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Anti-windup and the preservation of robustness against structured norm-bounded uncertainty

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Abstract: We consider robustness preserving anti-windup with structured norm-bounded uncertainty. A sufficient condition for the existence of such anti-windup is given, together with an expression for its construction. Existing results in the literature for additive unstructured uncertainty appear as a special case. The so-called IMC (internal model control) anti-windup does not necessarily preserve robustness for the general case.

1. INTRODUCTION

One of the primary purposes of feedback is to deal with uncertainty. It has become widely accepted that the appropriate framework for representing uncertain dynamics for linear control systems is structured norm-bounded uncertainty (e.g. McFarlane and Glover, 1990; Vinnicombe, 2001; Skogestad and Postlethwaite, 2005); i.e. the input-output map from u to y of the plant to be controlled should be modeled as (Fig 1)

$$\begin{bmatrix} p_\Delta \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} q_\Delta \\ u \end{bmatrix}$$
$$q_\Delta = \Delta p_\Delta \quad (1)$$

where G_{22} is the nominal plant model, G_{12}, G_{21} and G_{11} are known transfer function matrices and Δ satisfies

$$\|\Delta p_\Delta\|_\Gamma^2 \leq \|p_\Delta\|_\Gamma^2/\gamma_\Delta^2 \text{ for all } p_\Delta \in L_2^n[0,\infty) \quad (2)$$

Here Γ belongs to some specified class of positive definite symmetric matrix and γ_Δ is some positive scalar. Note that it is standard to set $\gamma_\Delta = 1$ without loss of generality, as its role may be subsumed within G_{11}, G_{12} and G_{21}; however we are interested in comparing the robustness of various control schemes, and hence will find it a useful measure of uncertainty given a fixed structure defined by G_{11}, G_{12}, G_{21} and the class of Γ.

A common nonlinearity encountered in practical control systems is actuator saturation. Anti-windup describes a control strategy for dealing with such actuator constraints. One definition of the anti-windup problem (e.g. Mulder et al., 2001) is that the linear controller should be designed *a priori*, and the anti-windup should be designed *a posteori.* Although there has been considerable recent interest in robust anti-windup (e.g. Tarbouriech and Garcia, 1997; Grimm et al., 2004), most studies do not consider norm-bounded uncertainty. There are two notable exceptions:

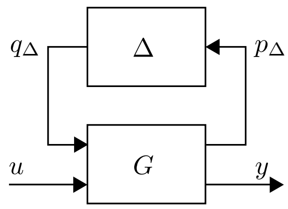

Fig. 1. Plant model with structured norm-bounded uncertainty.

(1) The IQC (integral quadratic constraint) framework of Megretski and Rantzer (1997) is naturally suited to anti-windup with norm-bounded uncertainty. A case study is discussed in Jönsson and Rantzer (2000).
(2) Turner et al. (2007) consider the robustness of anti-windup schemes to additive unstructured norm-bounded uncertainty—i.e. where $G_{11} = 0$, $G_{12} = I$, $G_{21} = I$ and $\Gamma = I$. In particular they derive a sufficient condition for robust stability and show that the so-called IMC (internal model control) anti-windup is optimally robust in the sense that it preserves the robustness of the unsaturated loop. They then suggest an anti-windup design procedure that takes into account the trade-off between performance and robustness.

Note that Saeki and Wada (2002) take an approach similar to that of IQC analysis for static anti-windup compensators, based on the theory of passivity. Meanwhile the work of Turner et al. (2007) is founded on a tradition of approaches which aim to limit the $\mathcal{L}_2$ gain of the anti-windup loop (e.g Mulder et al., 2001; Crawshaw and Vinnicombe, 2000)

In this paper we consider the *analytical* results of Turner et al. (2007) in the IQC framework of Megretski and Rantzer (1997). In particular we show that the stability condition of Turner et al. (2007) is sufficient for the standard IQC stability condition; i.e. the IQC condition

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is more general and potentially less conservative. Furthermore the IQC framework allows us to generalize the results of Turner et al. (2007) to structured norm-bounded uncertainty. Our main result is to find an expression for a robustness preserving anti-windup scheme with more general uncertainty structure. The results of Turner et al. (2007) emerge as a special case; in particular the robustness preserving anti-windup scheme does *not* necessarily correspond to the IMC anti-windup scheme for the general case.

Throughout this paper we assume that both the nominal plant and the actual plant are open loop stable. We also assume that the closed loop system without saturation is both nominally and robustly stable.

2. ANTI-WINDUP SCHEMES

2.1 The scheme of Turner et al. (2007)

Turner et al. (2007) propose the anti-windup scheme depicted in Fig 2. The control input is given by

$$\begin{aligned} u &= \text{sat}(u_{lin} - u_d) \\ u_{lin} &= K_1 r + K_2 y_{lin} \\ y_{lin} &= y + G_{22} M \tilde{u} \\ u_d &= (M - I)\tilde{u} \\ \tilde{u} &= u_{lin} - u_d - u \end{aligned} \tag{3}$$

The operator {sat} denotes term-by-term saturation so that the ith element of $u = \text{sat}(x)$ is given by

$$u_i = \begin{cases} -1 \text{ for } x_i \le -1 \\ x_i \text{ for } -1 < x_i < 1 \\ 1 \text{ for } 1 \le x_i \end{cases} \tag{4}$$

We have scaled the maximum and minimum values of u_i to $+1$ and -1 respectively without loss of generality. The transfer function matrix G_{22} is the nominal plant (assumed stable) and M is a transfer function matrix that determines the anti-windup scheme. The signal r denotes the set point, and the plant dynamics are represented as

$$y = \tilde{G} \begin{bmatrix} d \\ u \end{bmatrix} \tag{5}$$

where d denotes a disturbance signal.

Note that we must have $M(\infty) = I$ for the feedback around the nonlinearity to be strictly proper. If $M(\infty) \neq I$ then the feedback loop is equivalent to the solution of a quadratic program (Syaichu-Rohman et al., 2003). Thus it may be reasonable to allow $M(\infty) \neq I$ for discrete controllers, provided the computational implications can be addressed. For this paper we will assume that $M(\infty) = I$ is a requirement.

For the purposes of stability analysis it is not necessary to consider either the reference signal r or the exogenous disturbance signal d. Thus we let u_{lin} be given by

$$u_{lin} = K_2 y_{lin} \tag{6}$$

and we replace (5) with the input-output relation

$$y = \tilde{G}_u u \tag{7}$$

where $\tilde{G}_u$ is determined by the feedback uncertainty structure (1). In this case, simple block diagram manipulation allows us to reduce the system to that depicted in Fig 3 where

$$H = (I - K_2 G_{22})^{-1} K_2 \tag{8}$$

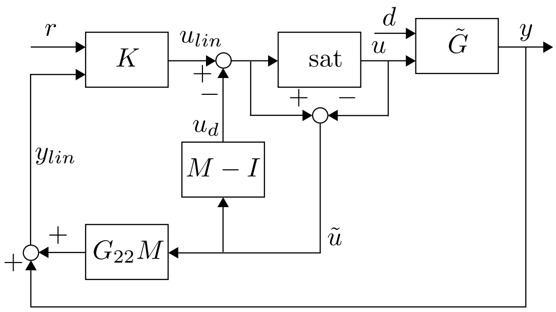

Fig. 2. The anti-windup scheme of Turner et al. (2007). The figure is closely based on one in their paper.

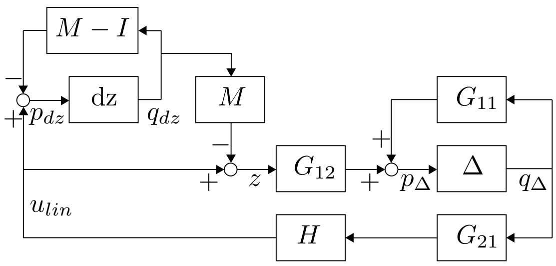

Fig. 3. Equivalent scheme for stability analysis.

and we have the relations

$$\begin{aligned} u_{lin} &= H G_{21} q_\Delta \\ q_\Delta &= \Delta p_\Delta \\ p_\Delta &= G_{11} q_\Delta + G_{12} z \\ z &= u_{lin} - M q_{dz} \\ q_{dz} &= \text{dz}(p_{dz}) \\ p_{dz} &= u_{lin} - (M - I) q_{dz} \end{aligned} \tag{9}$$

Here {dz} denotes the deadzone operator defined as

$$\text{dz}(x) = x - \text{sat}(x) \tag{10}$$

Note that we have substituted the notation q_{dz} in (9) for $\tilde{u}$ in (3).

2.2 The IMC scheme

Zheng et al. (1994) propose the anti-windup scheme depicted in Fig 4. Here

$$\begin{aligned} u &= \text{sat}(u_p) \\ u_p &= -Q_f y + Q_f G_{22} u - Q_b u \end{aligned} \tag{11}$$

where G_{22} is the nominal model and we have ignored (for the sake of stability analysis) the set point and any feedforward signals. The scheme is based on internal model control (with no saturation) where the input is given by

$$u = -Qy + Q G_{22} u \tag{12}$$

When there is no saturation (11) and (12) are equivalent provided Q_f and Q_b are constrained so that

$$Q = (I + Q_b)^{-1} Q_f \tag{13}$$

Furthermore, it is well-understood (e.g. Kothare et al., 1994) that many anti-windup schemes are formally equivalent (of course, different structures are appropriate for different design procedures). The schemes of Turner et al. (2007) and Zheng et al. (1994) are equivalent if M is invertible and we set

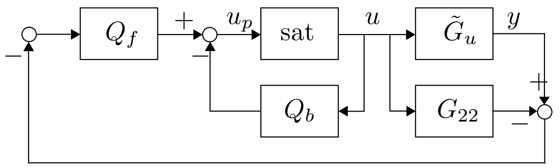

Fig. 4. The anti-windup scheme of Zheng et al. (1994). Following Turner et al. (2007) we refer to the case when $Q_b = 0$ as IMC anti-windup.

$$\begin{aligned} Q &= -(I - K_2 G_{22})^{-1} K_2 \\ Q_f &= M^{-1} Q \\ Q_b &= M^{-1} - I \end{aligned} \tag{14}$$

A special case occurs when $Q_b = 0$. This corresponds to the choice $M = I$, and Turner et al. (2007) refer to this specific case as "IMC anti-windup." We will follow their nomenclature in this contribution.

3. IQC NOTATION AND RESULTS

The material in this section is based on Megretski and Rantzer (1997) and Jönsson (2000). Consider the closed-loop system depicted in Fig 5 with

$$\begin{aligned} q &= \Phi p \\ p &= P q \end{aligned} \tag{15}$$

Here P is some linear time invariant transfer matrix and Φ is an operator that encapsulates any nonlinearity or uncertainty in the loop. We say $\Phi \in \text{IQC}(\Pi)$ for some self-adjoint Π if for all $p, q \in L_2^n[0, \infty)$ with $q = \Phi(p)$ we have the relation

$$\left\langle \begin{bmatrix} p \\ q \end{bmatrix}, \Pi \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle \geq 0 \tag{16}$$

Then under certain technical restrictions the loop is stable provided

$$\begin{bmatrix} P(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} P(j\omega) \\ I \end{bmatrix} < 0 \text{ for all } \omega \tag{17}$$

Suppose

$$\Phi = \begin{bmatrix} \Phi^a & \\ & \Phi^b \end{bmatrix} \tag{18}$$

for some Φ_a, Φ_b satisfying $\Phi^a \in \text{IQC}(\Pi_a)$, $\Phi^b \in \text{IQC}(\Pi_b)$. Then if Π_a and Π_b are structured as

$$\Pi_a = \begin{bmatrix} \Pi_{11}^a & \Pi_{12}^a \\ \Pi_{21}^a & \Pi_{22}^a \end{bmatrix}, \; \Pi_b = \begin{bmatrix} \Pi_{11}^b & \Pi_{12}^b \\ \Pi_{21}^b & \Pi_{22}^b \end{bmatrix} \tag{19}$$

we find $\Phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} \Pi_{11}^a & 0 & \Pi_{12}^a & 0 \\ 0 & \Pi_{11}^b & 0 & \Pi_{12}^b \\ \Pi_{21}^a & 0 & \Pi_{22}^a & 0 \\ 0 & \Pi_{21}^b & 0 & \Pi_{22}^b \end{bmatrix} \tag{20}$$

4. STABILITY ANALYSIS

4.1 Additive uncertainty

Turner et al. (2007) consider the case with unstructured additive uncertainty, i.e. where $G_{11} = 0$, $G_{12} = G_{21} = I$ and $\Gamma = I$. In this case they derive the following condition for determining robust stability.

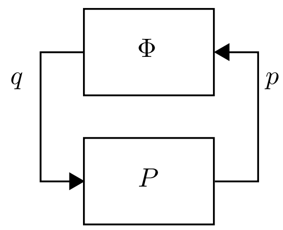

Fig. 5. IQC setup.

Statement 1:
Suppose $\|H\|_\infty = \gamma_h$ and $\|\Delta\|_\infty \leq 1/\gamma_\Delta$. Suppose further M is chosen such that there exist some symmetric $P > 0$ and diagonal $W > 0$ satisfying

$$\frac{d}{dt} x^T P x + \|z\|^2 - \gamma_J^2 \|u_{lin}\|^2 + 2\tilde{u}^T W (u_{lin} - u_d - \tilde{u}) < 0 \tag{21}$$

where x is the state associated with M. Suppose finally that

$$\gamma_J \gamma_h < \gamma_\Delta \tag{22}$$

Then the closed-loop is stable.

Proof:
See Turner et al. (2007). □

Turner et al. (2007) find an LMI that corresponds to this condition (since they are concerned with control synthesis). We will find it more useful to express our results in the frequency domain; equivalence can be established via the KYP lemma (e.g. Boyd et al., 1994).

Statement 2:
Condition (21) is equivalent to the condition $L_J(j\omega) < 0$ for all ω where

$$L_J = \begin{bmatrix} M^*M - WM - M^*W & (W - M^*)H \\ H^*(W - M) & H^*H(1 - \gamma_J^2) \end{bmatrix} \tag{23}$$

Proof:
Let M have state space representation

$$M \sim \left[\begin{array}{c|c} A & B \\ \hline C & I \end{array} \right] \tag{24}$$

Then noting that

$$\begin{aligned} \frac{d}{dt} x &= Ax + B\tilde{u} \\ z &= u_{lin} - Cx - \tilde{u} \\ u_d &= Cx \end{aligned} \tag{25}$$

we obtain the equivalent LMI condition that

$$\begin{bmatrix} PA + A^TP + C^TC & PB + C^T - C^TW & -C^T \\ B^TP + C - WC & I - 2W & W - I \\ -C & W - I & I(1 - \gamma_J^2) \end{bmatrix} < 0 \tag{26}$$

By the KYP lemma this is equivalent to the condition

$$\begin{bmatrix} (j\omega I - A)^{-1}B & 0 \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} C^TC & C^T - C^TW & -C^T \\ C - WC & I - 2W & W - I \\ -C & W - I & I(1 - \gamma_J^2) \end{bmatrix} \times \begin{bmatrix} (j\omega I - A)^{-1}B & 0 \\ I & 0 \\ 0 & I \end{bmatrix} < 0 \text{ for all } \omega \tag{27}$$

which reduces to the condition

$$\begin{bmatrix} M^*M - WM - M^*W & W - M^* \\ W - M & I(1-\gamma_J^2) \end{bmatrix} < 0 \text{ for all } \omega \quad (28)$$

Pre-multiplying by $\text{diag}(I, H^*)$ and post-multiplying by $\text{diag}(I, H)$ gives the result. □

Statement 3:
The result may also be obtained via IQC analysis, if we take the loop nonlinearity to be $\{\text{dz}\}$ and the loop uncertainty to be $(H\Delta)$.

Proof:
The loop takes the form (15) with

$$p = \begin{bmatrix} p_{dz} \\ z \end{bmatrix}, \; q = \begin{bmatrix} q_{dz} \\ u_{lin} \end{bmatrix}$$
$$\Phi = \begin{bmatrix} \text{dz} & \\ & (H\Delta) \end{bmatrix}, \; P = \begin{bmatrix} I - M & I \\ -M & I \end{bmatrix} \quad (29)$$

Since $\|(H\Delta)\|_\infty \le \gamma_h/\gamma_\Delta$ we find $\Phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & 0 & W & 0 \\ 0 & I & 0 & 0 \\ W & 0 & -2W & 0 \\ 0 & 0 & 0 & -\gamma_\Delta^2/\gamma_h^2 I \end{bmatrix} \quad (30)$$

and the stability condition (17) reduces to

$$\begin{bmatrix} M^*M - WM - M^*W & W - M^* \\ W - M & I(1-\gamma_\Delta^2/\gamma_h^2) \end{bmatrix} < 0 \quad (31)$$

□

This immediately suggests that we might obtain a stronger result if we take the loop uncertainty to be Δ alone and include H in the linear time invariant part of the feedback loop. The IQC stability result is then obtained as:

Statement 4:
The condition $L_{IQC}(j\omega) < 0$ for all ω is sufficient for stability where

$$L_{IQC} = \begin{bmatrix} M^*M - M^*W - WM & (W - M^*)H \\ H^*(W - M) & H^*H - \gamma_\Delta^2 I \end{bmatrix} \quad (32)$$

Proof:
The loop takes the form (15) with

$$p = \begin{bmatrix} p_{dz} \\ p_\Delta \end{bmatrix}, \; q = \begin{bmatrix} q_{dz} \\ q_\Delta \end{bmatrix}$$
$$\Phi = \begin{bmatrix} \text{dz} & \\ & \Delta \end{bmatrix}, \; P = \begin{bmatrix} I - M & H \\ -M & H \end{bmatrix} \quad (33)$$

We have $\Phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & 0 & W & 0 \\ 0 & I & 0 & 0 \\ W & 0 & -2W & 0 \\ 0 & 0 & 0 & -\gamma_\Delta^2 I \end{bmatrix} \quad (34)$$

Condition (17) reduces to $L_{IQC} < 0$ in this case. □

We may now observe that the IQC result is less conservative than that of Turner et al. (2007):

Result 1:
If $L_J(j\omega) < 0$ and $\gamma_J\gamma_h < \gamma_\Delta$ then $L_{IQC}(j\omega) < 0$.

Proof:
We have

$$\begin{aligned} L_{IQC} - L_J &= \begin{bmatrix} 0 & 0 \\ 0 & \gamma_J^2 H^*H - \gamma_\Delta^2 I \end{bmatrix} \\ &\le \begin{bmatrix} 0 & 0 \\ 0 & (\gamma_J^2\gamma_h^2 - \gamma_\Delta^2)I \end{bmatrix} \end{aligned} \quad (35)$$

Hence the result. □

Note that although the *analytical* result of Turner et al. (2007) is more conservative, its appeal lies in its simplicity which may be advantageous for anti-windup *synthesis.*

4.2 Structured uncertainty

We now allow more general G_{11}, G_{12}, G_{21} and Γ. We will also admit multipliers in the sector bound for the deadzone; i.e. we will exploit the IQC

$$\left\langle \begin{bmatrix} p_{dz} \\ q_{dz} \end{bmatrix}, \begin{bmatrix} 0 & W^* \\ W & -W - W^* \end{bmatrix} \begin{bmatrix} p_{dz} \\ q_{dz} \end{bmatrix} \right\rangle \ge 0 \quad (36)$$

for all p_{dz}, q_{dz} such that $q_{dz} = \text{dz}(p_{dz})$. The class of admissible W was considered for the single-input single-output case by Zames and Falb (1968), and for the multivariable case by D'Amato et al. (2001) and Mancera and Safonov (2005).

Statement 5:
Set

$$L_S = \begin{bmatrix} L_{11} & L_{21}^* \\ L_{21} & L_{22} \end{bmatrix} \quad (37)$$

with

$$\begin{aligned} L_{11} &= M^*G_{12}^*\Gamma G_{12}M - M^*W^* - WM \\ L_{21} &= G_{21}^*H^*W^* - \Theta^*\Gamma G_{12}M \\ L_{22} &= \Theta^*\Gamma\Theta - \gamma_\Delta^2\Gamma \end{aligned} \quad (38)$$

and

$$\Theta = G_{11} + G_{12}HG_{21} \quad (39)$$

Then $L_S(j\omega) < 0$ for all ω is a sufficient condition for stability.

Proof:
The loop takes the form (15) with

$$p = \begin{bmatrix} p_{dz} \\ p_\Delta \end{bmatrix}, \; q = \begin{bmatrix} q_{dz} \\ q_\Delta \end{bmatrix}$$
$$\Phi = \begin{bmatrix} \text{dz} & \\ & \Delta \end{bmatrix}, \; P = \begin{bmatrix} I - M & HG_{21} \\ -G_{12}M & \Theta \end{bmatrix} \quad (40)$$

We have $\Phi \in \text{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & 0 & W^* & 0 \\ 0 & \Gamma & 0 & 0 \\ W & 0 & -W - W^* & 0 \\ 0 & 0 & 0 & -\gamma_\Delta^2\Gamma \end{bmatrix} \quad (41)$$

Condition (17) reduces to $L_S < 0$ in this case. □

Statement 6:
With IQC analysis it is more common to exploit the sector bound of the saturation directly rather than the sector bound of the implicitly associated deadzone. However it makes no difference to the stability analysis provided both

M and M^{-1} exist and are stable (of course one or the other may be more conveniently structured for control synthesis).

Proof:
Consider the loop (15) with

$$p = \begin{bmatrix} p_{sat} \\ p_\Delta \end{bmatrix}, \; q = \begin{bmatrix} q_{sat} \\ q_\Delta \end{bmatrix}$$
$$\Phi = \begin{bmatrix} \text{sat} & \\ & \Delta \end{bmatrix}, \; P = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \quad (42)$$

We have $\Phi \in \text{IQC}(\Pi)$ with Π given by (41). Condition (17) reduces to $H_{sat} < 0$ for all ω with

$$H_{sat} = \begin{bmatrix} F_{11}^* W^* + W F_{11} - W - W^* & W F_{12} \\ F_{12}^* W^* & 0 \end{bmatrix} + \begin{bmatrix} F_{21}^* \Gamma F_{21} & F_{21} \Gamma F_{22} \\ F_{22}^* \Gamma F_{21} & F_{22}^* \Gamma F_{22} - \gamma_\Delta^2 \Gamma \end{bmatrix} \quad (43)$$

Suppose instead we consider stability for the loop (15) with

$$p = \begin{bmatrix} p_{dz} \\ p_\Delta \end{bmatrix}, \; q = \begin{bmatrix} q_{dz} \\ q_\Delta \end{bmatrix}$$
$$\Phi = \begin{bmatrix} \text{dz} & \\ & \Delta \end{bmatrix}, \; P = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} \quad (44)$$

As before $\Phi \in \text{IQC}(\Pi)$ with Π given by (41). Condition (17) reduces to $H_{dz} < 0$ for all ω with

$$H_{dz} = \begin{bmatrix} \tilde{F}_{11}^* W^* + W \tilde{F}_{11} - W - W^* & W \tilde{F}_{12} \\ \tilde{F}_{12}^* W^* & 0 \end{bmatrix} + \begin{bmatrix} \tilde{F}_{21}^* \Gamma \tilde{F}_{21} & \tilde{F}_{21} \Gamma \tilde{F}_{22} \\ \tilde{F}_{22}^* \Gamma \tilde{F}_{21} & \tilde{F}_{22}^* \Gamma \tilde{F}_{22} - \gamma_\Delta^2 \Gamma \end{bmatrix} \quad (45)$$

Since

$$p_{sat} = p_{dz} \text{ and } q_{sat} = p_{sat} - q_{dz} \quad (46)$$

we find

$$\tilde{F}_{11} = I - (I - F_{11})^{-1}$$
$$\tilde{F}_{12} = (I - F_{11})^{-1} F_{12}$$
$$\tilde{F}_{21} = -F_{21}(I - F_{11})^{-1}$$
$$\tilde{F}_{22} = F_{21}(I - F_{11})^{-1} F_{12} + F_{22} \quad (47)$$

A tedious calculation shows that

$$H_{dz} = \begin{bmatrix} -(I - F_{11})^{-1} & F_{12}(I - F_{11})^{-1} \\ 0 & I \end{bmatrix}^* H_{sat} \times \begin{bmatrix} -(I - F_{11})^{-1} & F_{12}(I - F_{11})^{-1} \\ 0 & I \end{bmatrix} \quad (48)$$

so that $H_{sat} < 0 \Leftrightarrow H_{dz} < 0$ provided $(I - F_{11})^{-1}$ exists. □

5. PRESERVATION OF ROBUSTNESS

For the case of unstructured additive uncertainty, Turner et al. (2007) show that the IMC anti-windup (where $M = I$) is optimally robust, in the sense that if the unsaturated loop is robust to uncertainty satisfying

$$\|\Delta p\|^2 \leq \|p\|^2 / \gamma_\Delta^2 \quad (49)$$

then setting $M = I$ ensures the loop with anti-windup is also robust to such uncertainty.

We are now in a position to generalize the result to structured norm-bounded uncertainty. In particular we allow uncertainty of the form (1) where Δ satisfies (2).

When there is no saturation we have

$$p_\Delta = \Theta q_\Delta \quad (50)$$

with Θ given by (39). It follows that the unsaturated loop is robustly stable provided

$$\Theta^* \Gamma \Theta - \gamma_\Delta^2 \Gamma < 0 \quad (51)$$

for all ω. We may write this condition $L_{22}(j\omega) < 0$ for all ω where L_{22} is defined in (38). So our anti-windup scheme defined by M preserves robustness if $L_{22}(j\omega) < 0$ for all ω implies there exists an admissible W such that $L_S(j\omega) < 0$. One possible way of achieving this is constructing M as follows:

Result 2:
Suppose we can write

$$G_{11} = \bar{G}_{11} H G_{21} \quad (52)$$

for some $\bar{G}_{11}$ and find some stable and inverse stable M satisfying $M(\infty) = I$ such that the choice

$$W = M^* G_{12}^* \Gamma (G_{12} + \bar{G}_{11}) \quad (53)$$

is admissible. Then M preserves robustness provided

$$G_{12}^* \Gamma G_{12} + \bar{G}_{11}^* \Gamma G_{12} + G_{12}^* \Gamma \bar{G}_{11} > 0 \text{ for all } \omega \quad (54)$$

Proof:
This choice of W gives

$$L_{21} = 0$$
$$L_{11} = -M^* \left(G_{12}^* \Gamma G_{12} + \bar{G}_{11}^* \Gamma G_{12} + G_{12}^* \Gamma \bar{G}_{11} \right) M \quad (55)$$

□

Corollary:
If HG_{21} is invertible then condition (54) is equivalent to the condition

$$\Theta^* \Gamma \Theta - G_{11}^* \Gamma G_{11} > 0 \text{ for all } \omega \quad (56)$$

□

Example 1:
Let $G_{11} = 0$ and suppose $G_{12}^* \Gamma G_{12} > 0$ for all ω. Suppose we take the spectral factorization

$$G_{12}^* \Gamma G_{12} = \Phi^* \Phi \quad (57)$$

where Φ is causal, stable and minimum phase and set

$$W = \Phi^*(\infty) \Phi \quad (58)$$

If W is an admissible multiplier then we can choose

$$M = \Phi^{-1} \Phi(\infty) \quad (59)$$

In particular we find $M(\infty) = I$, M is both stable and minimum phase, and with this choice $L_{11} = -\Phi^*(\infty)\Phi(\infty)$ and $L_{21} = 0$. So $L_S < 0$ for all ω. □

Example 2:
If $G_{11} = 0$, G_{12} is diagonal with scalar elements and Γ is diagonal then the construction of Example 1 yields W with scalar diagonal elements. Hence W is admissible. Furthermore $M = I$, so IMC anti-windup preserves robustness. □

Note that the result of Turner et al. (2007) is a special case of Example 2 with $G_{12} = I$, $G_{21} = I$ and $\Gamma = I$. Turner et al. (2007) state that their results can be generalised in a straightforward manner to output multiplicative uncertainty (where $G_{11} = 0$, $G_{21} = I$, $G_{12} = G_{22}$ and $\Gamma = I$) and input multiplicative uncertainty (where $G_{11} = 0$, $G_{21} = G_{22}$, $G_{12} = I$ and $\Gamma = I$). Example 2 confirms their statement for input multiplicative uncertainty. However their statement for output multiplicative uncertainty is not true: IMC anti-windup is not guaranteed to preserve robustness in this case (we have no guarantee that $L_S < 0$ for all ω). Furthermore, if the construction of W in Example 1 yields an admissible multiplier then there exists a robustness preserving anti-windup which is not necessarily the IMC anti-windup.

Finally we remark that there need not exist any anti-windup that preserves the robustness of the unsaturated loop. Nevertheless there will exist an anti-windup that is robust to a smaller class of uncertainty.

6. CONCLUSION

We have re-examined the analytical results of Turner et al. (2007) using the IQC framework of Megretski and Rantzer (1997). We have two main results:

(1) The stability result of Turner et al. (2007) may be subsumed within the IQC framework, which gives a more general result.
(2) We have established conditions for the existence of robustness preserving anti-windup with more general (and practical) uncertainty structures than those considered by Turner et al. (2007). In particular, the IMC anti-windup need not preserve robustness except for very specific uncertainty structures.

Although our results may seem to suggest that IQC methods are superior to those of Turner et al. (2007), we have only addressed robust stability. Anti-windup design involves a trade-off between (robust) stability and performance (Turner et al., 2007; Galeani and Teel, 2004), and choice of structure is likely to be as important as analytical method. Indeed the analytical results of this contribution are inspired by the insights offered by the structure of Turner et al. (2007). We believe that such insights may be fruitfully combined with the IQC framework of Megretski and Rantzer (1997) for design of anti-windup with structured norm-bounded uncertainty; this is the subject of current research.

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