Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Analysis of periodic solutions in piecewise linear feedback systems via a complementarity approach

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Abstract: Periodic solutions in piecewise linear feedback systems in Luré form, composed by a linear time invariant dynamical system closed in feedback through a static piecewise linear mapping are analyzed. A procedure for representing the closed loop system in complementarity form, based on resistors-diode-sources equivalent circuits of the piecewise linear characteristic is presented. Conditions for the existence of discrete–time periodic solutions in terms of solvability of a suitable static linear complementarity problem are obtained. Numerical results for a chaotic circuit show that the proposed approach can be also used to predict periodic solutions of continuous–time systems in Luré form provided that consistency of the discretization is assumed.

1. INTRODUCTION

We consider piecewise linear feedback systems in the Luré form, i.e. representable as the feedback interconnection of a linear time invariant dynamical system Σ_d with a piecewise linear static characteristic (φ, λ), as reported in Fig. 1. Σ_d represents the linear system with a minimal state space realization. The static characteristic (φ, λ) is a piecewise linear multi–valued mapping, which includes piecewise linear functions (e.g. saturation), set–valued functions (e.g. relay, quantizer) and unbounded characteristics (e.g. ideal diode and Zener diode characteristics). The analysis of periodic solutions in this class of nonsmooth dynamical systems has attracted a wide interest in the literature, see among others Gonçalves (2005); Stan and Sepulchre (2007). A strong motivation for such interest is the relevance of oscillations in the behavior of several practical systems representable in Luré form. Interesting classes of such systems are nonlinear circuits and power electronics converters Frasca (2007). The existence of a periodic solution is often assumed for local and global stability analysis of oscillations. Dealing with limit cycles conditions on the existence are typically obtained by constructing a Poincaré map and by imposing, given a sequence of switches per cycle, the switching conditions Gonçalves (2005); Galias (2005); di Bernardo and Vasca (2000). However such Poincaré maps need the assumption on the structure of the limit cycle and closed form solutions can only be given for very special cases. In Jonsson and Megretski (2006) some conditions for existence and uniqueness of limit cycles were obtained by assuming Lipschitz continuity of the feedback characteristic and its derivative.

In this paper we propose the use of the complementarity formalism for obtaining conditions on the existence of periodic solutions of known period in feedback systems in Luré form forced by periodic inputs. The complementarity formalism has been shown to be useful for well–posedness and stability analysis of piecewise linear feedback systems Brogliato (2004); Çamlıbel et al. (2002, 2006); Iannelli et al. (2006). A complementarity model of the system in Fig. 1 is obtained by using resistors-diode-sources circuits representing the piecewise linear characteristic Chua (1984); Stern (1956). By assuming consistency of the discretization the problem of the existence of a periodic solution is than reformulated as a static linear complementarity problem for which one can use conditions on the existence of a solution. Numerical results with a chaotic circuit show the effectiveness of the proposed approach for detecting both stable and unstable periodic solutions.

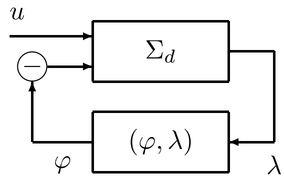

Fig. 1. The class of systems under consideration.

2. PRELIMINARIES

This section presents some facts about linear complementarity problems Cottle et al. (1992).

***Problem** 1.* (LCP(q,L)). Given a real vector q and a real matrix L, find a real vector z such that

$$z \geqslant 0 \tag{1a}$$
$$q + Lz \geqslant 0 \tag{1b}$$
$$z^T(q+Lz) = 0, \tag{1c}$$

where the inequalities are considered componentwise.

In the sequel (1) will be more compactly indicated by means of the complementarity condition

$$0 \leqslant (q + Lz) \perp z \geqslant 0. \tag{2}$$

The meaning of (2) is that if two variables, say $(q + Lz)$ and z, are in a complementarity relation, then at least one of them must be zero and the other will be nonnegative (componentwise interpretation).

***Definition** 1.* A matrix L is called a P-matrix if all its principal minors are strictly positive.

According to the definition, every positive definite matrix is a P-matrix but the converse is not true.

***Theorem** 1.* (Cottle et al., 1992, Theorem 3.3.7) Let L be a real matrix. Then the LCP(q,L) has a unique solution for any real vector q if and only if L is a P-matrix.

We now introduce the concept of a complementarity system, i.e. a linear system whose dynamics must satisfy a LCP for each time instant.

***Definition** 2.* A discrete–time linear complementarity system (LCS) is the following linear system subject to complementarity constraints on z and w variables:

$$x_k = Ax_{k-1} + Bz_k + Eu_k \quad (3a)$$
$$w_k = Cx_{k-1} + Dz_k + Fu_k \quad (3b)$$
$$0 \leqslant w_k \perp z_k \geqslant 0, \quad (3c)$$

where x, u, z and w are real vectors (w and z of the same dimension), k is the discrete–time variable and A, B, C, D, E, F are matrices of suitable dimensions.

It appears evident how the LCP($Cx_{k-1}+Fu_k$, D) must be feasible for each discrete step k since its solution z_k affects the dynamics.

3. PIECEWISE LINEAR CHARACTERISTICS

In this section we detail a possible procedure for representing piecewise linear characteristics (φ, λ) (examples are reported in Fig. 2) in the following complementarity formalism

$$\varphi = A_s\lambda + B_s z + g_s \quad (4a)$$
$$w = C_s\lambda + D_s z + h_s \quad (4b)$$
$$0 \leq w \perp z \geq 0, \quad (4c)$$

where the real matrices A_s, B_s, g_s, C_s, D_s and h_s have suitable dimensions. Such representation can be considered really general for describing set–valued piecewise linear mappings Leenaerts and Bokhoven (1998). Different techniques can be used to obtain the model (4). Here we propose to obtain the representation by using Resistor-Diode-Source (RDS) equivalent circuits corresponding to a given piecewise linear characteristic Stern (1956); Chua (1984). The interesting feature of the proposed representation is that, if Σ_d is passive and the characteristic is nondecreasing, the representation preserves the passivity of the closed loop system which is an important property for obtaining well-posedness and stability results Iannelli et al. (2006). In order to obtain the complementarity model (4) it is useful to analyze first the characteristic of the so-called Ideal Diode (ID), see Fig. 3. Such (φ, λ) characteristic can be written in the complementarity form (4) by choosing $\varphi = z$ and $w = -\lambda$, i.e. $A_s = 0$, $B_s = 1$, $g_s = 0$, $C_s = -1$, $D_s = 0$, $h_s = 0$. The ID characteristic is a particular case of a piecewise–affine nondecreasing convex single breaking point characteristic, see Fig. 4. By using the ID behavior it is possible to show that the (φ, λ) characteristic depicted in Fig. 4 represents the current–voltage characteristic for the circuit shown in the same figure. Note that in Chua (1984) the series of a resistor, an ID and a voltage source is called

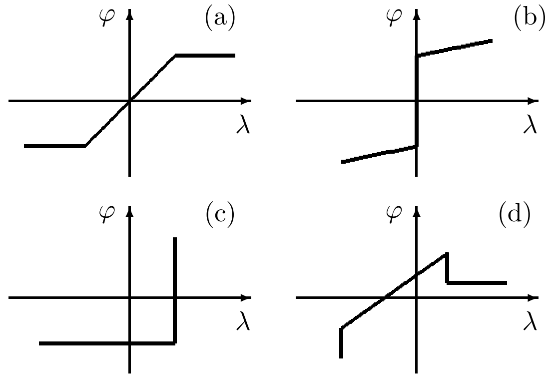

Fig. 2. Piecewise linear mappings (one-dimensional case): (a) piecewise linear function, (b-c-d) set–valued functions, (b-c-d) unbounded characteristics.

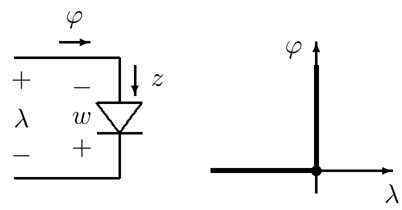

Fig. 3. Ideal Diode symbol with the corresponding (φ, λ) current–voltage characteristic and the indication of a possible pair of complementarity variables.

a *concave resistor*. For $\lambda < \Lambda_1$ the ID is in the blocking state, i.e. $z_1 = 0$. By applying the Kirchhoff current law (KCL) one can write $\varphi = g_0\lambda + \Phi_0$, which by choosing $g_0 \triangleq \sigma_0 \geqslant 0$ is the equation of the lowest affine piece of the characteristic. For $\lambda > \Lambda_1$ the ID will be in the conducting state, then $w_1 = 0$ and $\varphi = g_0\lambda + z_1 + \Phi_0$. By substituting $z_1 = g_1(\lambda - \Lambda_1)$, which is obtained by applying the Kirchhoff voltage law (KVL) to the circuit in Fig. 4, we get $\varphi = (g_0+g_1)\lambda - g_1\Lambda_1 + \Phi_0$. Then by choosing $g_1 \triangleq \sigma_1 - \sigma_0 > 0$ we get $\varphi = \sigma_1\lambda - (\sigma_1 - \sigma_0)\Lambda_1 + \Phi_0$ which is the equation of the second piece of the characteristic in Fig. 4. The (φ, λ) characteristic in Fig. 4 can be represented by integrating both blocking and conducting states of the ID into the following complementarity model:

$$\varphi = g_0\lambda + z_1 + \Phi_0 \quad (5a)$$
$$w_1 = -\lambda + \frac{1}{g_1}z_1 + \Lambda_1 \quad (5b)$$
$$0 \leqslant w_1 \perp z_1 \geqslant 0. \quad (5c)$$

Let us consider now the piecewise–affine nondecreasing concave single breaking point characteristic in Fig. 5. Note that in Chua (1984) the parallel of a resistor, an ID and a current source is also called a *convex resistor*. By using the ID behavior and arguments similar to those presented above, it is simple to show that the (φ, λ) characteristic represents the current–voltage characteristic for the circuit depicted in the same figure. By applying the KCL and KVL to the circuit the (φ, λ) characteristic can be represented in the following complementarity form:

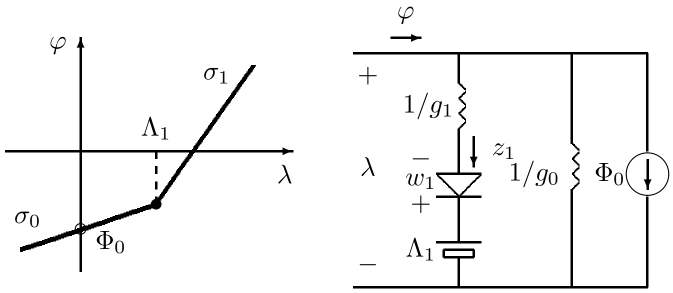

Fig. 4. Piecewise–affine nondecreasing convex single breaking point (φ, λ) characteristic and a corresponding RDS circuit; σ_0 and σ_1 are the slopes of the affine pieces; $1/g_0$ and $1/g_1$ are resistances.

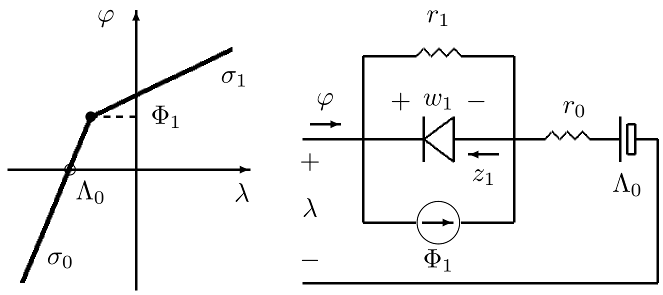

Fig. 5. Piecewise–affine nondecreasing concave single breaking point (φ, λ) characteristic and a corresponding RDS circuit; here r_0 and r_1 are resistances.

$$\varphi = \frac{1}{r_0 + r_1}\lambda - \frac{r_1}{r_0 + r_1} z_1 - \frac{1}{r_0 + r_1}\Lambda_0 + \frac{r_1}{r_0 + r_1}\Phi_1 \tag{6a}$$

$$w_1 = \frac{r_1}{r_0 + r_1}\lambda + \frac{r_0 r_1}{r_0 + r_1} z_1 - \frac{r_1}{r_0 + r_1}\Lambda_0 - \frac{r_0 r_1}{r_0 + r_1}\Phi_1 \tag{6b}$$

$$0 \leqslant w_1 \perp z_1 \geqslant 0, \tag{6c}$$

where $r_0 \triangleq \frac{1}{\sigma_0} \geqslant 0$ and $r_1 \triangleq \frac{1}{\sigma_1} - \frac{1}{\sigma_0} > 0$.

By generalizing the procedure presented above it is possible to obtain a complementarity model (4) for any nondecreasing piecewise affine characteristic (φ, λ), Vasca et al. (2007).

4. COMPLEMENTARITY MODEL

Let us assume that Σ_d in Fig. 1 is a discrete–time linear time invariant system:

$$x_k = A_d x_{k-1} + B_d(-\varphi_k) + E_d u_k \tag{7a}$$

$$\lambda_k = C_d x_k + D_d(-\varphi_k) + F_d u_k, \tag{7b}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and (A_d, B_d, C_d) minimal.

We will consider the representation (4) for the static piecewise linear characteristics (φ, λ) in Fig. 1. By substituting (4) in (7)

$$x_k = A_d x_{k-1} - B_d\left[A_s\lambda_k + B_s z_k + g_s\right] + E_d u_k \tag{8a}$$

$$\lambda_k = C_d x_{k-1} - D_d\left[A_s\lambda_k + B_s z_k + g_s\right] + F_d u_k \tag{8b}$$

$$w_k = C_s\lambda_k + D_s z_k + h_s \tag{8c}$$

$$0 \leqslant w_k \perp z_k \geqslant 0. \tag{8d}$$

By looking at (8b), if the matrix $D_d A_s$ has no eigenvalues in -1, the matrix $M \triangleq I + D_d A_s \in \mathbb{R}^{m\times m}$ is invertible and

$$\lambda_k = M^{-1}\left[C_d x_{k-1} - D_d B_s z_k - D_d g_s + F_d u_k\right]. \tag{9}$$

Now system (8) can be written as (3) with

$$A := A_d - B_d A_s M^{-1} C_d, \tag{10a}$$

$$B := B_d A_s M^{-1} D_d B_s - B_d B_s, \tag{10b}$$

$$C := C_s M^{-1} C_d, \tag{10c}$$

$$D := D_s - C_s M^{-1} D_d B_s, \tag{10d}$$

$$E := \begin{bmatrix} E_d - B_d A_s M^{-1} F_d & g \end{bmatrix}, \tag{10e}$$

$$F := \begin{bmatrix} C_s M^{-1} F_d & h \end{bmatrix}, \tag{10f}$$

where we have included in the vector u also the ones needed to represent the constant term coming from g_s and h_s, i.e.

$$g := B_d\left[A_s M^{-1} D_d - I\right] g_s, \tag{11a}$$

$$h := h_s - C_s M^{-1} D_d g_s. \tag{11b}$$

Note that being M singular, it means that the feedback structure has an algebraic loop not solvable and we get an ill-posed problem. Note that in the case $D_d > 0$ and $A_s \geqslant 0$, the matrix $M = (I + D_d A_s)$ is invertible Haddad and Bernstein (1990). The same can be proved if $D_d \geqslant 0$ and $A_s \geqslant 0$ and diagonal, which is the case for the matrix A_s in the proposed complementarity model.

In next section we analyze the existence of periodic solution of the discrete–time system (3). If Σ_d is a continuous–time linear time invariant system representable as

$$\dot{x} = \tilde{A}_d x + \tilde{B}(-\varphi) + \tilde{E}_d u \tag{12a}$$

$$\lambda = \tilde{C}_d x + \tilde{D}_d(-\varphi) + \tilde{F}_d u, \tag{12b}$$

one can also conclude the existence of periodic solutions of such system by assuming consistency of the disretization. In fact, following a procedure similar to the one presented above, the closed loop system would be

$$\dot{x} = A_c x + B_c z + E_c u \tag{13a}$$

$$w = C_c x + D_c z + F_c u \tag{13b}$$

$$0 \leqslant w \perp z \geqslant 0, \tag{13c}$$

where the matrices are given by (10) with A_d, B_d, C_d, D_d, E_d and F_d replaced by the corresponding matrices with the tilde, respectively. By discretizing (13) by using the backward Euler method with sampling period T_s it is possible to get the following discrete–time linear complementarity system:

$$x_k = x_{k-1} + T_s A_c x_k + T_s B_c z_k + T_s E_c u_k \tag{14a}$$

$$w_k = C_c x_k + D_c z_k + F_c u_k \tag{14b}$$

$$0 \leqslant w_k \perp z_k \geqslant 0 \tag{14c}$$

and thus one obtains (3) with

$$A := (I - T_s A_c)^{-1}, \tag{15a}$$

$$B := (I - T_s A_c)^{-1} T_s B_c, \tag{15b}$$

$$C := C_c (I - T_s A_c)^{-1}, \tag{15c}$$

$$D := D_c + C_c (I - T_s A_c)^{-1} T_s B_c, \tag{15d}$$

$$E := (I - T_s A_c)^{-1} T_s E_c, \tag{15e}$$

$$F := F_c + C_c (I - T_s A_c)^{-1} T_s E_c. \tag{15f}$$

Let us assume that the continuous–time complementarity system (13) has a periodic trajectory of period T. It is natural to assume for the discretization $T_s = T/N$ with N

a positive integer. Note that the continuous–time instants at which conditions (13c) change, i.e. when one or more components of w or z become zero, do not need to be known a priori and do not need to be sampling time instants. In other words the shape of the periodic solution is not fixed a priori. By assuming consistency of the discretization, i.e. the discrete–time system approximates the continuous–time system, the discrete–time complementarity system (3) will have a periodic trajectory of period N. Such arguments are often valid from a practical point of view, so as it will be shown by our numerical results. However from a more theoretical point of view one should prove consistency of the discretization which is a non trivial task in the complementarity framework Çamlıbel (2001); Frasca (2007).

5. EXISTENCE OF PERIODIC SOLUTIONS

Let us consider system (3) which is here repeated for the sake of readability:

$$x_k = Ax_{k-1} + Bz_k + Eu_k \tag{16a}$$
$$w_k = Cx_{k-1} + Dz_k + Fu_k \tag{16b}$$
$$0 \leqslant w_k \perp z_k \geqslant 0 \tag{16c}$$

If (16) has a periodic solution of period N that means $x_N = x_0$. The state evolution gives

$$x_N = A^N x_0 + \sum_{i=1}^{N} A^{N-i}\left(Bz_i + Eu_i\right) = x_0 \tag{17}$$

By solving with respect to x_0 and defining $\Pi_N \triangleq \left(I - A^N\right)^{-1}$:

$$x_0 = \Pi_N \sum_{i=1}^{N} A^{N-i}\left(Bz_i + Eu_i\right). \tag{18}$$

Note that Π_N satisfies the following properties:

$$A\Pi_N = \Pi_N A \tag{19a}$$
$$\Pi_N A^N = \Pi_N - I \tag{19b}$$

By writing (16b) for $k = 1, \dots, N$

$$w_1 = Cx_0 + Dz_1 + Fu_1 \tag{20a}$$
$$w_2 = CAx_0 + CBz_1 + Dz_2 + CEu_1 + Fu_2 \tag{20b}$$
$$\begin{aligned} w_3 = &CA^2x_0 + CABz_1 + CBz_2 + Dz_3 \\ &+ CAEu_1 + CEu_2 + Fu_3 \end{aligned} \tag{20c}$$
$$\vdots$$
$$\begin{aligned} w_N = &CA^{N-1}x_0 + \sum_{i=1}^{N-1} CA^{N-1-i}Bz_i + Dz_N \\ &+ \sum_{i=1}^{N-1} CA^{N-1-i}Eu_i + Fu_N \end{aligned} \tag{20d}$$

By substituting (18) in (20)

$$\begin{aligned} w_k = &CA^{k-1}\Pi_N \left(\sum_{i=1}^{N} A^{N-i}\left(Bz_i + Eu_i\right)\right) \\ &+ \sum_{i=1}^{k-1} CA^{k-1-i}Bz_i + Dz_k + \sum_{i=1}^{k-1} CA^{k-1-i}Eu_i + Fu_k \end{aligned} \tag{21}$$

for $k = 1, \dots, N$.

By collecting all the terms z_i and u_i and by using properties (19) one can write

$$\begin{aligned} w_1 = &C\Pi_N A^{N-1}Bz_1 + Dz_1 + C\Pi_N A^{N-2}Bz_2 + \dots \\ &+ C\Pi_N A^{N-1}Eu_1 + Fu_1 + C\Pi_N A^{N-2}Eu_2 + \dots \end{aligned} \tag{22a}$$
$$\begin{aligned} w_2 = &\left(C(\Pi_N - I)B + CB\right)z_1 \\ &+ \left(C\Pi_N A^{N-1}B\right)z_2 + Dz_2 + \dots + \\ &+ \left(C(\Pi_N - I)E + CE\right)u_1 \\ &+ \left(C\Pi_N A^{N-1}E\right)u_2 + Fu_2 + \dots \end{aligned} \tag{22b}$$

and so on. The equations above together with the complementarity conditions (16c) can be rewritten as the following LCP:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = M_N \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} + q_N \tag{23a}$$
$$0 \leqslant w_k \perp z_k \geqslant 0, \quad k = 1, \dots, N \tag{23b}$$

where

$$\begin{aligned} M_N = &\begin{pmatrix} C\Pi_N A^{N-1}B & C\Pi_N A^{N-2}B & \cdots & C\Pi_N B \\ C\Pi_N B & C\Pi_N A^{N-1}B & \cdots & C\Pi_N AB \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi_N A^{N-2}B & C\Pi_N A^{N-3}B & \cdots & C\Pi_N A^{N-1}B \end{pmatrix} \\ &+ \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix} \end{aligned} \tag{24}$$

and q_N is reported in (25). Note that M_N is a circulant matrix whose first row is

$$\left(C\Pi_N A^{N-1}B + D \;\; C\Pi_N A^{N-2}B \;\cdots\; C\Pi_N B\right)$$

Then, by using *Theorem 1* one can conclude that the system (16) has a unique periodic solution of period N if and only if M_N is a P-matrix. Moreover, the periodic solution is given by the solution of the LCP (23). If M_N is not a P-matrix the LCP (23) could have no solution or multiple solutions. In any case any solution of the LCP will correspond to a periodic solution of the system (16).

6. NONLINEAR CIRCUIT EXAMPLE

Let us consider the electrical circuit (Wolfram-Research, 2007) reported in Fig. 6, where the capacitor value depend on the charge x_1: $C = C_1$ for $x_1 > 0$ and $C = C_2 < C_1$ for $x_1 < 0$. By considering as state variables the charge on

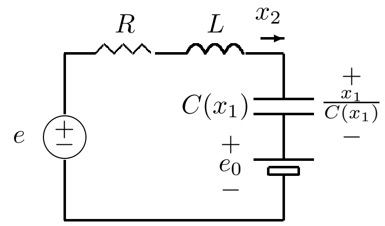

Fig. 6. A chaotic circuit. The input voltage is $e = V_0 \sin(2\pi f)$ and the dynamic behavior of the system depends on the input voltage amplitude V_0.

$$q_N = \begin{pmatrix} C\Pi_N A^{N-1}E + F & C\Pi_N A^{N-2}E & \cdots & C\Pi_N E \\ C\Pi_N E & C\Pi_N A^{N-1}E + F & \cdots & C\Pi_N AE \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi_N A^{N-2}E & C\Pi_N A^{N-3}E & \cdots & C\Pi_N A^{N-1}E + F \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \tag{25}$$

the capacitor x_1 and the current in the inductance x_2 the circuit can be modeled as:

$$\dot{x}_1 = x_2, \tag{26a}$$
$$L\dot{x}_2 = -Rx_2 + e - e_0 - \varphi(\lambda), \tag{26b}$$
$$\lambda = x_1. \tag{26c}$$

and the characteristic $\varphi(\lambda)$ can be represented as in Fig. 7.

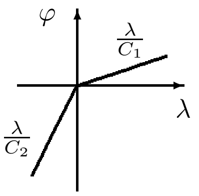

Fig. 7. Piecewise linear characteristic $\varphi(\lambda) = \frac{\lambda}{C(\lambda)}$ with $C(\lambda) = C_1$ for $\lambda > 0$ and $C(\lambda) = C_2 < C_1$ for $\lambda < 0$.

The model (26) can be represented in the form (12) with $u = e - e_0 = V_0 \sin(2\pi f t) - e_0$. By following the procedure described in Section 3 the parameters of the representation in Fig. 5 are $\Lambda_0 = 0$, $\Phi_1 = 0$, $\sigma_0 = 1/C_2$, $\sigma_1 = 1/C_1$, and from (6) $r_0 = C_2$ and $r_1 = C_1 - C_2$. Then the static model (4) will have the following matrices:

$$A_s = \frac{1}{C_2}, \qquad B_s = -\frac{1}{C_2}, \qquad g_s = 0 \tag{27a}$$
$$C_s = -\frac{1}{C_2}, \qquad D_s = \frac{1}{C_2} + \frac{1}{C_1 - C_2}, \qquad h_s = 0. \tag{27b}$$

Let us assume the following values for the circuit parameters:

$$C_1 = 0.1\mu\text{F}, \qquad C_2 = 400\text{pF}, \qquad R = 60\Omega$$
$$L = 100\mu\text{H}, \qquad f = 700\text{kHz}, \qquad e_0 = 0.1\text{V}.$$

It is possible to show that for small values of V_0 (like $V_0 = 0.1\,\text{V}$) there exists a steady state solution of the same period of the forcing signal ($T = 1/f$). By choosing the sampling period $T_s = 2.86\,\text{ns}$, constructing the model (16) and solving the corresponding LCP (23) with $N = 500 = round[1/(T_s f)]$ the results reported in Fig. 8 are obtained. The LCP solution formulated starting from the discrete–time system captures the periodic solution of the continuous–time system. The amplitude V_0 of the sinusoidal input e is the bifurcation parameter. By increasing the amplitude V_0 first it occurs a period doubling bifurcation, i.e. the continuous–time system will exhibit a stable periodic solution with period $T = 2/f$. The sampling period is still fixed to $T_s = 2.86\,\text{ns}$ and the simulation results obtained for $V_0 = 0.2\,\text{V}$ are reported in Fig. 9. By choosing $N = 500$ the solution of the LCP (23) provides the solution reported with dashed line in Fig. 9. Indeed for $V_0 = 0.2\,\text{V}$ it exists not only the solution of period $T = 2/f$ but also an unstable periodic solution of period $1/f$. By choosing (with the same sampling period) $N = 1000$ samples the solution of the LCP (23) provides

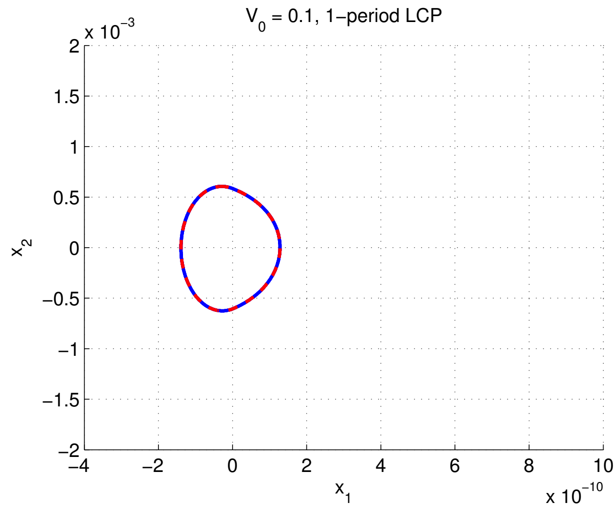

Fig. 8. Steady state solutions computed by simulation, i.e. by using Matlab/Simulink scheme or equivalently by using the step by step solutions of LCPs (16) (*solid* line), and computed by the proposed LCP approach (*dashed* line), i.e. by using (23).

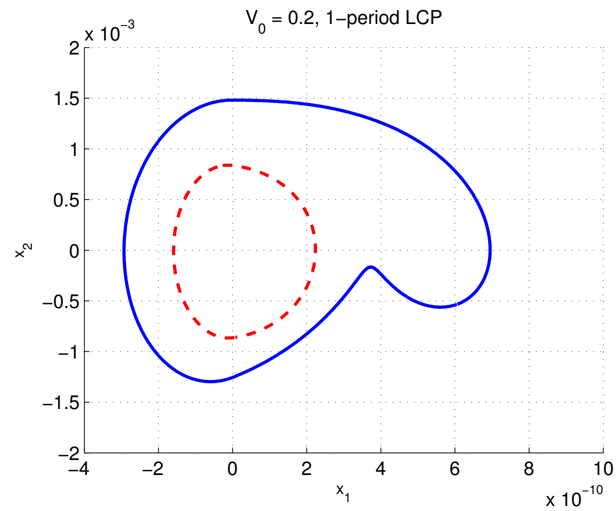

Fig. 9. Steady state solutions computed by simulation (*solid* line) and computed by the proposed LCP approach (*dashed* line).

the stable solution reported in Fig. 9 with solid line. The same arguments can be repeated for $V_0 = 0.26\,\text{V}$ for which it exists a stable periodic solution of period four times the forcing $T = 4/f$ computable from the LCP (23) with $N = 2000$, and an unstable periodic solution with period $1/f$ computable from the LCP (23) with $N = 500$, see Fig. 10. The time evolution of the complementarity variables for the stable periodic solution (including the transient) is reported in Fig. 11. Also, for $V_0 = 0.26\,\text{V}$ it exists an unstable periodic solution of period $T = 2/f$

which can be detected by the proposed complementarity approach by solving the LCP (23) with $N = 1000$.

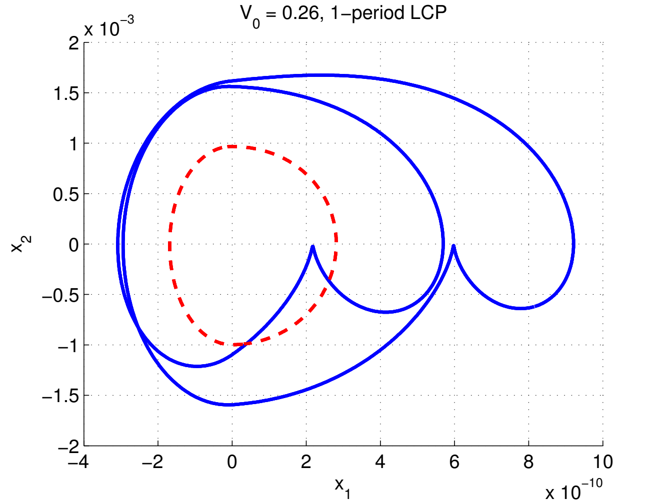

Fig. 10. Steady state solutions computed by simulation (*solid* line) and computed by the proposed LCP approach (*dashed* line).

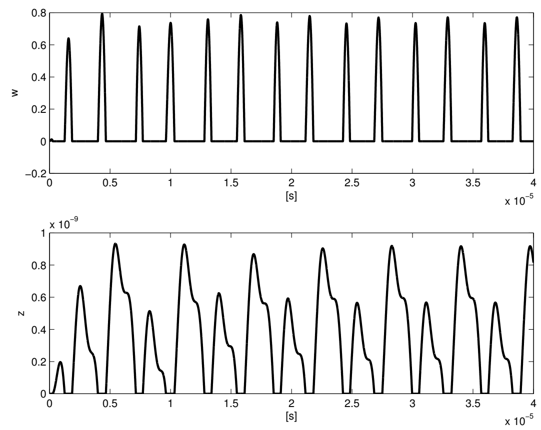

Fig. 11. Time evolution of the complementarity variables starting from zero initial conditions and computed by using the step by step solutions of LCPs (16).

7. CONCLUSION

The complementarity framework has been used to prove existence of periodic solutions for linear time invariant systems connected in feedback through a piecewise linear static mapping. By assuming consistency of the discretization, conditions for the existence of periodic solutions in terms of solvability of a suitable static linear complementarity problem are obtained. The conditions do not need to fix a priori the shape of the periodic trajectory and the LCP allows to compute both stable and unstable periodic solutions.

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