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Fault Estimation for Single Output Nonlinear Systems Using an Adaptive Sliding Mode Observer

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Abstract: In this paper, a class of single output nonlinear systems with an uncertain parameter is considered. A diffeomorphism is first introduced to simplify the system structure, then, by employing an adaptive approach to identify the unknown parameter, a sliding mode observer is developed to estimate the system state variables. Based on the observer, a fault estimation scheme is proposed based on the minimisation of a weighted $\mathcal{L}_2$ norm of the fault estimation error. A simulation example is given to demonstrate the proposed scheme.

1. INTRODUCTION

In recent decades, some control inspired approaches — for instance sliding mode techniques (Edwards, Spurgeon & Patton 2000), modern differential geometric approaches (Persis & Isidori 2001) and adaptive schemes (Zhang, Polycarpou & Parisini 2004) have been successfully incorporated with the observer-based FDI approach. In particular, modern geometric approaches have been shown to be effective in the investigation of a class of nonlinear systems (Isidori 1995, Marino & Tomei 1995).

Single output systems have received much attention becuase the structure of such a class of systems is simple (see e.g. (Marino & Tomei 2000, Jo & Seo 2002, Zhang et al. 2004, Chen & Saif 2007)). When a system has unknown parameters, adaptive control techniques can be employed to estimate the unknown parameters (Xu & Zhang 2004, Zhang et al. 2004). In Shafai, Pi, Bas & Linder (2001), a PI adaptive observer was proposed for the purpose of FDI for SISO linear systems. An adaptive FDI scheme was given in Wang & Daley (1996) for linear systems which requires that the observation error dynamics are strictly positive real. More recently, Chen and Saif (Chen & Saif 2007) considered a FDI problem for a class of SISO linear systems and an adaptive technique was used to identify the unknown parameters. However, in reality, most systems exhibit varieties of nonlinearities and therefore it is necessary to deal with nonlinear systems.

Recently, sliding mode techniques have been successfully used in fault detection and isolation (Floquet, Barbot, Perruquetti & Djemai 2004, Edwards et al. 2000, Tan & Edwards 2002, Yan & Edwards 2007) and it has been proved to be an effective way to estimate/reconstruct system faults. A 'precise' fault reconstruction approach is proposed in Edwards et al. (2000) based on an equivalent output error injection. Based on the work in Edwards et al. (2000), a sensor fault reconstruction scheme is given in Tan & Edwards (2002). It should be noted that in (Edwards et al. 2000, Tan & Edwards 2002), only linear systems are considered and uncertainty is not involved. A fault estimation approach for linear systems with uncertainty was proposed by Tan & Edwards (2003). Later, a robust actuator fault reconstruction scheme was presented in Yan & Edwards (2007) using the characteristics of the uncertainty structure and the fault distribution. Jiang, Staroswiecki & Cocquempot (2004) proposed a fault estimation scheme for a class of systems with uncertainty, and a robust fault detection method for nonlinear systems with disturbances was also considered in Floquet et al. (2004). However, in these papers involving uncertainty, the reconstruction/estimation signal is explicitly dependent on the bounds on uncertainties, and they all require that the uncertainties are bounded with known upper bounds.

It should be noted that in contrast to multi-output systems, a single output system only has one output which can be employed to estimate a fault and identify a parameter. Therefore in order to make estimation and identification possible simultaneously, both the fault and the uncertain parameter must appear in the output channel of the dynamics. Since the uncertain parameter and the fault both exist in the same channel and can not be separated, 'precise' fault reconstruction as given in (Edwards et al. 2000, Yan & Edwards 2007) is, generally speaking, not possible. However, it is possible to estimate the fault affecting the system and then to detect the fault by establishing an appropriate threshold. In this paper, fault estimation is considered for a class of single output nonlinear systems with an uncertain parameter. Both the fault distribution vector and the distribution vector for the uncertain parameter are allowed to be functions of the system output and input. It is not required that the uncertainty/fault is matched. Under the assumption that the time derivative of the system output is measurable, a novel adaptive update law is proposed to identify the unknown parameter and then an observer is established using sliding mode techniques. It is shown that the trajectories of the error system dynamics enter a domain around the operating point in finite time and remain inside thereafter. Based on the sliding mode observer, a fault estimation scheme is proposed to estimate the fault signal. Bounds for the fault estimation error are also given, which are independent of the uncertain parameter. Moreover, the uncertain parame-

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ter is allowed to be arbitrarily large, and a bound on the parameter is not required. This is different in comparison with the existing related work in (Jiang et al. 2004, Floquet et al. 2004, Tan & Edwards 2003, Yan & Edwards 2007) where the uncertain bounds must be used in the observer design and fault estimation/reconstruction.

Notation: The symbol $\|\cdot\|$ denotes the Euclidean norm or its induced norm, and in particular, $\|\cdot\|_2$ is used to represent the $\mathcal{L}_2$ norm. The symbols used in Section 2 are the same as those in (Marino & Tomei 1995).

2. SYSTEM DESCRIPTION AND ANALYSIS

Consider a nonlinear system described by

$$\dot{x} = F(x) + G(x,u) + \Phi(x,u)\theta + \Psi(x,u)f(t) \quad (1)$$

$$y = h(x), \qquad x_0 := x(0) \quad (2)$$

where $x \in \Omega \subset \mathcal{R}^n$ (Ω is a neighbourhood of the origin), $u \in \mathcal{U} \subset \mathcal{R}^m$ ($\mathcal{U}$ is an admissible control set) and $y \in \mathcal{Y} \subset \mathcal{R}$ ($\mathcal{Y}$ is output space) are the state variables, the inputs and the output respectively. The vector fields $F(x) \in \mathcal{R}^n$, $G(x,u) \in \mathcal{R}^n$, $\Phi(x,u) \in \mathcal{R}^n$ and $\Psi(x,u) \in \mathcal{R}^n$ are assumed to be smooth. The scalar $\theta \in \mathcal{R}$ represents an unknown constant parameter. The unknown function $f(t) \in \mathcal{R}$ is a fault affecting the system which satisfies

$$|f(t)| \le \rho(t) \quad (3)$$

where $\rho(\cdot)$ is assumed to be known.

In this paper, all the assumptions are only assumed to hold in a domain of the operating point instead of the whole state-space. Without loss of generality, the domain is assumed to be a compact neighbourhood of the origin.

Assumption 1 The pair $(F(x), h(x))$ satisfies

i) $\text{rank}\begin{bmatrix} dh, & d(L_F h), & \cdots, & d(L_F^{n-1}h)\end{bmatrix} = n$;
ii) $[ad_F^i \xi, ad_F^j \xi] = 0, \qquad i,j = 0,1,2,\dots,n-1$;

where $\xi(x)$ is a vector field solution of the equation

$$\begin{bmatrix} \langle dh, \xi\rangle \\ \langle d(L_F h), \xi)\rangle \\ \vdots \\ \langle d(L_F^{n-1}h), \xi\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

For convenience, the notation $\mathcal{O}_{(F,h)}$ is used throughout the paper and denotes a set of n-dimensional vector fields defined by

$$\mathcal{O}_{(F,h)} := \left\{ \zeta(x,u) \;\middle|\; \begin{matrix} [\zeta,\ ad_F^j \xi(x)] = 0 \\ u \in \mathcal{U},\ j = 0,1,2,\dots,n-2 \end{matrix} \right\}$$

where $\xi(\cdot)$ is determined by (4).

Assumption 2. The vector fields $G(x,u)$, $\Phi(x,u)$ and $\Psi(x,u)$ satisfy

$$G(x,u), \Phi(x,u), \Psi(x,u) \in \mathcal{O}_{(F,h)}$$

in the domain $(x,u) \in \Omega \times \mathcal{U}$

Under Assumptions 1 and 2, it follows from Chapter 5 in (Marino & Tomei 1995) that there exists a diffeomorphism $w = T_1(x)$ such that in the new coordinates w, system (1)–(2) can be described by

$$\dot{w} = Aw + \tilde{G}(y,u) + \tilde{\Phi}(y,u)\theta + \tilde{\Psi}(y,u)f(t) \quad (5)$$

$$y = Cw, \quad (6)$$

where

$$A = \left[\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right] := \left[\begin{array}{c|c} A_1 & 0 \\ \hline A_2 & 0 \end{array}\right] \in \mathcal{R}^{n\times n} \quad (7)$$

$$C = [0 \quad \cdots \quad 0 \quad 1] \in \mathcal{R}^{1\times n} \quad (8)$$

Clearly, the matrix pair (A_1, A_2) is observable and thus there exists a matrix $L \in \mathcal{R}^{(n-1)\times 1}$ such that $A_1 - LA_2$ is stable.

For system (5)–(6), introduce a further linear coordinate transformation $z = T_2 w$ with

$$T_2 := \begin{bmatrix} I_{n-1} & -L \\ 0 & 1 \end{bmatrix} \quad (9)$$

Let $T = T_2 \circ T_1$. It is obvious that $T : x \mapsto z$ is a diffeomorphism defined in a domain of the origin. It follows that in the new coordinate system z defined by $z = T(x)$, system (1)–(2) has the following form

$$\dot{z} = T_2 \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} T_2^{-1} z + T_2\tilde{G}(y,u) + T_2\tilde{\Phi}(y,u)\theta$$
$$+T_2\tilde{\Psi}(y,u)f(t) \quad (10)$$

$$y = Cz \quad (11)$$

since $CT_2^{-1} = C$. The system above can be further decomposed as

$$\dot{z}_1 = (A_1 - LA_2)z_1 + (A_1 - LA_2)Lz_2 + g_1(y,u)$$
$$+\phi_1(y,u)\theta + \psi_1(y,u)f(t) \quad (12)$$

$$\dot{z}_2 = A_2 z_1 + A_2 L z_2 + g_2(y,u) + \phi_2(y,u)\theta$$
$$+\psi_2(y,u)f(t) \quad (13)$$

$$y = z_2 \quad (14)$$

where $z = \text{col}(z_1, z_2)$ with $z_1 \in \mathcal{R}^{n-1}$ and $z_2 \in \mathcal{R}$ and

$$\begin{bmatrix} g_1(y,u) \\ g_2(y,u) \end{bmatrix} := T_2\tilde{G}(y,u) = \left[\frac{\partial T(x)}{\partial x} G(x,u)\right]_{x=T^{-1}(z)} \quad (15)$$

$$\begin{bmatrix} \phi_1(y,u) \\ \phi_2(y,u) \end{bmatrix} := T_2\tilde{\Phi}(y,u) = \left[\frac{\partial T(x)}{\partial x} \Phi(x,u)\right]_{x=T^{-1}(z)} \quad (16)$$

$$\begin{bmatrix} \psi_1(y,u) \\ \psi_2(y,u) \end{bmatrix} := T_2\tilde{\Psi}(y,u) = \left[\frac{\partial T(x)}{\partial x} \Psi(x,u)\right]_{x=T^{-1}(z)} \quad (17)$$

with $g_1(\cdot)$, $\phi_1(\cdot)$, $\psi_1(\cdot) \in \mathcal{R}^{n-1}$ and $g_2(\cdot)$, $\phi_2(\cdot)$, $\psi_2(\cdot) \in \mathcal{R}$.

Remark 1. Since T_1 is available in (Marino & Tomei 1995) and the matrix T_2 has been given in (9), the transformation T is available. Therefore, the vectors $g_i(\cdot)$, $\phi_i(\cdot)$ and $\psi_i(\cdot)$ for $i = 1,2$ can be obtained directly using the coordinate transformation $z = T(x)$ and thus system (12)–(14) is well defined and can be obtained from (1)–(2).

Assumption 3 The functions ϕ_2 and ψ_2 satisfy $\phi_2(\cdot) \neq 0$ and $\psi_2(\cdot) \neq 0$ in the domain $\mathcal{Y} \times \mathcal{U}$. Furthermore $\frac{1}{\psi_2(\cdot)}$ is assumed to be bounded in $\mathcal{Y} \times \mathcal{U}$.

Remark 2. Assumption 3 is a limitation on the fault and uncertain parameter distributions. It should be noted that $\phi_2 \neq 0$ and $\psi_2 \neq 0$ are necessary for the parameter θ to be identified and the fault f to be estimated, since in system (12)–(13), only z_2 is measurable and z_1 is not available.

3. ADAPTIVE SLIDING MODE OBSERVER DESIGN

In this section, as in (Fu & Liao 1990), it is assumed that $\dot{y}$ is measurable[1]. A sliding mode observer will now be proposed employing an adaptive law. For system (12)–(14), consider the dynamical observer system

$$\dot{\hat{z}}_1 = (A_1 - LA_2)\hat{z}_1 + \big(A_1 - LA_2\big)Ly + g_1(y,u) + \phi_1(y,u)\hat{\theta} \tag{18}$$

$$\dot{\hat{z}}_2 = A_2\hat{z}_1 + A_2Ly - \Lambda\left(\hat{z}_2 - y\right) + g_2(y,u) + \phi_2(y,u)\hat{\theta} + \nu \tag{19}$$

$$\hat{y} = \hat{z}_2 \tag{20}$$

where $\hat{y} \in \mathcal{R}$ is the observer output, $\Lambda > 0$ is a design scalar, and the estimate of θ denoted by $\hat{\theta}$ is given by the following adaptive law

$$\dot{\hat{\theta}} = -\phi_2(\cdot)\left(\phi_2(y,u)\hat{\theta} + A_2\hat{z}_1 + A_2Ly + g_2(y,u) - \dot{y}\right) \tag{21}$$

The output error injection term ν is defined by

$$\nu = k\left(t,y,u\right)\operatorname{sgn}(y-\hat{y}) \tag{22}$$

where sgn denotes the usual signum function. The gain $k(\cdot)$ is to be determined later and only depends on known information.

Let $e_z = z_1 - \hat{z}_1$, $e_y = y - \hat{y}$ and $e_\theta = \theta - \hat{\theta}$. It follows from (12)–(14) and (18)–(20) that the dynamical error system is described by

$$\dot{e}_z = (A_1 - LA_2)e_z + \phi_1(y,u)e_\theta + \psi_1(y,u)f(t) \tag{23}$$

$$\dot{e}_\theta = -\phi_2(y,u)A_2e_z - \phi_2^2(y,u)e_\theta - \phi_2(y,u)\psi_2(y,u)f(t) \tag{24}$$

$$\dot{e}_y = A_2e_z - \Lambda e_y + \phi_2(y,u)e_\theta + \psi_2(y,u)f(t) - \nu \tag{25}$$

The following conclusion is now ready to be presented:

Theorem 1. Assume that there exists a positive definite matrix P and a positive constant ε such that the matrix

$$W := \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix}$$

is positive definite for $(y,u) \in \mathcal{Y}\times\mathcal{U}$ where

$$W_{11} := -\Big((A_1 - LA_2)^TP + P(A_1 - LA_2) + \frac{\|\psi_1(\cdot)\|^2}{\varepsilon\psi_2^2(\cdot)}P^2\Big)$$

$$W_{12} := -P\frac{\phi_1(y,u)}{\phi_2(y,u)} + P\frac{\psi_1(y,u)}{\varepsilon\psi_2(y,u)} + A_2^T$$

$$W_{22} := 2 - \frac{1}{\varepsilon}$$

Then, under Assumptions 2 and 3, the trajectories of (23)–(24) will enter a bounded domain of the origin in finite time if $\psi_2(y,u)\rho(t)$ is bounded in $\mathcal{Y}\times\mathcal{U}\times\mathcal{R}^+$ and $\gamma := \inf_{(y,u)\in\mathcal{Y}\times\mathcal{U}}\{\lambda_{\min}(W(y,u))\} > 0$.

[1] Otherwise the approach proposed in (Levant 1998) can be employed to compute the time derivative of the system output.

Proof: Consider a candidate Lyapunov function V for the error dynamics (23)–(24) as follows

$$V(e_z, e_\theta) = e_z^TPe_z + e_\theta^2 \tag{26}$$

Then the time derivative of V along the trajectories of system (23)–(24) are given by

$$\begin{aligned} &\dot{V}\,|_{(23)-(24)} \\ &= e_z^T\left((A_1 - LA_2)^TP + P(A_1 - LA_2)^T\right)e_z + 2e_z^TP\phi_1(\cdot)e_\theta \\ &\quad + 2e_z^TP\psi_1(y,u)f(t) - 2\phi_2(y,u)e_\theta A_2e_z \\ &\quad - 2\phi_2^2(y,u)e_\theta^2 - 2\phi_2(y,u)\psi_2(y,u)e_\theta f(t) \end{aligned} \tag{27}$$

From Young's inequality and $\psi_2 \neq 0$, it follows that for any positive constant ε

$$\begin{aligned} &2e_z^TP\psi_1(y,u)f(t) - 2\phi_2(y,u)\psi_2(y,u)e_\theta f(t) \\ &\le \frac{1}{\varepsilon}\left\|e_z^TP\frac{\psi_1(y,u)}{\psi_2(y,u)} - \phi_2(y,u)e_\theta\right\|^2 + \varepsilon\left\|\psi_2(y,u)f(t)\right\|^2 \\ &= \frac{1}{\varepsilon}e_z^T\Big(\frac{\|\psi_1(y,u)\|^2}{\psi_2^2(y,u)}P^2\Big)e_z + \frac{1}{\varepsilon}\phi_2^2(y,u)e_\theta^2 \\ &\quad - 2\frac{\phi_2(y,u)}{\varepsilon\psi_2(y,u)}e_z^TP\psi_1(y,u)e_\theta + \varepsilon\left\|\psi_2(y,u)f(t)\right\|^2 \end{aligned} \tag{28}$$

Substituting (28) into (27), it follows that

$$\begin{aligned} &\dot{V}\,|_{(23)-(24)} \\ &= e_z^T\Big((A_1 - LA_2)^TP + P(A_1 - LA_2) + \frac{\|\psi_1(\cdot)\|^2}{\varepsilon\psi_2^2(\cdot)}P^2\Big)e_z \\ &\quad - 2e_z^T\left(-P\frac{\phi_1(y,u)}{\phi_2(y,u)} + P\frac{\psi_1(y,u)}{\varepsilon\psi_2(y,u)} + A_2^T\right)\phi_2(y,u)e_\theta \\ &\quad - \Big(2 - \frac{1}{\varepsilon}\Big)\phi_2^2(y,u)e_\theta^2 + \varepsilon\left\|\psi_2(y,u)f(t)\right\|^2 \\ &= -\left[\,e_z^T \quad \phi_2(\cdot)e_\theta\,\right]W\begin{bmatrix} e_z \\ \phi_2(\cdot)e_\theta \end{bmatrix} + \varepsilon\left\|\psi_2(\cdot)f(t)\right\|^2 \end{aligned} \tag{29}$$

Since by assumption $\psi_2(y,u)\rho(t)$ is bounded in $\mathcal{Y}\times\mathcal{U}\times\mathcal{R}^+$, suppose that $|\psi_2(y,u)\rho(t)|^2 \le M$. Then, it follows from (29) that when $\|e_z\|^2 + \phi_2^2(y,u)e_\theta^2 > \frac{\varepsilon M}{\gamma(1-\gamma_0)}$ with $0 < \gamma_0 < 1$,

$$\begin{aligned} \dot{V} &\le -\gamma\left(\|e_z\|^2 + \phi_2^2(y,u)e_\theta^2\right) + \varepsilon\left\|\psi_2(y,u)f(t)\right\|^2 \\ &\le -\gamma\left(\|e_z\|^2 + \phi_2^2(y,u)e_\theta^2\right) + \varepsilon M \\ &< -\gamma\left(\|e_z\|^2 + \phi_2^2(\cdot)e_\theta^2\right) + \gamma(1-\gamma_0)\left(\|e_z\|^2 + \phi_2^2(\cdot)e_\theta^2\right) \\ &= -\gamma\gamma_0\left(e_z^Te_z + \phi_2^2(y,u)e_\theta^2\right) \le 0 \end{aligned}$$

Since $\phi_2^2(\cdot) \neq 0$ in the considered domain, it follows that the trajectories of system (23)–(24) will enter a bounded domain in finite time ((Khalil 2002)). Hence the conclusion follows. #

Remark 3. Theorem 1 shows that under certain conditions both e_z and e_θ are bounded. Throughout the paper, it is assumed that for any $(t,y,u) \in \mathcal{R}^+\times\mathcal{Y}\times\mathcal{U}$ there exist χ_1 and χ_2 such that

$$\|e_z(t)\| \le \chi_1, \qquad |e_\theta| \le \chi_2 \tag{30}$$

Now, for the error dynamical equation (23)–(25), consider the following sliding surface

$$S = \{(e_z, e_\theta, e_y) \mid e_y = 0\} \tag{31}$$

The reduced-order sliding mode associated with the surface in (31) can be described by (23)–(24). Theorem 1 has shown that under certain conditions, when the sliding motion takes place, the trajectories of the sliding mode dynamics will enter a bounded domain in finite time. A reachability condition will be developed so that the error dynamical system (23)–(25) is driven to the sliding surface (31) in finite time and a sliding motion maintained thereafter.

Theorem 2. Assume that (30) holds. Then the trajectories of the dynamical system (23)–(25) can be driven to the sliding surface (31) in finite time if the gain $k(t,y,u)$ in (22) satisfies

$$k(\cdot) \geq \chi_1 + \Lambda\,|e_y| + |\phi_2(y,u)|\chi_2 + |\psi_2(y,u)|\rho(t) + \eta \quad (32)$$

for some $\eta > 0$.

Proof: Substituting (22) into (25), it follows that

$$e_y\dot{e}_y = e_yA_2e_z - e_y\Lambda e_y + e_y\phi_2(y,u)e_\theta + e_y\psi_2(y,u)f(t) - e_yk(\cdot)\mathrm{sgn}(e_y)$$

From (3), (30) and the fact that "$e_y\mathrm{sgn}(e_y) = |e_y|$",

$$\begin{aligned} e_y\dot{e}_y &\leq |e_y|\,\|A_2\|\,\chi_1 + \Lambda\,e_y^2 + |\phi_2(y,u)|\,|e_y|\chi_2 + \\ &\quad |e_y|\,|\psi_2(y,u)|\rho(t) - k(\cdot)|e_y| \\ &= \Big(\chi_1 + \Lambda\,|e_y| + |\phi_2(\cdot)|\chi_2 + |\psi_2(\cdot)|\rho(\cdot) - k(\cdot)\Big)|e_y| \end{aligned} \quad (33)$$

(In establishing (33) the fact that $\|A_2\| = 1$ has been used). Then, using (32), it is easy to see that (33) becomes

$$e_y\dot{e}_y \leq -\eta|e_y|$$

This shows that the traditional reachability condition (Utkin 1992) is satisfied and thus $e_y \to 0$ in finite time. Hence the conclusion follows. #

Remark 4 From sliding mode theory, Theorems 1 and 2 show that system (18)–(20) is an observer for the system (12)–(14) and the estimation error enters a bounded domain in finite time.

4. FAULT ESTIMATION

In this section, it is assumed that an adaptive sliding mode observer (18)–(20) has been properly designed and that the conditions in Theorems 1 and 2 are satisfied. When a sliding motion takes place,

$$e_y(t) = 0, \qquad \text{and} \qquad \dot{e}_y(t) = 0 \quad (34)$$

and in this case the output error injection signal ν in (25) can be replaced by an equivalent output error injection signal ν_{eq} (Utkin 1992). By applying (34) to (25), it follows that when the sliding motion takes place

$$A_2e_z + \phi_2(y,u)e_\theta + \psi_2(y,u)f(t) - \nu_{eq} = 0 \quad (35)$$

From Assumption 3, $\psi_2(y,u) \neq 0$. Then, from (35) the fault signal satisfies

$$f(t) = \frac{1}{\psi_2(y,u)}\left(-A_2e_z - \phi_2(y,u)e_\theta + \nu_{eq}\right) \quad (36)$$

In order to estimate the fault $f(t)$, it is necessary to recover the equivalent output error injection signal ν_{eq}. In the work described in (Utkin 1992), it was obtained using a low-pass filter. Here a modification to the approach given in (Edwards et al. 2000) will be employed. From (22), the equivalent output error injection signal ν_{eq} in (36) can be approximated to any accuracy by

$$\nu_\sigma := k\,(t,y,u)\,\zeta(e_y) \quad (37)$$

where $k(\cdot)$ satisfies (32) and

$$\zeta(e_y) = \frac{e_y}{\|e_y\| + \delta_1\exp\{-\delta_2 t\}} \quad (38)$$

where δ_1 and δ_2 are positive design constants.

Construct a signal

$$\hat{f}(t) = \frac{1}{\psi_2(y,u)}\nu_\sigma \quad (39)$$

where the function ν_σ is defined by (37). It will be shown that $\hat{f}(t)$ is a reasonable estimate for the fault $f(t)$. Define

$$\mathcal{Z} := A_2e_z + \phi_2(y,u)e_\theta \quad (40)$$

Then, the following conclusion is ready to be presented:

Theorem 3. Consider the system described in (1)–(2). Assume that the conditions of Theorem 1 are satisfied and that the matrix

$$\widetilde{W} := \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{12}^T & \widetilde{W}_{22} \end{bmatrix}$$

defined by

$$\begin{aligned} \widetilde{W}_{11} &:= -\Big((A_1 - LA_2)^TP + P(A_1 - LA_2) + \\ &\qquad \frac{\|\psi_1(y,u)\|^2}{\varepsilon\psi_2^2(y,u)}P^2 + A_2^TA_2\Big) \\ \widetilde{W}_{12} &:= -P\frac{\phi_1(y,u)}{\phi_2(y,u)} + P\frac{\psi_1(y,u)}{\varepsilon\psi_2(y,u)} \\ \widetilde{W}_{22} &:= 1 - \frac{1}{\varepsilon} \end{aligned}$$

is semi-positive definite where ε is a positive scalar. Then,

i) $\|\mathcal{Z}\|_2 \leq V(0) + \varepsilon\|\psi_2(y,u)f(t)\|_2$ with V and $\mathcal{Z}$ defined by (26) and (40) respectively;

ii) the $\mathcal{L}_2$ norm of the estimation error $f(t) - \hat{f}(t)$ where $\hat{f}(t)$ is defined by (39), satisfies

$$\|\psi_2(f(t) - \hat{f}(t))\|_2 \leq V(0) + \varepsilon\|\psi_2 f(t)\|_2 + \varepsilon_1 \quad (41)$$

where ε_1 is an arbitrary small positive constant.

Proof: From the definition of $\mathcal{Z}$ in (40), it follows that

$$\|\mathcal{Z}\|^2 = e_z^TA_2^TA_2e_z + 2e_z^TA_2^T\phi_2(y,u)e_\theta + \phi_2^2(y,u)e_\theta^2 \quad (42)$$

Then, from the proof of Theorem 1 and (29),

$$\begin{aligned} \dot{V}\,|_{(23)-(24)} &\leq -\left[\,e_z^T \quad \phi_2(y,u)e_\theta\,\right]\widetilde{W}\begin{bmatrix} e_z \\ \phi_2(y,u)e_\theta \end{bmatrix} \\ &\quad -\|\mathcal{Z}\|^2 + \varepsilon\,\|\psi_2(y,u)f(t)\|^2 \end{aligned} \quad (43)$$

Since $\widetilde{W}$ is semi-positive, it follows from (43) that for $t \in \mathcal{R}^+$

$$V(t) - V(0) \leq -\int_0^t \|\mathcal{Z}\|^2dt + \varepsilon\int_0^t \|\psi_2(y,u)f(t)\|^2\,dt$$

Because $V(t) > 0$,

$$\int_0^t \|\mathcal{Z}\|^2 dt \le V(0) + \varepsilon \int_0^t \|\psi_2(y,u) f(t)\|^2 dt$$

Hence conclusion i) follows.

Since ν_σ can approximate ν_{eq} to any accuracy, and from Assumption 3 $\frac{1}{\psi_2(\cdot)}$ is bounded in $\mathcal{Y} \times \mathcal{U}$, it follows that for any $\varepsilon_1 > 0$ there exist σ_1 and σ_2 such that

$$\|\nu_{eq} - \nu_\sigma\|_2 < \varepsilon_1$$

Consequently, from (36), (39) and conclusion i),

$$\begin{aligned}\|\psi_2(f(t) - \hat{f}(t))\|_2 &\le V(0) + \| - \mathcal{Z} + \nu_{eq} - \nu_\sigma\|_2 \\ &\le V(0) + \varepsilon\|\psi_2 f(t)\|_2 + \varepsilon_1 \end{aligned} \quad (44)$$

Hence, conclusion ii) follows. #

Remark 5. Theorem 3 shows that $\hat{f}$ is an estimate for the fault f and the estimation error is given by (41). Clearly, the conservatism of the estimation error $\|\psi_2(f - \hat{f})\|_2$ can be reduced by optimizing ε.

Remark 6. From the proof of Theorems 2 and 3, it is easy to see that if $\widetilde{W}$ is positive definite then this implies that W is positive definite. However, the fact that $\widetilde{W}$ is semi-positive definite does not imply W is positive definite.

In the generic case it is hard to give a general approach for optimising the parameter ε. However, it is possible to do this for some special cases. An interesting case is when

$$\phi_1(y,u) = d_1\phi_2(y,u), \qquad \psi_1(y,u) = d_2\psi_2(y,u) \quad (45)$$

where $d_1, d_2 \in \mathcal{R}^{n-1}$ are constant vectors. In this case the matrices W and $\widetilde{W}$ have the following forms respectively

$$W = \begin{bmatrix} -\Gamma & -\Xi + A_2^T \\ -\Xi^T + A_2 & 2 - \frac{1}{\varepsilon} \end{bmatrix}$$

$$\widetilde{W} = \begin{bmatrix} -\Gamma - A_2^T A_2 & -\Xi \\ -\Xi^T & 1 - \frac{1}{\varepsilon} \end{bmatrix}$$

where

$$\Gamma := (A_1 - LA_2)^T P + P(A_1 - LA_2) + \frac{\|d_2\|^2}{\varepsilon} P^2$$

$$\Xi := P\left(d_1 - \frac{d_2}{\varepsilon}\right)$$

In order to get the least conservative bound for the fault estimation error $f - \hat{f}$, one approach is to minimize ε such that W is positive definite and $\widetilde{W}$ is semi-positive definite.

5. SIMULATION EXAMPLE

Consider a DC-to-DC boost power converter ((Ortega, Schaft, Maschke & G. 2002)). The dynamical equation describing this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\mathcal{D} \\ \mathcal{D} & -\frac{1}{\mathcal{R}_L} \end{bmatrix} \begin{bmatrix} \frac{1}{\mathcal{L}} x_1 \\ \frac{1}{\mathcal{C}} x_2 \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} V_{in} \quad (46)$$

where x_1 is the induction flux, x_2 is the charge in the capacitor and V_{in} represents the input voltage. The scalar $\mathcal{C}$ is the capacitance, $\mathcal{L}$ is the inductance and the constant $\mathcal{D}$ $(0 < D < 1)$ is the duty ratio. $\mathcal{R}_L$ is the output load resistance and E is the DC voltage. The system in (46) can be expressed in the form of (1)–(2) as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\mathcal{D}}{\mathcal{C}} x_2 \\ \frac{\mathcal{D}}{\mathcal{L}} x_1 - \frac{1}{\mathcal{R}_L \mathcal{C}} x_2 \end{bmatrix}}_{F(x)} + \underbrace{\begin{bmatrix} Eu \\ 0 \end{bmatrix}}_{G(x,u)} + \underbrace{\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}_{\Phi(x,u)} \theta + \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix} k_f}_{\Psi(x,u)} f$$

$$y = -\frac{\mathcal{C}}{\mathcal{D}} x_1$$

where $u := V_{in}$ is the input and $y = -\frac{\mathcal{C}}{\mathcal{D}} x_1$ is the output. The parameter θ is assumed to be a constant disturbance affecting the system and $f(t)$ is an 'actuator fault' added to illustrate the results obtained in this paper (and is not a feature of Ortega et al. (2002)). By direct computation, it follows that Assumptions 1 and 2 are both satisfied. Then introduce the coordinate transformations

$$\begin{aligned} w_1 &= x_2 - \frac{1}{\mathcal{R}_L \mathcal{D}} x_1 \\ w_2 &= -\frac{\mathcal{C}}{\mathcal{D}} x_1 \end{aligned} \quad \text{and} \quad z = \begin{bmatrix} 1 & -l \\ 0 & 1 \end{bmatrix} w$$

where $z := \mathrm{col}(z_1, z_2)$. Then the system can be described in the coordinates z in the canonical form (12)–(14) as

$$\begin{aligned} \dot{z}_1 = &-lz_1 - l^2 z_2 + \underbrace{(\frac{l}{\mathcal{R}_L} - \frac{D^2}{\mathcal{L}})\frac{y}{\mathcal{C}} + (l\mathcal{C} - \frac{1}{\mathcal{R}_L})\frac{E}{\mathcal{D}} u}_{g_1} \\ &+ \underbrace{(k_2 - \frac{k_1}{\mathcal{R}_L \mathcal{D}} + \frac{\mathcal{C} l}{\mathcal{D}} k_1)}_{\phi_1} \theta + \underbrace{(l\mathcal{C} - \frac{1}{\mathcal{R}_L})\frac{E}{\mathcal{D}} k_f}_{\psi_1} f(t) \end{aligned} \quad (47)$$

$$\begin{aligned} \dot{z}_2 = &z_1 + lz_2 + \underbrace{(-\frac{1}{\mathcal{R}_L \mathcal{C}} y - \frac{\mathcal{C} E}{\mathcal{D}} u)}_{g_2} + \underbrace{(-\frac{\mathcal{C}}{\mathcal{D}} k_1)}_{\phi_2} \theta \\ &+ \underbrace{(-\frac{\mathcal{C}}{\mathcal{D}} E) k_f}_{\psi_2} f(t) \end{aligned} \quad (48)$$

$$y = z_2 \quad (49)$$

From (18)–(19), the corresponding candidate observer for system (47)–(49) is given by

$$\begin{aligned} \dot{\hat{z}}_1 = &-l\hat{z}_1 - l^2 y + (\frac{l}{\mathcal{R}_L} - \frac{D^2}{\mathcal{L}})\frac{y}{\mathcal{C}} + (l\mathcal{C} - \frac{1}{\mathcal{R}_L})\frac{E}{\mathcal{D}} u \\ &+ (k_2 - \frac{k_1}{\mathcal{R}_L \mathcal{D}} + \frac{\mathcal{C} l}{\mathcal{D}} k_1)\hat{\theta} \end{aligned} \quad (50)$$

$$\begin{aligned} \dot{\hat{z}}_2 = &\hat{z}_1 + ly - \Lambda(\hat{z}_2 - y) + (-\frac{1}{\mathcal{R}_L \mathcal{C}} y - \frac{\mathcal{C} E}{\mathcal{D}} u) \\ &+ (-\frac{\mathcal{C}}{\mathcal{D}} k_1)\hat{\theta} + \nu \end{aligned} \quad (51)$$

$$\hat{y} = \hat{z}_2 \quad (52)$$

where ν is defined by (22) and $\hat{\theta}$ is given by

$$\dot{\hat{\theta}} = \frac{\mathcal{C}}{\mathcal{D}} k_1 \left(-\frac{\mathcal{C}}{\mathcal{D}} k_1 \hat{\theta} + \hat{z}_1 + ly - \frac{1}{\mathcal{R}_L \mathcal{C}} y - \frac{\mathcal{C} E}{\mathcal{D}} u - \dot{y} \right) \quad (53)$$

As in (Sira-Ramirez, Perez-Moreno, Ortega & Garcia-Esteban 1997), the parameters are chosen as: $\mathcal{C} = 20\mu F$, $\mathcal{L} = 20\text{mH}$, $\mathcal{R}_L = 0.05\Omega$, $\mathcal{D} = 0.5$ and $E = 15\text{V}$. If $k_1 = 0.2$, $k_2 = 0$, $k_f = 0.02$ and the design parameters are chosen as $l = 1$, $P = 10$, $\Lambda = 1.8$, $\eta = 2.5$ and $\varepsilon = 1.1$, by direct computation, it follows that

$$W = \begin{bmatrix} 20 & 1.0000 \\ 1 & 1.0909 \end{bmatrix} \quad \text{and} \quad \widetilde{W} = \begin{bmatrix} 19 & 0 \\ 0 & 0.0909 \end{bmatrix}$$

Clearly, both matrices are positive definite and thus (50)–(53) is a sliding mode observer of the system in (47)–(49). The corresponding estimates for x_1 and x_2 are given by $\hat{x}_1 = -\frac{D}{C}\hat{z}_2$ and $\hat{x}_2 = \hat{z}_1 + (L - \frac{1}{R_L C})\hat{z}_2$ respectively.

For simulation purposes, choose $\theta = -1.5$, $x_{10}(0) = 8$, $x_{20}(0) = 14$, $\hat{x}_{10}(0) = -5$, $\hat{x}_{20}(0) = -12$ and $\hat{\theta}_0(0) = 15$. A simple feedback controller $u = [0.0133 \quad 0.0077]x$ has been introduced to stabilise the nominal system for demonstration purposes. The simulation results in Figures 1 and 2 show that the proposed approach is effective.

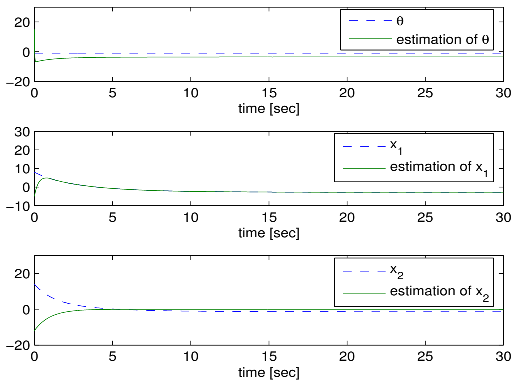

Fig. 1. The responses of the system and the corresponding observer with the adaptive estimation

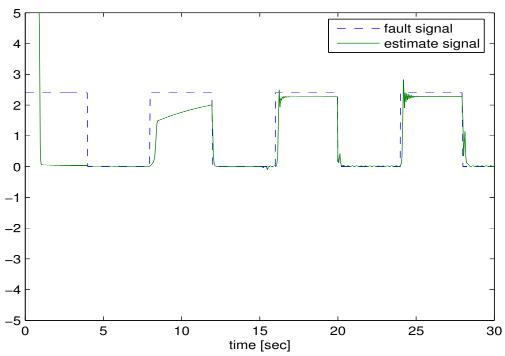

Fig. 2. The fault signal and its estimation

6. CONCLUSION

A class of single output nonlinear systems has been considered in this paper. By using a geometric approach and sliding mode techniques, a sliding mode observer has been established in the presence of a fault and an uncertain parameter. Using the observer, a fault estimation signal has been proposed using the output estimation error injection signal. An approach based on the $\mathcal{L}_2$ norm has been suggested as a measure of the fault estimation error.

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