)$$

is minimized. Such an optimal control problem reduces to the standard LQR problem when $\gamma = 0$ and many of its properties can be derived from the results on the standard LQR problem. Some of the key results on this problem are summarized in the following proposition. The proof is omitted due to space limitation.

Proposition 2. Consider the linear system (2) and the cost function (3). Assume that (A, B) is stabilizable and (A, C) is detectable. Then, $J(u)$ is minimized with

$$u^*(k) = -\left(R + B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}PAx(k), \quad (4)$$

where P is the unique positive definite solution to the discrete-time algebraic Riccati equation (DARE)

$$(1-\gamma)P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB\left(R+B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}PA, \quad (5)$$

and

$$\lim_{k\to\infty}(1-\gamma)^{-\frac{k}{2}}x(k) = 0. \quad (6)$$

Moreover, the closed-loop system (2) and (4) is globally exponentially stable if

$$\frac{1}{\sqrt{1-\gamma}}\left|\lambda\left(A - B\left(R+B^{\mathrm{T}}P(\gamma)B\right)^{-1}B^{\mathrm{T}}P(\gamma)A\right)\right|_{\max} < 1, \quad (7)$$

where $|\lambda(\cdot)|_{\max}$ denotes the largest modul of eigenvalue of a matrix. The condition (7) holds for all $\gamma \in (-\gamma^*, 1)$ for some small enough $\gamma^* > 0$.

We note here that equation (6) implies that, for any $\gamma \in (-\gamma^*, 1)$, the convergence rate of the closed-loop system is faster than $\left(\sqrt{1-\gamma}\right)^k$. In this paper we are interested in the special case where $Q = 0$. In this case, the DARE (5) becomes

$$(1-\gamma)P = A^{\mathrm{T}}PA - A^{\mathrm{T}}PB\left(R + B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}PA, \quad (8)$$

which corresponds to the DARE for the "minimal energy control with guaranteed convergence rate"problem.

In what follows we establish some key properties of the DARE (8) and the feedback law that results from it.

Theorem 3. Let A be nonsingular and (A, B) be controllable.

(1) The DARE (8) has a unique positive definite solution $P(\gamma)$ if and only if
$$1 - |\lambda(A)|^2_{\min} < \gamma < 1, \quad (9)$$
where $|\lambda(A)|_{\min}$ denotes the minimal modulus of eigenvalue of matrix A. Moreover, this unique positive definite solution is given by $P(\gamma) = W^{-1}(\gamma)$, where $W(\gamma)$, a rational matrix in γ, is the unique positive definite solution to the parametric discrete-time Lyapunov matrix equation
$$W - \frac{1}{1-\gamma}AWA^{\mathrm{T}} = -BR^{-1}B^{\mathrm{T}}. \quad (10)$$
(2) Let $P(\gamma)$ be the unique positive definite solution to the DARE (8). Denote
$$A_c(\gamma) = A - B(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A.$$
Then the eigenvalues of $A_c(\gamma)$ and those of A are mirror images of each other with respect to the circle $|z|^2 = 1-\gamma$. Moreover, $A_c(\gamma)$ is Schur stable if and only if
$$1 - |\lambda(A)|_{\min} < \gamma. \quad (11)$$
(3) Let (9) hold. Then the positive definite matrix $P(\gamma)$ to the DARE (8) is differentiable and monotonically increasing with respect to γ, *i.e.*,
$$\frac{dP(\gamma)}{d\gamma} > 0. \quad (12)$$

Proof. 1. We first show that for the DARE (8) to have a positive definite solution, $\gamma < 1$ must be true. Assume that $P(\gamma) > 0$ is a solution to the DARE (8). Using the identity (1) of Lemma 1, we have

$$\begin{aligned} &P^{-1}(\gamma) - B(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}} \\ &= P^{-1}(\gamma)(P^{-1}(\gamma) + BR^{-1}B^{\mathrm{T}})^{-1}P^{-1}, \end{aligned}$$

substituting of which into the DARE (8) gives

$$(1-\gamma)P(\gamma) = A^{\mathrm{T}}(P^{-1}(\gamma) + BR^{-1}B^{\mathrm{T}})^{-1}A. \quad (13)$$

If $\gamma = 1$, since A is nonsingular, it follows from (13) that

$$(P^{-1}(\gamma) + BR^{-1}B^{\mathrm{T}})^{-1} = 0,$$

which is impossible. Now consider the case of $\gamma \neq 1$. Since A is invertible, taking inverse of both sides of (13) and rearranging the terms gives

$$A^{-1}P^{-1}(\gamma)A^{-\mathrm{T}} + \frac{1}{\gamma-1}P^{-1}(\gamma) = -A^{-1}BR^{-1}B^{\mathrm{T}}A^{-\mathrm{T}}. \quad (14)$$

Now suppose that $\gamma > 1$, then the left hand side of the above equation is positive definite while the right hand side is semi-negative definite. This is a contradiction. Therefore, a positive definite solution may exist only when $\gamma < 1$. In this case equation (14) can be rewritten as

$$\breve{A}(\gamma)W\breve{A}^{\mathrm{T}}(\gamma) - W = -\breve{A}(\gamma)BR^{-1}B^{\mathrm{T}}\breve{A}^{\mathrm{T}}(\gamma), \quad (15)$$

where $\breve{A}(\gamma) = \sqrt{1-\gamma}A^{-1}$ and $W = P^{-1}$. Equation (15) is equivalent to equation (10). We note that $(\breve{A}(\gamma), B)$ is controllable if and only if (A, B) is controllable. Moreover, we have

$$\left|\lambda(\breve{A}(\gamma))\right|_{\max} = \left|\lambda\left(\sqrt{1-\gamma}A^{-1}\right)\right|_{\max} = \frac{\sqrt{1-\gamma}}{|\lambda(A)|_{\min}},$$

which implies that the matrix $\breve{A}(\gamma)$ is Schur stable if and only if

$$\gamma > 1 - |\lambda(A)|^2_{\min}. \quad (16)$$

We next proceed to establish that condition (16) is necessary and sufficient for the DARE (8) to have a unique positive definite solution.

Sufficiency. Assume that (16) is satisfied. Then, $\breve{A}(\gamma)$ is Schur stable. It is well known that there is a unique positive definite solution $W(\gamma)$ to equation (15). Consequently, $P(\gamma) = W^{-1}(\gamma)$ is the unique positive definite solution to the DARE (8).

Necessary. Suppose that a positive definite solution to (15) exists. If (16) does not hold. Then matrix $\breve{A}(\gamma)$ has at least one eigenvalue λ such that $|\lambda| \geq 1$. Let z^{H} be the corresponding left eigenvector of λ, *i.e.*, $z^{\mathrm{H}}\breve{A}(\gamma) = \lambda z^{\mathrm{H}}$. Multiplying the equation (15) from left by z^{H} and from right by z gives

$$\left(|\lambda|^2 - 1\right)z^{\mathrm{H}}W(\gamma)z = -|\lambda|^2 z^{\mathrm{H}}BR^{-1}B^{\mathrm{T}}z. \quad (17)$$

Since $\left(\breve{A}(\gamma), B\right)$ is controllable, by the PBH test (Kailath (1980)), $z^{\mathrm{H}}B \neq 0$. Consequently, it follows from (17) that $|\lambda| < 1$, which is a contradiction.

2. It follows from the DARE (8) that

$$(1-\gamma)P(\gamma) = A^{\mathrm{T}}P(\gamma)\left(A - B(R+B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A\right),$$

which, by the non-singularity of both A and $P(\gamma)$, is equivalent to

$$A_c(\gamma) = (1-\gamma)P^{-1}(\gamma)A^{-\mathrm{T}}P(\gamma).$$

Consequently, for any eigenvalue of $A_c(\gamma)$, $\lambda(A_c(\gamma))$, there exists an eigenvalue of A, $\lambda(A)$, such that $\lambda(A_c(\gamma))\lambda(A) = 1-\gamma$. By definition, eigenvalues of $A_c(\gamma)$ are mirror images of those of A with respect to the circle $|z|^2 = 1-\gamma$. It then follows that $A_c(\gamma)$ is Schur stable if and only if (11) holds.

3. By taking derivative of both sides of (8) with respect to γ, denoting $\bar{A}(\gamma) = \frac{1}{\sqrt{1-\gamma}}A$ and $\bar{A}_{c1}(\gamma)$ as

$$\bar{A}_{c1}(\gamma) = \bar{A}(\gamma) - B(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)\bar{A}(\gamma).$$

then we can obtain the following equation

$$\bar{A}^{\mathrm{T}}_{c1}(\gamma)\frac{dP(\gamma)}{d\gamma}\bar{A}_{c1}(\gamma) - \frac{dP(\gamma)}{d\gamma} = -\frac{P(\gamma)}{1-\gamma}. \quad (18)$$

We next show that $\bar{A}_{c1}(\gamma)$ is Schur stable. To this end, we note that

$$\begin{aligned}
&\left|\lambda\left(\bar{A}_{c1}(\gamma)\right)\right|_{\max} \\
&= \left|\lambda\left(\bar{A}(\gamma) - B(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)\bar{A}(\gamma)\right)\right|_{\max} \\
&= \frac{1}{\sqrt{1-\gamma}}\left|\lambda\left(A - B(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A\right)\right|_{\max} \\
&= \frac{1}{\sqrt{1-\gamma}}\left|\lambda\left((1-\gamma)A^{-\mathrm{T}}\right)\right|_{\max} \\
&= \frac{\sqrt{1-\gamma}}{|\lambda\left(A\right)|_{\min}} < 1.
\end{aligned}$$

The last inequality is guaranteed by (9). It then follows that the discrete-time Lyapunov matrix equation (18) has a unique positive definite solution, *i.e.*, (12) holds. ■

Remark 4. This result in Item 2 can be viewed as a generalization of the result given in Mori & Shimemura (1980), where it was shown that when A is Schur anti-stable, *i.e.*, all the eigenvalues of A have moduls strictly bigger than 1, and (A, B) is controllable, then the eigenvalues of $A - B(I + B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}PA$ with P being the unique positive definite solution to

$$P = A^{\mathrm{T}}PA - A^{\mathrm{T}}PB(I + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA,$$

and those of A are mirror images of each other with respect to the unite circle. The result in continuous-time setting corresponding to the above result can be found in Molinari (1977).

Remark 5. Theorem 3 extends several aspects of the results in Rousant & Sawan (1992), where the following DARE

$$P = \alpha A^{\mathrm{T}}\left(P - PB\left(I + B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}P\right)\alpha A, \qquad (19)$$

is used to shift the poles of the closed-loop system. Note that (19) is equivalent to (8) with $\alpha^2 = \frac{1}{1-\gamma}$. First, it identifies and corrects some error in Rousant & Sawan (1992). It was established in Rousant & Sawan (1992) that a positive definite solution to (19) exists if

$$\alpha^2\left|\lambda\left(A\right)\right| > 1. \qquad (20)$$

It turns out that this statement is incorrect according to Theorem 3. To see this, consider

$$A = \begin{bmatrix} 0 & 1 \\ -\beta^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |\beta| < 1.$$

The unique solution to (8) can be found as

$$P(\gamma) = \begin{bmatrix} \dfrac{(\gamma-1)^2 - \beta^4}{\gamma - 1} & 0 \\ 0 & \dfrac{\beta^4 - (\gamma-1)^2}{(\gamma-1)^2} \end{bmatrix}, \qquad (21)$$

which is positive definite if and only if

$$1 - \left|\lambda\left(A\right)\right|_{\min}^2 = 1 - \beta^2 < \gamma < 1.$$

The above inequality coincides with (9), but not (20). For example, let $\beta = 0.5$ and $\gamma = 0.6$. Then, by the explicit solution (21), the DARE does not have a positive solution. Yet, the condition (20) is satisfied, which falsely indicates the existence of a positive definite solution.

Second, we provide necessary and sufficient condition for the existence of a unique positive definite solution to the DARE (8). Third, we proposed to solve the DARE through the solution of a parametric discrete-time Lyapunov equation and thus are able to obtain the explicit solution as a rational matrix in the parameter γ. Fourth, we obtain a geometric interpretation of the pole shifting property by introducing the notion of mirror image with respect to a circle. Finally, Item 3 is new and essential in developing the low gain feedback design to be presented next.

We next establish some further properties of the solution $P(\gamma)$ to the DARE (8) in the situation when all eigenvalues of A are on the unit circle, *i.e.*, $|\lambda(A)| = 1$. In this case, the inequalities (9) and (11) are reduced to the single one

$$0 < \gamma < 1. \qquad (22)$$

Theorem 6. Assume that all the eigenvalues of A are on the unit circle, (A, B) is controllable and γ satisfies (22). Let $P(\gamma)$ be the unique positive definite solution to (8). Then $\lim_{\gamma \to 0^+} P(\gamma)$ exists and

$$\lim_{\gamma \to 0^+} P(\gamma) = 0.$$

Proof. Omitted due to space limitation. ■

It is because of this property of $P(\gamma)$, the resulting feedback law

$$u(k) = -(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)Ax(k), \quad \gamma \in (0, 1),$$

is referred to as a low gain feedback, as the gain $-(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A$ decreases to zero as the value of γ does.

3. SEMI-GLOBAL STABILIZATION OF LINEAR SYSTEMS UNDER ACTUATOR SATURATION

Consider the following discrete-time linear system subject to actuator saturation

$$\begin{cases} x\left(k+1\right) = Ax\left(k\right) + B\sigma(u\left(k\right)) \\ \quad\;\; y\left(k\right) = Cx\left(k\right) \end{cases}, \qquad (23)$$

where $x \in \mathrm{R}^n$, $u \in \mathrm{R}^m$ and $y \in \mathrm{R}^p$ are respectively the state, input and output vectors and $\sigma : \mathrm{R}^m \to \mathrm{R}^m$ is a saturation function, *i.e.*,

$$\sigma(u) = \left[\,\sigma(u_1)\ \sigma(u_2)\ \cdots\ \sigma(u_m)\,\right]^{\mathrm{T}},$$

and for each $i = 1, 2, \cdots, m$, $\sigma(u_i) = \mathrm{sign}(u_i)\min\{1, |u_i|\}$. Here, we have slightly abused the notation by using σ to denote both the scalar valued and vector valued function. We have also assumed without loss of generality, the unity saturation level. Non-unity saturation level can be absorbed by the matrix B and the feedback gain.

As an application of the low gain feedback design of the previous section, we will show how it can be used to achieve semi-global stabilization for (23). As is well-known (see, for example, Sussmann et al. (1994)) that such a system can be semi-globally stabilized if and only if (A, B) is stabilizable, (A, C) is detectable, and all the eigenvalues of A are in the closed unit circle. Without loss of generality, we assume that (A, B) are given in the following form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_- \end{bmatrix}, \quad B = \begin{bmatrix} B_0 \\ B_- \end{bmatrix},$$

where A_- contains all eigenvalues of A that have moduls strictly less than 1 and A_0 contains all eigenvalues of A that have a modulus 1. The stabilizability of (A, B) then implies that (A_0, B_0) is controllable.

Clearly, the subsystem (A_-, B_-) does not affect the stabilizability property of the system. In what follows, we will further assume that (A, B) is controllable with all the eigenvalues of A on the unit circle.

We have the following results on semi-global stabilization of the system (23), either by state feedback or by output feedback.

Theorem 7. Let (A, B) be controllable and all eigenvalues of A be on the unit circle. Then, the family of feedback laws

$$u(k) = -F(\gamma)x(k),\ F(\gamma) = -(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A, \tag{24}$$

semi-globally stabilizes the system (23), where $P(\gamma) = W^{-1}(\gamma)$ with $W(\gamma)$ being the unique positive definite solution to the parametric discrete-time Lyapunov equation (10). More specifically, for any a priorly given (arbitrary large) bounded set $\mathcal{X}_0 \subset \mathrm{R}^n$, there exists a $\gamma^* \in (0,1)$ such that for any $\gamma \in (0, \gamma^*]$, the equilibrium $x = 0$ of the closed-loop system is locally exponentially stable with $\mathcal{X}_0$ contained in the domain of attraction. Furthermore, the convergence to the origin is no slower than $(1-\gamma)^{k/2}$.

Proof. Under the state feedback law (24), the closed-loop system is given by

$$x(k+1) = Ax(k) + B\sigma(F(\gamma)x(k)), \quad \gamma \in (0,1). \tag{25}$$

We will adopt a Lyapunov function $V(x) = x^{\mathrm{T}}P(\gamma)x(k)$ and consider its level set of the form

$$L(V) = \left\{x \in \mathrm{R}^n : x^{\mathrm{T}}P(\gamma)x \le 1\right\}.$$

Let $\gamma^* \in (0,1)$ be such that

$$\mathcal{X}_0 \subset L(V) \subset \mathcal{L}(F(\gamma)), \quad \forall \gamma \in (0, \gamma^*],$$

where

$$\mathcal{L}(F(\gamma)) = \{x \in \mathrm{R}^n : \|F(\gamma)x\|_\infty \le 1\}$$

is the area in the state space where the actuator does not saturate. Such a γ^* exists as $\mathcal{X}_0$ is bounded and $\lim_{\gamma \to 0^+} P(\gamma) = 0$. We now consider any $\gamma \in (0, \gamma^*]$. For any $x \in L(V)$, the actuator does not saturate and the closed-loop system simplifies to

$$x(k+1) = \left(A - (R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A\right)x(k),$$

for $\forall x \in L(V)$. Thus, in view of (8), the difference of V along the trajectories of the closed-loop system (25) within $L(V)$ can be evaluated as follows,

$$\begin{aligned}
&\Delta V(x(k) \\
&= V(x(k+1)) - V(x(k)) \\
&= x^{\mathrm{T}}(k)\left[-\gamma P(\gamma) - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}P(\gamma)B)^{-1}R\right. \\
&\quad \left.\times (R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A\right]x(k) \\
&\le -\gamma V(x(k)), \quad \forall x \in L(V).
\end{aligned}$$

This indicates that, for any $\gamma \in (0, \gamma^*]$, the closed-loop system is asymptotically stable at $x = 0$ with $\mathcal{X} \subset L(V)$ contained in its domain of attraction, and the convergence rate to $x = 0$ is no slower than $(1-\gamma)^{k/2}$. ■

Theorem 8. Let (A, B) be controllable, (A, C) be detectable, and all eigenvalues of A be on the unit circle. Then, the family of output feedback laws

$$\begin{cases}
\hat{x}(k+1) = (A + LC + BF(\gamma))\,\hat{x}(k) - Ly(k), \\
u(k) = F(\gamma)\hat{x}(k), \\
F(\gamma) = -(R + B^{\mathrm{T}}P(\gamma)B)^{-1}B^{\mathrm{T}}P(\gamma)A,
\end{cases}$$

semi-globally stabilizes the system (23), where $P(\gamma) = W^{-1}(\gamma)$ with $W(\gamma)$ being the unique positive definite solution to the parametric discrete-time Lyapunov matrix equation (10) and $L \in \mathrm{R}^{n \times p}$ is any matrix such that $A + LC$ is Schur stable. That is, for any given arbitrarily large bounded set $\mathcal{X}_0 \subset \mathrm{R}^{2n}$, there exists a $\gamma^* \in (0,1)$ such that, for any $\gamma \in (0, \gamma^*]$, the closed-loop system is asymptotically stable with $\mathcal{X}_0$ contained in the domain of attraction.

Proof. Omitted for space limitation. ■

4. AN EXAMPLE

In this section we use a simple example to illustrate the results presented in this paper. Consider the following system (Lin & Saberi (1995)),

$$x(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \sigma(u(k)).$$

The open loop system has repeated poles at $\left\{\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}j\right\}$. To construct the low gain feedback law, we choose $R = I$ and solve the parametric discrete-time Lyapunov matrix equation (10) to give $W(\gamma)$. Then $P(\gamma) = W^{-1}(\gamma)$ (shown on the top of the next page). The family of low gain feedback laws can be constructed as

$$\begin{aligned}
F(\gamma) = -\,[\,&\gamma(\gamma-2)\left(\gamma^2 - 2\gamma + 2\right) \quad 2\sqrt{2}\gamma\left(\gamma^2 - 3\gamma + 3\right) \\
&4\gamma(\gamma-2) \quad 2\sqrt{2}\gamma\,],
\end{aligned}$$

which is the same as given in Lin & Saberi (1995). It is easy to verify that the eigenvalues of the matrix $A + BF(\gamma)$ are given by

$$\lambda(A + BF(\gamma)) = \left\{\left(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}j\right)(1-\gamma)\right\}.$$

For two values of γ, $\gamma_1 = 0.005$ and $\gamma_2 = 0.01$, the resulting feedback gain are given by

$$F_1 = -\,[\,-0.0199\ 0.0422\ -0.0399\ 0.0141\,],$$

and

$$F_2 = -\,[\,-0.0394\ 0.0840\ -0.0796\ 0.0283\,].$$

Shown in Fig. 2 are the simulation results for the initial condition $x_0 = [4, -4, 4, -4]^{\mathrm{T}}$. The simulation results clearly show that the magnitudes of the control input decreases as the value of γ does which indicates that the semi-global stabilization can be achieved.

5. CONCLUSIONS

This paper considered a parametric discrete-time Lyapunov matrix equation. Several properties of the solution to such an equation and the resulting feedback gain were presented. By using such a design technique, an alternative approach to the design of low gain feedback is revealed. This new approach possesses the advantages of the two existing approaches, namely eigenstructure assignment approach and ARE-based approach for low gain feedback design. The problem of semi-global stabilization for linear systems subject to actuator saturation is used as an example of the application of this low gain feedback design approach.

$$W(\gamma)=\begin{bmatrix}
\frac{(\gamma-1)(\gamma-2)\rho(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & 2\sqrt{2}\frac{(\gamma-1)^2\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & 4\frac{(\gamma-2)(\gamma-1)^3}{(\gamma^2-2\gamma+2)^3\gamma} & 2\sqrt{2}\frac{(\gamma^2+2\gamma-2)(\gamma-1)^4}{(\gamma^2-2\gamma+2)^3\gamma^3}\\
2\sqrt{2}\frac{(\gamma-1)^2\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & \frac{-(\gamma-2)(\gamma-1)^2\rho(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & -2\sqrt{2}\frac{(\gamma-1)^3\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & -4\frac{(\gamma-2)(\gamma-1)^4}{(\gamma^2-2\gamma+2)^3\gamma}\\
\frac{4(\gamma-2)(\gamma-1)^3}{(\gamma^2-2\gamma+2)^3\gamma} & -2\sqrt{2}\frac{(\gamma-1)^3\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & \frac{(\gamma-2)(\gamma-1)^3\rho(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & 2\sqrt{2}\frac{(\gamma-1)^4\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3}\\
2\sqrt{2}\frac{(\gamma^2+2\gamma-2)(\gamma-1)^4}{(\gamma^2-2\gamma+2)^3\gamma^3} & -4\frac{(\gamma-2)(\gamma-1)^4}{(\gamma^2-2\gamma+2)^3\gamma} & 2\sqrt{2}\frac{(\gamma-1)^4\phi(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3} & \frac{-(\gamma-2)(\gamma-1)^4\rho(\gamma)}{(\gamma^2-2\gamma+2)^3\gamma^3}
\end{bmatrix},$$

where

$$\begin{cases}\rho(\gamma)=\gamma^4-8\gamma^3+12\gamma^2-8\gamma+4,\\ \phi(\gamma)=\gamma^4-5\gamma^3+7\gamma^2-4\gamma+2.\end{cases}$$

The matrix $P(\gamma)=W^{-1}(\gamma)$ is given by

$$P(\gamma)=\begin{bmatrix}
\frac{\gamma(\gamma-2)(\gamma^2-2\gamma+2)}{\gamma-1} & \frac{2\sqrt{2}(\gamma^2-3\gamma+3)\gamma}{\gamma-1} & \frac{4\gamma(\gamma-2)}{\gamma-1} & \frac{2\sqrt{2}\gamma}{\gamma-1}\\
\frac{2\sqrt{2}(\gamma^2-3\gamma+3)\gamma}{\gamma-1} & -\frac{(\gamma^2-10\gamma+10)\gamma(\gamma-2)}{(\gamma-1)^2} & -\frac{2\sqrt{2}(\gamma^2-7\gamma+7)\gamma}{(\gamma-1)^2} & -\frac{4\gamma(\gamma-2)}{(\gamma-1)^2}\\
\frac{4\gamma(\gamma-2)}{\gamma-1} & -\frac{2\sqrt{2}(\gamma^2-7\gamma+7)\gamma}{(\gamma-1)^2} & \frac{(\gamma^2-10\gamma+10)\gamma(\gamma-2)}{(\gamma-1)^3} & \frac{2\sqrt{2}(\gamma^2-3\gamma+3)\gamma}{(\gamma-1)^3}\\
\frac{2\sqrt{2}\gamma}{\gamma-1} & -\frac{4\gamma(\gamma-2)}{(\gamma-1)^2} & \frac{2\sqrt{2}(\gamma^2-3\gamma+3)\gamma}{(\gamma-1)^3} & -\frac{\gamma(\gamma-2)(\gamma^2-2\gamma+2)}{(\gamma-1)^4}
\end{bmatrix}.$$

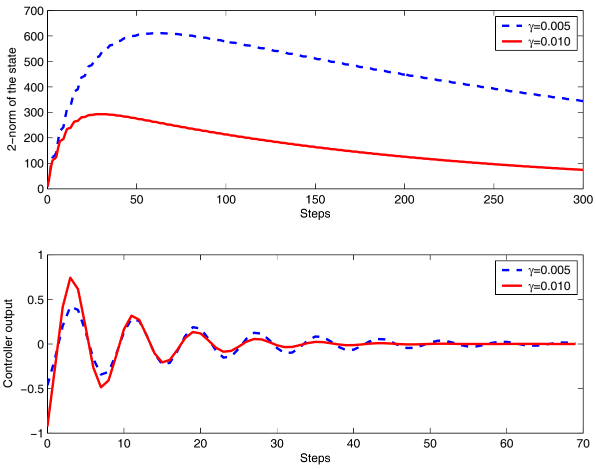

Fig. 2. State responses and control signal for $\gamma=0.005$ and $\gamma=0.010$.

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