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Improved multipliers for input-constrained model predictive control

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Abstract: The stability and robustness of input-constrained model predictive control can be analyzed using the theory of integral quadratic constraints. We demonstrate the existence of improved multipliers when there are only stage constraints. This can significantly reduce the conservatism of any stability analysis, and we illustrate the improved performance with a simple numerical example.

1. INTRODUCTION

1.1 Overview

Model predictive control (MPC) has found widespread use and success in the process industries (Qin and Badgwell, 2003). Despite this success, it remains hard to guarantee that a controller is robust without introducing prohibitive complexity (Mayne et al., 2000). One reason is that standard approaches address controllers with general nonlinear models and state constraints (see Magni and Scattolini, 2007, for a useful survey). Many practical problems involve only linear stable models with input constraints. For this case it is straightforward to find output-feedback controllers with arbitrary horizon that are robust to structured and unstructured uncertainty (Heath et al., 2006).

The approach of Heath et al. (2006) uses the framework of integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997), and is based on the observation that the associated quadratic program is sector bounded (Heath et al., 2005). The conservativeness sometimes associated with such analysis can be reduced in many ways. One approach is to interpret the results in terms of the theory of dissipative systems (Hill and Moylan, 1977); modifying the implicitly associated storage function can lead to significantly improved results (Løvaas et al., 2007). Alternatively, if the constraints are static then results can be improved by the use of Zames-Falb multipliers (Heath and Wills, 2007).

In this paper we assume the input constraints are restricted to the class of stage constraints:

$$u_i \in \mathbb{U}_i \text{ for } i = 0, \ldots, N-1 \tag{1}$$

where u_i is the predicted i-step-ahead control input, $N-1$ is the control horizon and each $\mathbb{U}_i$ is some convex polytopic constraint set which includes the origin, *but is independent of* u_j *for* $j = 0, \ldots, N-1$. The class includes simple bounds on the actuators, and also constraints where several actuators are constrained to move in a partially coordinated manner. In general it *excludes* rate constraints, which can only be expressed as stage constraints if additional state constraints are introduced. A typical example of input stage constraints occurs in cross-directional control for web processes where adjacent actuators are constrained so as not to cause excessive bending to the slice lip (Van Antwerp et al., 2007; Heath, 1996). We show that for such stage constraints a wider class of multiplier is applicable than reported by Heath and Wills (2007). A special case occurs when there are only simple bounds on the actuators, and was reported by Heath (2006).

The key idea is to represent the quadratic program ϕ associated with the model predictive control itself as an equivalent feedback structure. The structure has a modified quadratic program ψ in the forward path together with a linear feedback term. With certain constraint structures and by construction the modified quadratic program ψ can be separated into several smaller quadratic programs θ_i acting in parallel. Multipliers can then be associated with each quadratic program θ_i; furthermore when the quadratic programs θ_i are identical we can exploit the results of Mancera and Safonov (2005).

The feedback structure has been considered before in the special case of box constraints (Soroush and Muske, 2000; Syaichu-Rohman et al., 2003; Heath, 2006); here the nonlinearity in the forward path can be reduced to several saturation functions in parallel. Note that Soroush and Muske (2000) and Syaichu-Rohman et al. (2003) are both primarily concerned with computation and not with stability analysis. The idea of generalizing the class of multipliers by use of feedback structures was suggested by D'Amato et al. (2001) in the context of repeated SISO nonlinearities; in this paper the nonlinearities θ_i appear as (not necessarily repeated) MIMO nonlinearities.

The paper is structured as follows. In Section 2 we demonstrate the equivalent feedback structures. In Section 3 we consider the application to model predictive control with stage input constraints. We illustrate a simple numerical application in Section 4 where an improved stability result is obtained.

1.2 Notation

As in Heath (2006) we follow the notation of D'Amato et al. (2001) and Heath and Wills (2007) with the following exceptions:

(1) We use lower and upper case letters to distinguish scalar (or vector) and matrix valued functions respectively. Thus we use $\tilde{h}$ to denote the Fourier transform of h and $\tilde{H}$ to denote the Fourier transform of H.
(2) We use Γ to denote the Hessian in a quadratic program.

Following Megretski and Rantzer (1997) we write

$$f \in \mathrm{IQC}(\Pi) \tag{2}$$

or more loosely we say f satisfies the IQC defined by Π to denote

$$\left\langle \begin{bmatrix} x \\ f(x) \end{bmatrix}, \Pi \begin{bmatrix} x \\ f(x) \end{bmatrix} \right\rangle \geq 0 \text{ for all } x \tag{3}$$

On occasion we will find it useful to determine Π in terms of its four block entries:

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \tag{4}$$

Following Jönsson (2000) we define the diagonal augmentation of $\Pi^1, \ldots, \Pi^N$ as

$$\mathrm{daug}(\Pi^1, \ldots, \Pi^N) = \begin{bmatrix} \Pi^1_{11} & & & \Pi^1_{12} & & \\ & \ddots & & & \ddots & \\ & & \Pi^N_{11} & & & \Pi^N_{12} \\ \Pi^1_{21} & & & \Pi^1_{22} & & \\ & \ddots & & & \ddots & \\ & & \Pi^N_{21} & & & \Pi^N_{22} \end{bmatrix} \tag{5}$$

For any $M \in \mathbb{R}^{m,n}$ with rank r define $M^c \in \mathbb{R}^{n-r,n}$ such that $M^c M^T = 0$, $M^c M^{cT} = I$ and $[M^T, M^{cT}]^T \in \mathbb{R}^{m+n-r,n}$ has rank n. Also define $\bar{M} \in \mathbb{R}^{r,n}$ as

$$\bar{M} = \begin{cases} M^{cc} \text{ when } r < n \\ I \text{ when } r = n \end{cases} \tag{6}$$

so that $\bar{M}\bar{M}^T = I$ and the rows of $\bar{M}$ form an orthonormal basis of the space spanned by the rows of M.

We use $\preceq$ to denote row-wise non-strict inequality.

2. BLOCK STRUCTURE RESULTS

Let $\phi(y)$ be the quadratic program

$$\phi(y) = \arg\min_u \frac{1}{2} u^T \Gamma u + u^T y \text{ subject to } Lu \preceq b \tag{7}$$

with $\Gamma > 0$ fixed and $b \succeq 0$.

2.1 Available IQCs

We begin by recalling the various IQCs satisfied by ϕ:

Lemma 1:

(1) Suppose ϕ takes the form (7). Then $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} 0 & -I \\ -I & -2\Gamma \end{bmatrix} \tag{8}$$

(2) Suppose further that L and b are fixed. Let $h \in L_1$ (or $h \in l_1$) satisfy $||h||_1 < 1$ and either let ϕ be odd or let $h(t) \geq 0$ for all t. Let $\tilde{h}$ be the continuous (or discrete) Fourier transform of h.
(a) $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} 0 & (\tilde{h}^* - 1)I \\ (\tilde{h} - 1)I & 0 \end{bmatrix} \tag{9}$$

(b) $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} 0 & (1+\varepsilon)(\tilde{h}^* - 1)I \\ (1+\varepsilon)(\tilde{h} - 1)I & (\tilde{h} + \tilde{h}^* - 2)\Gamma \end{bmatrix} \tag{10}$$

for any $\varepsilon > 0$.

Proof:
For 1) see Heath et al. (2005); for 2) see Heath and Wills (2007). □

2.2 Equivalent feedback structure

We now show that the quadratic program ϕ is equivalent to a feedback circuit with a related quadratic program in the feedback loop. The following lemma is a generalization of results of Soroush and Muske (2000) and Syaichu-Rohman et al. (2003) where the constraints are box constraints and it can be shown that ϕ is equivalent to a number of parallel saturation functions together with a linear feedback. Here we require a more general nonlinearity in the forward path. Let $\psi(y)$ be the quadratic program

$$\psi(y) = \arg\min_u \frac{1}{2} u^T \Gamma_\psi u + u^T y \text{ subject to } Lu \preceq b \tag{11}$$

Lemma 2:
Setting $u = \phi(y)$ in (7) is equivalent to the feedback structure $u = \psi(x)$ given by (11) with $x = y - (\Gamma_\psi - \Gamma)u$.

Proof:
See Appendix. □

Since

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I & (\Gamma - \Gamma_\psi) \\ & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{12}$$

the following Corollary follows immediately:

Corollary:
If $\psi \in \mathrm{IQC}(\Pi_\psi)$ for some Π_ψ then $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} I & \\ (\Gamma - \Gamma_\psi) & I \end{bmatrix} \Pi_\psi \begin{bmatrix} I & (\Gamma - \Gamma_\psi) \\ & I \end{bmatrix} \tag{13}$$

□

The equivalent Corollary was exploited by Heath (2006) to generalize the class of available multipliers for the case where there are only box constraints. In this paper we will endeavor to find forms which allow us to increase the available set of multipliers when the constraints are more general.

2.3 Constraint analysis

Our results require L to be structured in a certain manner. Suppose we can partition L and b as

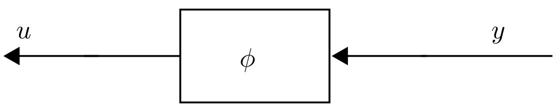

Fig. 1. The nonlinearity ϕ.

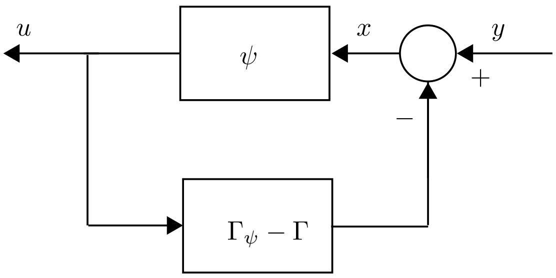

Fig. 2. The nonlinearity ϕ expressed as the nonlinearity ψ with linear feedback.

$$L = \begin{bmatrix} L_0 \\ \vdots \\ L_{N_L-1} \end{bmatrix} \text{ and } b = \begin{bmatrix} b_0 \\ \vdots \\ b_{N_L-1} \end{bmatrix} \tag{14}$$

with

$$L_i L_j^T = 0 \text{ for } i \neq j \tag{15}$$

Set

$$\Gamma_\psi = I \tag{16}$$

We can exploit the orthogonality of the terms L_i to break the quadratic program ψ into several smaller quadratic programs.

Lemma 3:
Suppose ψ takes the form (11). Then we may write $u = \psi(x)$ as

$$u = \sum_{i=0}^{N_L} u_i \tag{17}$$

with

$$\begin{aligned} u_i = \arg\min_{\tilde{u}_i} \frac{1}{2}\tilde{u}_i^T \tilde{u}_i + \tilde{u}_i^T x \text{ subject to } L_i \tilde{u}_i \preceq b_i \\ \text{and } L_i^c \tilde{u}_i = 0 \end{aligned} \tag{18}$$

for $i = 0, \ldots, N_L - 1$ and

$$\begin{aligned} u_{N_L} = \arg\min_{\tilde{u}_{N_L}} \frac{1}{2}\tilde{u}_{N_L}^T \tilde{u}_{N_L} + \tilde{u}_{N_L}^T x \\ \text{subject to } L\tilde{u}_{N_L} = 0 \end{aligned} \tag{19}$$

Proof:
See Appendix. □

Let $\theta_i(p)$ be the quadratic program

$$\theta_i(p) = \arg\min_q \frac{1}{2} q^T q + q^T p \text{ subject to } L_i \bar{L}_i^T q \preceq b_i \tag{20}$$

Then we can write each u_i (for $i = 0, \ldots N_L - 1$) as

$$u_i(x) = \bar{L}_i^T \theta_i(\bar{L}_i x) \tag{21}$$

We may also write u_{N_L} in this form if we define (with some abuse of notation) $L_{N_L} = 0$ and $\bar{L}_{N_L} = L^c$, since

$$u_{N_L} = -L^{cT} L^c x \tag{22}$$

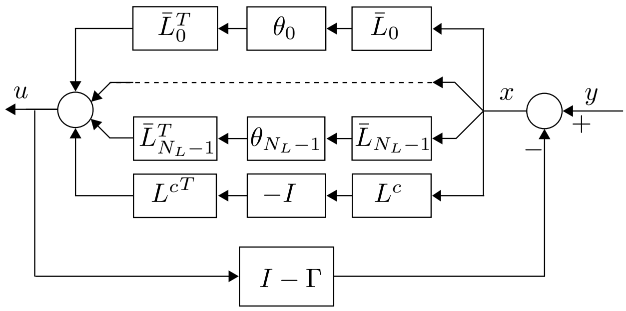

Fig. 3. The nonlinearity ϕ expressed as the nonlinearities θ_i with linear feedback.

Suppose $\theta_i \in \mathrm{IQC}(\Pi_i)$ for some Π_i. Since the $\bar{L}_i$'s are orthogonal we have $\psi \in \mathrm{IQC}(\Pi_\psi)$ with

$$\Pi_\psi = \sum_{i=0}^{N_L} \lambda_i \begin{bmatrix} \bar{L}_i^T & \\ & \bar{L}_i^T \end{bmatrix} \Pi_i \begin{bmatrix} \bar{L}_i & \\ & \bar{L}_i \end{bmatrix} \tag{23}$$

for any $\lambda_i \geq 0$ with $i = 0, \ldots, N_L$.

Remark: Since θ_{N_L} is linear, it may be better in the general case to find an IQC for the nonlinearity $\psi(x) - L^{cT}\theta_{N_L}(L^c x)$, and subsume θ_{N_L} within the linear parts of the feedback loop. However for the cases we consider we usually have $L^c = 0$, so the issue does not arise.

Remark: We have the three important special cases:

(1) A special case occurs when L has block diagonal structure
$$L = \begin{bmatrix} \tilde{L}_0 & & \\ & \ddots & \\ & & \tilde{L}_{N_L-1} \end{bmatrix} \tag{24}$$
so that
$$L_i = \begin{bmatrix} 0 & \cdots & \tilde{L}_i & \cdots & 0 \end{bmatrix} \tag{25}$$
If each $\tilde{L}_i$ has full column rank then $\bar{L}_i = [\, 0 \ \cdots \ I \ \cdots \ 0 \,]$ for $i = 0, \ldots, N_L - 1$ and $L^c = 0$. In this case
$$\Pi_\psi = \mathrm{daug}\left[\lambda_0 \Pi_0, \ldots, \lambda_{N_L-1}\Pi_{N_L-1}\right]$$
(2) If each $\tilde{L}_i$ is identical (i.e. $\tilde{L}_i = \tilde{L}$ for some $\tilde{L}$) and each corresponding b_i is identical ($b_i = b$ for some b) then each θ_i is an identical nonlinearity. In this case we can use the results of Mancera and Safonov (2005) where the symmetry is exploited to generalize the class of available multiplier.
(3) A further special case occurs where there are only box constraints. In this case each $\tilde{L}_i$ takes the form
$$\tilde{L}_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{26}$$
This last special case is considered by Heath (2006).

3. APPLICATION TO MPC

The application of the quadratic program in the form (7) to input-constrained MPC is standard, e.g. (Maciejowski, 2002). For completeness we briefly describe the relation for state space MPC without integral action. The generalization to other forms of MPC is straightforward. Note that some forms of MPC may require more than one quadratic program per operation (Muske and Rawlings, 1993), but this need be no impediment to the use of IQC analysis (Heath et al., 2006).

Given a horizon N, let $J = J(X, U)$ describe the cost function

$$J = \frac{1}{2}||x_N - x_{ss}||_P^2 + \frac{1}{2}\sum_{i=1}^{N-1}||x_i - x_{ss}||_Q^2 + \frac{1}{2}\sum_{i=0}^{N-1}||u_i - u_{ss}||_R^2 \tag{27}$$

Here X and U are sequences of predicted states and inputs $X = (x_1, x_2, \ldots, x_N)$ with $x_i \in \mathbb{R}^{n_x}$ and $U = (u_0, u_1, \ldots, u_{N-1})$ with $u_i \in \mathbb{R}^{n_u}$. Where convenient we will consider X and U to be stacked vectors $X \in \mathbb{R}^{Nn_x}$ and $U \in \mathbb{R}^{Nn_u}$ without change of notation. The terms x_{ss} and u_{ss} correspond to desired steady state values. The weighting matrices P and Q are positive semi-definite while R is positive definite.

Let the state evolution model be $x_{i+1} = Ax_i + Bu_i$ and let the input constraint sets be $u_i \in \mathbb{U}_i$ for some polytopic sets $\mathbb{U}_i$ containing the origin. We may express $u_i \in \mathbb{U}_i$ as $\tilde{L}_i u_i \preceq b_i$ with $b_i \succeq 0$. The MPC law may then be defined to be:

MPC: Set $u(t)$ to $u(t) = \bar{E}U^*$ where $\bar{E} = [\,I\ 0 \ldots 0\,]$ and

$$\begin{aligned}[X^*, U^*] = \arg \min_{X,U}\ & J(X, U) \\ \text{s. t.}\ & x_{i+1} = Ax_i + Bu_i, \\ & \tilde{L}_i u_i \preceq b_i \\ & \text{for } i = 0, \ldots, N-1 \end{aligned} \tag{28}$$

The MPC may be expressed in implicit form by projecting onto the equality constraints defined by the model. Introduce the matrices

$$\bar{P} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P \end{bmatrix}, \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$
$$\Phi = \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \Lambda = \begin{bmatrix} A \\ \vdots \\ A^N \end{bmatrix} \tag{29}$$

Note that $\bar{P} = P$ when $N = 1$. Define

$$\begin{aligned} I_x &= [\,I \cdots I\,]^T \text{ with } I_x \in \mathbb{R}^{n_x, Nn_x}, \\ I_u &= [\,I \cdots I\,]^T \text{ with } I_u \in \mathbb{R}^{n_u, Nn_u} \end{aligned} \tag{30}$$

Define the implicit cost

$$\begin{aligned} J_I(U) =& \frac{1}{2}U^T(\bar{R} + \Phi^T\bar{P}\Phi)U \\ &+ U^T\left(\Phi^T\bar{P}\Lambda x_0 - \Phi^T\bar{P}I_x x_{ss} - \bar{R}I_u u_{ss}\right) \end{aligned} \tag{31}$$

We can then replace (28) in the MPC law by expressing U^* as

$$\begin{aligned} U^* = \arg \min_U\ & J_I(U) \\ \text{s. t.}\ & LU \preceq b \end{aligned} \tag{32}$$

where

$$L = \begin{bmatrix} \tilde{L}_0 & & \\ & \ddots & \\ & & \tilde{L}_{N-1} \end{bmatrix} \text{ and } \begin{bmatrix} b_0 \\ \vdots \\ b_{N-1} \end{bmatrix}$$

This is exactly the form we discussed in Section 2; furthermore L takes the structure of the first special case we considered (24).

If further each $\mathbb{U}_i$ is identical (i.e. $\mathbb{U}_i = \mathbb{U}$ for some $\mathbb{U}$) then L takes the structure of the second special case. Similarly if each $\mathbb{U}_i$ consists of only simple bounds on the elements of u_i then L takes the structure of the third special case.

4. SIMULATION EXAMPLE

In this section, we use a simple numerical example to illustrate the application to the MPC developed in Section 3.

Consider a two-input two-output discrete plant with transfer function matrix

$$G(z) = \begin{bmatrix} \dfrac{1.11}{z-0.21} & \dfrac{0.66}{z-0.52} \\ \dfrac{0.94}{z-0.93} & \dfrac{0.53}{z-0.26} \end{bmatrix}$$

We may write $G(z)$ in state space from as

$$G(z) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{33}$$

with

$$A = \begin{bmatrix} 0.21 & & & \\ & 0.93 & & \\ & & 0.52 & \\ & & & 0.26 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1.11 & 0 & 0.66 & 0 \\ 0 & 0.94 & 0 & 0.53 \end{bmatrix} \tag{34}$$

We assume full state information is available. The plant is controlled by an MPC law with cost function (27) where the input horizon $(N-1) = 2$ and the weighting matrices are $Q = I$ and $R = kI$ for some $k \geq 0$. The weighting matrix P is given as the solution of the discrete algebraic Riccati equation

$$P = A^TPA - A^TP(B^TPB + R)^{-1}A^TPB + Q \tag{35}$$

The input is expressed as $u_i = [u_i^{(1)}, u_i^{(2)}]^T$. Suppose that this system is subject to the input constraints of the form $|u_i^{(1)}| \leq 1$, $|u_i^{(2)}| \leq 1$ and $|u_i^{(1)} + u_i^{(2)}| \leq 1$ with $i = 0, \ldots, N-1$. These can be expressed as $\tilde{L}u_i \preceq \tilde{b}$ with

$$\tilde{L} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{36}$$

Hence the constraints for the whole horizon can be expressed as $Lu \preceq b$ with

$$L = \begin{bmatrix} \tilde{L} & & \\ & \ddots & \\ & & \tilde{L} \end{bmatrix} \qquad b = \begin{bmatrix} \tilde{b} \\ \vdots \\ \tilde{b} \end{bmatrix} \tag{37}$$

Then

$$\begin{aligned} L_0 &= \left[\, \tilde{L} \;\; 0 \;\; 0 \,\right] \\ L_1 &= \left[\, 0 \;\; \tilde{L} \;\; 0 \,\right] \\ L_2 &= \left[\, 0 \;\; 0 \;\; \tilde{L} \,\right] \end{aligned} \tag{38}$$

If we let I_2 denote the 2×2 identity matrix and 0 denote the 2×2 zero matrix then

$$\begin{aligned} L_0^c &= \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \\ L_1^c &= \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \\ L_2^c &= \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} \end{aligned} \tag{39}$$

and

$$\begin{aligned} \bar{L}_0 = L_0^{cc} &= [I_2 \;\; 0 \;\; 0] \\ \bar{L}_1 = L_1^{cc} &= [0 \;\; I_2 \;\; 0] \\ \bar{L}_2 = L_2^{cc} &= [0 \;\; 0 \;\; I_2] \end{aligned} \tag{40}$$

If we set $\Gamma_\psi = I$ then ψ can be written as

$$\psi(x) = \sum_{i=0}^{2} \theta_i(x) \tag{41}$$

with

$$\begin{aligned} \theta_i(x) = \arg\min_{u_i} \; & \frac{1}{2} u_i^T u_i + u_i^T x \\ \text{subject to } & L_i u_i \preceq b_i \\ \text{and } & L_i^c u_i = 0 \end{aligned} \tag{42}$$

Since the L_i's and b_i's are fixed both ϕ itself and each θ_i satisfy IQCs of the form (8), (9) and (10). For the sake of this demonstration we will only consider IQCs of the form (8). Thus $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} 0 & -I \\ -I & -2(\bar{R} + \Phi^T \bar{P} \Phi) \end{bmatrix} \tag{43}$$

Meanwhile each θ_i satisfies $\theta_i \in \mathrm{IQC}(\Pi_i)$ with

$$\Pi_i = \begin{bmatrix} 0 & -I_2 \\ -I_2 & -2I_2 \end{bmatrix} \tag{44}$$

Since each θ_i has identical structure, we may exploit the results of Mancera and Safonov (2005) to find IQCs for ψ. Once again for this demonstration we will limit ourselves to exploiting the relation (23). Hence $\psi \in \mathrm{IQC}(\Pi_\psi)$ with

$$\begin{aligned} \Pi_\psi &= \sum_{i=0}^{2} \lambda_i \begin{bmatrix} \bar{L}_i^T & \\ & \bar{L}_i^T \end{bmatrix} \Pi_i \begin{bmatrix} \bar{L}_i & \\ & \bar{L}_i \end{bmatrix} \\ &= \mathrm{daug}(\lambda_0 \Pi_0, \lambda_1 \Pi_1, \lambda_2 \Pi_2) \\ &= \begin{bmatrix} 0 & -\Pi_\lambda \\ -\Pi_\lambda & -2\Pi_\lambda \end{bmatrix} \end{aligned} \tag{45}$$

where 0 denotes the 6×6 zero matrix and Π_λ is defined as

$$\Pi_\lambda = \begin{bmatrix} \lambda_0 I_2 & & \\ & \lambda_1 I_2 & \\ & & \lambda_2 I_2 \end{bmatrix} \tag{46}$$

Suppose we set $x_{ss} = 0$ and $u_{ss} = 0$. Then the MPC cost function (27) can be expressed as

$$J_I(U) = \frac{1}{2} U^T (\bar{R} + \Phi^T \bar{P} \Phi) U + U^T \Phi^T \bar{P} \Lambda x_0 \tag{47}$$

Define $G_x(z) = (zI - A)^{-1} B$ and

$$M(z) = \Phi^T \bar{P} \Lambda G_x \bar{E} \sim \begin{bmatrix} A_M & B_M \\ C_M & D_M \end{bmatrix} \tag{48}$$

The QP for MPC controller can be expressed by an IQC as $\phi \in \mathrm{IQC}(\Pi_\phi)$ with

$$\Pi_\phi = \begin{bmatrix} I & \bar{R} + \Phi^T \bar{P} \Phi - I \\ 0 & I \end{bmatrix}^T \Pi_\psi \begin{bmatrix} I & \bar{R} + \Phi^T \bar{P} \Phi - I \\ 0 & I \end{bmatrix} \tag{49}$$

corresponding to the vector $[x^T, u^T]^T$.

Hence the system is stable if the following inequality can be satisfied

$$\begin{bmatrix} M(e^{j\omega}) \\ I \end{bmatrix}^T \Pi_\phi \begin{bmatrix} M(e^{j\omega}) \\ I \end{bmatrix} \le -\varepsilon I, \quad \forall \omega \in [-\pi, \pi] \tag{50}$$

with some $\varepsilon > 0$

By the KYP lemma, the above inequality can be converted into the following LMI

$$\begin{bmatrix} A_M^T P_M A_M - P_M & A_M^T P_M B_M \\ B_M^T P_M A_M & B_M^T P_M B_M \end{bmatrix} + \tilde{\Pi}_\phi \le -tI, \quad \text{for some } t > 0 \tag{51}$$

with $P_M = P_M^T > 0$ and

$$\tilde{\Pi}_\phi = \begin{bmatrix} C_M & D_M \\ 0 & I \end{bmatrix}^T \Pi_\phi \begin{bmatrix} C_M & D_M \\ 0 & I \end{bmatrix} \tag{52}$$

Recall that the input weighting is $R = kI$. We know (Heath and Wills, 2005) that for k sufficiently big the closed-loop system is stable. Exploiting the IQC (43) we find that the closed-loop system is stable for $k \ge 7.9$. Exploiting the IQC (49) and the results of this paper we find the closed-loop system is guaranteed stable for all k.

5. CONCLUSION

For input constrained MPC the nonlinearity in the controller satisfies certain IQCs. We have shown that it is possible to generalize the class of IQCs for MPC with only stage input constraints of the form $u_i \in \mathbb{U}_i$ where each $\mathbb{U}_i$ is a convex polytope. The results are derived by considering feedback structures to represent the nonlinearity. We have demonstrated with a simple numerical example that the results may lead to considerable reduction of conservativeness in the stability analysis of MPC.

APPENDIX: PROOF OF LEMMAS

Proof of Lemma 2:
The KKT conditions for ψ are

$$\begin{aligned} \Gamma_\psi u + x + L^T \lambda &= 0 \\ Lu - b + s &= 0 \\ s &\succeq 0 \\ \lambda &\succeq 0 \\ \lambda^T s &= 0 \end{aligned} \tag{53}$$

If we substitute for x we obtain

$$\begin{aligned} \Gamma u + y + L^T\lambda &= 0 \\ Lu - b + s &= 0 \\ s &\succeq 0 \\ \lambda &\succeq 0 \\ \lambda^T s &= 0 \end{aligned} \tag{54}$$

These are precisely the KKT conditions for ϕ. □

Proof of Lemma 3:
The KKT conditions for u_i with $i = 0, \ldots N_L - 1$ are

$$u_i + x + L_i^T \lambda_i + L_i^{cT} z_i = 0 \tag{55}$$

$$L_i^c u_i = 0 \tag{56}$$

and

$$\left.\begin{aligned} L_i u_i - b_i + s_i &= 0 \\ \lambda_i^T s_i &= 0 \\ \lambda_i &\succeq 0 \\ s_i &\succeq 0 \end{aligned}\right\} \tag{57}$$

Furthermore

$$u_{N_L} = -L^{cT} L^c x \tag{58}$$

From (55) and (56) we find

$$z_i = -L_i^c x \tag{59}$$

for $i = 0, \ldots, N_L - 1$. Summing (55) over i together with (58) and (59) gives

$$\begin{aligned} u + N_L x + \sum_{i=0}^{N_L-1} L_i^T \lambda_i &= \sum_{i=0}^{N_L-1} L_i^{cT} L_i^c x - L^{cT} L^c x \\ &= \sum_{i=0}^{N_L-1} (I - \bar{L}_i^T \bar{L}_i) x - L^{cT} L^c x \\ &= (N_L - 1)x \end{aligned} \tag{60}$$

Hence

$$u + x + L^T \lambda = 0 \tag{61}$$

with

$$\lambda = \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_{N_L-1} \end{bmatrix} \tag{62}$$

We may also write (57) as

$$\left.\begin{aligned} Lu - b + s &= 0 \\ \lambda^T s &= 0 \\ \lambda &\succeq 0 \\ s &\succeq 0 \end{aligned}\right\} \tag{63}$$

where

$$s = \begin{bmatrix} s_0 \\ \vdots \\ s_{N_L-1} \end{bmatrix} \tag{64}$$

□

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