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Direct Method of Manipulator Endpoint Control Synthesis

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Abstract: Manipulator endpoint location autonomous control procedures are suggested. A method of uplimited control hierarchy has been designed, which allows to provide a desired tracking precision under conditions of uncertainty of the control object operator and the effect of external unmeasured disturbances. Sliding mode state observers synthesis procedures have been designed, which allow, in a theoretically limited time interval, to obtain information on immeasurable variables of the state vector and available uncertainty. The results of the designed algorithms modeling are presented. *Copyright © 2008 IFAC*

1. INTRODUCTION

In this paper, program trajectory tracking problem by a endpoint manipulator is considered. The presently sufficiently studied manipulator planning and control methods in the general configuration variables space are not directly applied in this problem. The known control methods in the endpoint coordinates space almost always require a solution of reversed kinematics and dynamic problems in real time, which on rare occasions have an analytic and/or a single valued solution. In this work, a conceptually different approach this problem's solution is suggested based on the mechanical system's resulting image decomposition and not requiring a reversed problems solution in real time.

The paper has the following structure. In section 1 the plant model is presented. In section 2, a procedure of transforming the plant model into the block canonical control form is designed with respect to the output variables (BCCFO, Utkin, 2007) describing the space orientation of the endpoint. Such a transformation allows the use of the block principle (Drakunov *et al*., 1990; Krasnova, 2001) and to decompose the synthesis problem into independently solved elementary subproblems of lower dimension. In section 3, output variables autonomous control algorithms are designed under the conditions of certainty of the input control channels. The information support problems are solved with the help of a sliding mode state observer. It is significant that in this case being considered the obtained block controllable form is, at the same time, a block observed (Krasnova *et al*., 2001), i.e., the problems of the unmeasured variables control and observation are solved in the same transformed coordinates, which considerably facilitates the regulator synthesis. In section 4, a new stabilizing feedback type (which is new to mechanical systems) which is an uplimited realization of discontinuous control is suggested. A hierarchical principle for choosing of feedback coefficients is designed, which allows an autonomous control realization under the incomplete information on input control channels. In section 5, the results of the designed algorithms modeling in MATLAB for a two-link plain manipulator are presented.

2. PLANT MODEL AND PROBLEM STATEMENT

Let us consider the dynamic model of the rigid manipulator with n degrees of freedom

$$\dot{q}_1 = q_2,\, \dot{q}_2 = H^{-1}(q_1)[u - C(q_1,q_2)q_2 - G(q_1) + \eta(t)], \quad (1)$$

where $q_1 \in Q_1 \subset R^n$ is the vector of angular positions of the manipulator, $H(q_1) \in R^{n\times n}$ is the positive definite nonlinear symmetric matrix of inertia, $C(q_1,q_2) \in R^{n\times n}$ is the matrix of centripetal and Coriolis forces, $G(q_1) \in R^n$ is the vector of gravitation forces, $u \in R^n$ is the vector of generalized moments developed by actuators, $\eta(t) \in R^n$ is the vector of external unmeasurable bounded disturbances. The multilink manipulator construction ends with a replaceable working mechanism (endpoint). The endpoint space orientation vector is described by nonlinear smooth functions of angular positions $y_1 = h(q_1)$, $y_1 \in Y_1 \subset R^m$, $Q_1 \to Y_1$, $m \le n$. Program motion $y_{1d}(t) \in R^m$ tracking problem given in terms of the endpoint working space $y_{1d} \in Y_1$ is posed. It is supposed that the vector function y_{1d} and its derivations are restricted. The tracking problem is limited to the problem of stabilizing the mismatch $e_1(t) = y_1(t) - y_{1d}(t)$, $e_1 \in R^m$ and is solved depending on the technological requirements or asymptotically

$$\lim_{t\to\infty} e_1(t) = 0\,, \quad (2)$$

or with given accuracy

$$\|e_1\| \le \delta_1 = \text{const} \quad (3)$$

based on model (1) representation in block canonical control form with respect to output variables y_1.

3. MECHANICAL SYSTEM OUTPUT IMAGE

The peculiarity of the problem being considered is that the image $Q_1 \to Y_1$ is single valued, but not mutually single valued, because different manipulator configurations may correspond to a particular endpoint position. Besides, rank$J$, where $J(q_1)_{m\times n} = \partial h/\partial q_1$, may be different at different points of space Q_1. Let us assume that $\mathrm{rank}J \equiv m \quad \forall q_1 \in \overline{Q}_1 \subset Q_1$ with the exception of a finite number of particular points q_1^*. In case of redundant dimensionality of plant $m < n$ it is assumed that the same group of general basis variables $q_1^1 \in R^m$ ($q_1 = \mathrm{col}(q_1^1, q_1^2)$, $q_1^2 \in R^{n-m}$ are free variables) chosen out of constructive considerations can be matched with the basis minor of matrix $J(q_1)$ (i.e. the output variables y_1) with $q_1 \in \overline{Q}_1$. Otherwise, the program trajectory should be divided in an appropriate manner into sections to which a different closed loop structure will correspond.

The essence of model (1) imaging procedure in BCCFO consists of a two-time differentiation of output variables with account of system (1) and the assumptions made, in particular

$$\begin{aligned} &\dot y_1 = J(q_1)q_2 = y_2,\ \dot y_2 = J'(q_1,q_2)q_2 + J(q_1)\dot q_2 = \\ &= J'(q_1,q_2)q_2 + J(q_1)H^{-1}[u - Cq_2 - G + \eta] = \\ &= A(q_1,q_2) + B(q_1)u + B(q_1)\eta, \end{aligned} \qquad (4)$$

where $A = J'(q_1,q_2)q_2 - J(q_1)H^{-1}(q_1)[C(q_1,q_2)q_2 + G(q_1)]$, $B = J(q_1)H^{-1}(q_1)$, $\mathrm{rank}H^{-1}(q_1) = n \Rightarrow \mathrm{rank}B(q_1) = m$ $\forall q_1 \in \overline{Q}_1$; $J_{m\times n} = (J_{ij})$, $J'_{m\times n} = (J'_{ij})$, $J'_{ij} = (\partial J_{ij}/\partial q_1)q_2$.

The posed problem (2) and (3) is solved in terms of system (4) based on block approach (Drakunov *et al.*, 1990; Krasnova, 2001). A conjoint problem, which is not considered here, consists of free coordinates $q_{12} \in R^{n-m}$ control whose behavior is determined by the corresponding part of system (1).

4. AUTONOMOUS CONTROL BASIC ALGORITHMS

In this section, under the assumption about the certainty of the input channels, output variables autonomous control synthesis methods are designed implementing algorithms of various complexity, which is preconditioned by the possibility and the expediency of the setup of various measuring devices. System (4) consists of two elementary blocks in each one of which the state vector dimensionality coincides with the dimensionality of the virtual and true control vector, which allows for partitioning of the synthesis problem into independently solvable problems of dimensionality m. Using (4), let us rewrite the differential equation with respect to the mismatch

$$\dot e_1 = y_2 - \dot y_{1d}. \qquad (5)$$

In system (5), vector y_2 is treated as a virtual control chosen in the form of $y_2 = -K_1 e_1 + \dot y_{1d}$, where $K_1 = \mathrm{diag}\{k_{1i}\}$, $k_{1i} > 0$ ($i = \overline{1,m}$) are feedback coefficients providing the demanded rate of convergence (2). To provide for the chosen virtual control, it is required to solve the mismatch stabilization problem

$$e_2 = y_2 + K_1 e_1 - \dot y_{1d}. \qquad (6)$$

Taking (6) into account, equation (5) takes the form

$$\dot e_1 = -K_1 e_1 + e_2. \qquad (7)$$

Provision of asymptotic convergence of variable (6), i.e.

$$\lim_{t\to\infty} e_2(t) = 0 \qquad (8)$$

will lead to a solution of problem (2); provision of demanded precision

$$\|e_2\| \le \delta_2 = \mathrm{const} \qquad (9)$$

will lead to (3) within system (7). Problems (8) and (9) are solved within system

$$\dot e_2 = A(q_1,q_2) + B(q_1)u + B(q_1)\eta + K_1(-K_1 e_1 + e_2) - \ddot y_{1d} \quad (10)$$

with using of control u. Taking into account $q_1 = \mathrm{col}(q_1^1, q_1^2)$, let us split the control vector $Bu = B_1(q_1)u_1 + B_2(q_1)u_2$, $u_1 \in R^m$, $\det B_1 \ne 0 \quad \forall q_1 \in \overline{Q}_1$ and present system (10) in the form

$$\dot e_2 = \varphi(.) + B_1(q_1)u_1, \qquad (11)$$

$$\varphi(.) = A(q_1,q_2) + B_2(q_1)u_2 + B(q_1)\eta + K_1(-K_1 e_1 + e_2) - \ddot y_{1d}.$$

The widespread concept of system (11) stabilization problem synthesis consists of providing an autonomous control, i.e. decoupling common system motion into independently controlled subsystems describing the dynamics of separate output variables and subsequent independent stabilization problem synthesis in these subsystems. In this section, autonomous control methods within system (11) are designed under assumption that matrix $B_1(q_1)$ parameters are known. Then the general control law has the form

$$B_1(q_1)u_1 = U_0(e_2) - \phi(.), \qquad (12)$$

where $U_0(e_2) = \mathrm{col}(U_0(e_{21}),...,U_0(e_{2m}))$ is the stabilizing feedback compensating for intersecting links, $\phi(.)$ is a vector function whose role consists of compensating for existing uncertainties. Control moments (12) play a dual role in the control system, because they are the demanded effects to be treated by actuators at the same time, which imposes a set of restrictions on their choice. For control law (12) to take effect, it is required that the vector functions it consists of be restricted and continuously uninterrupted in general case k times over all of its arguments in the area being considered, where k is the relative degree of the actuator's dynamic model composed with respect to the moments applied the actuator's axis.

Let us introduce denotation of mismatch

$$e_3 = B_1(q_1)u_1 - U_0(e_2) + \phi(.),\ e_3 \in R^m. \qquad (13)$$

It is essential that under this approach realized is the possibility of choosing various standard actuators in which the control moments demanded values tracking problem is solved either asymptotically

$$\lim_{t\to\infty} e_3(t) = 0, \qquad (14)$$

or with demanded precision

$$\|e_3\| \le \delta_3 = \mathrm{const}. \qquad (15)$$

4.1 Linear stabilizing feedback

In system (11) let us form combined control

$$B_1(q_1)u_1 = -K_2 e_2 - \varphi(.), \qquad (16)$$

where $\varphi(.) = \mathrm{col}(\varphi_1,...,\varphi_m)$, $K_2 = \mathrm{diag}\{k_{2i}\}$, $k_{2i} > 0$, $i = \overline{1,m}$. Closed systems (11), (16) with account of (13) will take the form $\dot e_2 = -K_2 e_2 + e_3$. Provision of (14) within the actuator's control system will lead to the following relations:

$$e_3 \to 0 \Rightarrow e_2 \to 0 \Rightarrow e_1 \to 0 \Rightarrow y_1 \to y_{1d}.$$

To implement control (16), measurements of q_1, q_2 and real time evaluation with high precision and quick action of compensating component $\varphi(.)$ are required, which presupposes the plant operator's parametrical certainty, calculation of $\dot{y}_{1d}(t), \ddot{y}_{1d}(t)$ and a construction of the adequate model of disturbances $\eta(t)$ (under the assumption of their smoothness). Let us show that the requirements of the plant's and its functional environment's antecedent information volume and also of the number of calculations done in real time can be significantly reduced if a sliding mode state observer (Krasnova *et al.*, 2001) is added to the feedback loop. This observer (measurements q_1 are sufficient for its construction) allows, in a theoretically limited time interval, for obtaining an estimate of transformed variables, linear combinations of the plant's components (foregoing their immediate calculations) and existing uncertainties under the condition of their limitation, namely

$$|\varphi_i| \le F_i = \text{const} > 0\,,\ \bar{F} = \max\{F_i\}\,,\ i = \overline{1,m}\,. \qquad (17)$$

For systems (7), (11) the state observer has the form

$$\dot{z}_1 = -K_1 z_1 + z_2 + v_1,\ \dot{z}_2 = B_1(q_1)u_1 + v_2, \qquad (18)$$

where $z_1, z_2 \in R^m$ are the state vectors, v_1, v_2 are the observer's corrective effects which are chosen in the form of discontinuous functions so that the system stabilization problem written with respect to mismatches $\varepsilon_1 = h(q_1) - g - z_1 = e_1 - z_1$, $\varepsilon_2 = e_2 - z_2$, $\varepsilon_1, \varepsilon_2 \in R^m$ is solved and which, using (7), (11), and (18), has from

$$\dot{\varepsilon}_1 = -K_1\varepsilon_1 + \varepsilon_2 - v_1,\ \dot{\varepsilon}_2 = \varphi(.) - v_2. \qquad (19)$$

In the first equation of system (18) let us form discontinuous correcting actions $v_1 = M_1 \text{sign}\varepsilon_1$, where here and further $M_1 = \text{diag}\{m_{1i}\}$, $\text{sign}\varepsilon_1 = \text{col}(\text{sign}\varepsilon_{11},...,\text{sign}\varepsilon_{1m})$, which will lead to generation in finite time $t_1 > 0$ of a sliding mode over multiplicity $S_1 = \{\varepsilon_1 = 0\} \Rightarrow z_1 = e_1$ under the fulfillment of conditions $m_{1i} > |\varepsilon_{2i}|, i = \overline{1,m}$. According to equivalent control method (Utkin, 1992), with $t > t_1$ from a static equation we have estimates $\dot{\varepsilon}_1 = \varepsilon_2 - v_{1\text{eq}} = 0 \Rightarrow v_{1\text{eq}} = \varepsilon_2$ whose values are obtained from the outputs of the first order linear filters

$$\mu_1\dot{\tau}_1 = -\tau_1 + v_1\,,\ \tau_1 \in R^m\,,\ \mu_1 > 0\,,\ \lim_{\mu_1 \to 0} \tau_1 = v_{1\text{eq}}\,. \qquad (20)$$

Obtained values (20) are used to form discontinuous corrective effects in the second equation (18) which, with $v_2 = M_2 \text{sign}\varepsilon_2$, $m_{2i} > F_i, i = \overline{1,m}$, will lead to generation in finite time $t_2 > t_1$ of a sliding mode over multiplicity $S_2 = \{\varepsilon_2 = 0 \cap S_1\} \Rightarrow z_2 = e_2$. From the static equation we have estimates $\dot{\varepsilon}_2 = \varphi(.) - v_{2\text{eq}} = 0 \Rightarrow v_{2\text{eq}} = \varphi(.)$ whose values will be obtained from linear filters outputs

$$\mu_2\dot{\tau}_2 = -\tau_2 + v_2\,,\ \tau_2 \in R^m\,,\ \mu_2 > 0\,,\ \lim_{\mu_2 \to 0} \tau_2 = v_{2\text{eq}}\,. \ (21)$$

Control law (16), in the presence of state observer (18) and filters (20)–(21), is implemented in the form $B_1(q_1)u_1 = -K_2 z_2 - \tau_2$.

The volume of information support can be further reduced if the compensating component is not used in the law, i.e. accepted is control law

$$B_1(q_1)u_1 = -k_2 e_2\,,\ k_2 = \text{const} > 0\,. \qquad (22)$$

To implement (22), measurements q_1 and a reduced observer

$$\dot{z}_1 = -K_1 z_1 + v_1 \qquad (23)$$

are sufficient. Taking (7) into account, we have an equation with respect to mismatch $\dot{\varepsilon}_1 = -K_1\varepsilon_1 + e_2 - v_1$. Let us form discontinuous correcting effects $v_1 = M_1\text{sign}\varepsilon_1$, $m_{1i} > |e_{2i}|$, $i = \overline{1,m}$, which will lead to generation in finite time of a sliding mode over multiplicity $S_1 = \{\varepsilon_1 = 0\} \Rightarrow z_1 = e_1$. From the static equation we have estimates $\dot{\varepsilon}_1 = e_2 - v_{1\text{eq}} = 0 \Rightarrow v_{1\text{eq}} = e_2$ whose values will be obtained from filters outputs (20). Control (22), in the presence of state observer (23) and filter (20), is implemented as $B_1(q_1)u_1 = -k_2\tau_1$.

To implement control law (22), unlike (16), it is not necessary to define the compensating component $\varphi(.)$. It suffices to be sure that its components (17) are restricted. The role of coefficient k_2 in closed system (11), (22) consists of suppressing of the existing uncertainties which leads to (9) holding true and, accordingly, the solution of the tracking problem with given precision (3). For better fine tuning, let us introduce majoritating function

$$\|\varphi\| \le L_0 + L_1\|e_1\| + L_2\|e_2\|,\ L_0, L_1, L_2 = \text{const} > 0 \qquad (24)$$

for uncertain system (11).

Let us show that there exist such values $k_1 = k_{1i} = \text{const} > 0$, k_2, with which in closed system (7), (11) and (22) hold true relations (3) and (9). Let us introduce a quadratic form as a sum of quadratic forms

$$V = V_1 + V_2 = \tfrac{1}{2}e_1^T e_1 + \tfrac{1}{2}e_2^T e_2\,. \qquad (25)$$

For the derivative of the first summand of quadratic form (25), in view of (7), the following estimate is valid: $\dot{V}_1 = e_1^T\dot{e}_1 = e_1^T(e_2 - k_1 e_1) \le \|e_1\|(\|e_2\| - k_1\|e_1\|)$. The inequality $\dot{V}_1 < 0$ is ensured beyond the neighborhood $\|e_1\| \le \|e_2\|/k_1 \le \delta_1$ with the fulfillment of the condition $k_1 > \|e_2\|/\delta_1$. From this inequality at a fixed value of $k_1 = k_1^*$ we define accuracy (9) that is necessary to ensure in system (11), (22):

$$\|e_2\| \le k_1^*\delta_1 = \delta_2 = \text{const}. \qquad (26)$$

For the derivative of the second summand of quadratic form (25), in view of (11), (13), (22), (24), (26), the following estimate is valid: $\dot{V}_2 = e_2^T\dot{e}_2 = e_2^T(\varphi(.) + e_3 - k_2 e_2) \le \|e_2\|(L_0 + L_1\|e_2\|\frac{1}{k_1^*} + L_2\|e_2\| + \|e_3\| - k_2\|e_2\|) < 0$ beyond the neighborhood $\|e_2\| \le \delta_2$ with the fulfillment of the conditions $k_{22} > L_0/(k_1^*\delta_1) + L_1/k_1^* + L_2$, $k_{23} > \|e_3\|/(k_1^*\delta_1)$, $k_2 = k_{22} + k_{23}$. The first inequality is the lower estimate for the choice of $k_{22} = k_{22}^*$. From the second inequality at a fixed value of $k_{23} = k_{23}^*$ we define the accuracy (15) that is necessary to ensure in actuators control system synthesis ($k_{23}^* = 0$, if (14) is valid). Control (22) in closed tracking system (7) and (11) will lead to the following relations:

$$\|e_3\| \le \delta_3 \Rightarrow \|e_2\| \le \delta_2 \Rightarrow \|e_1\| \le \delta_1 .$$

Note that if control resources $u_1 = \mathrm{col}(u_{11},...,u_{1m})$ are bounded $|u_{1i}| \le U_{1i}$, $U_1 = \min\{U_{1i}\}$, then there may not be enough allowed reinforcement coefficients $k_{2\max}(U_1)$ for landing in given neighborhood (3). In practice, the presence of restrictions will lead to (22) realizing in view of piece-wise linear function $B_1(q_1)u_1 = -\mathrm{sat}(k_{2\max}(U_1)e_2)$.

The natural tendency would be to account for existing restrictions on the synthesis stage. This goal is served, for example, by systems with discontinuous controls.

4.2 Discontinuous stabilizing feedback

In system (11), (17) let us form a discontinuous control

$$B_1(q_1)u_1 = -M\mathrm{sign}e_2 , \quad (27)$$

which under conditions $F_i < m_i \le m_{i\max}(U_{1i})$, $i = \overline{1,m}$ will lead to the appearance in a finite time of a sliding mode over multiplicity $S = \{e_2 = 0\} \Rightarrow e_1 \to 0 \Rightarrow y_1 \to y_{1d}$. A combined control $B_1(q_1)u_1 = -M\mathrm{sign}e_2 - \varphi(.)$, where $0 < m_i \le m_{i\max} - F_i$, $i = \overline{1,m}$, will lead to a similar result.

Even though algorithm (27) is not realizable in the problem being considered due to physical limitations imposed on controlling moments, it is important from theoretical point of view as a limited case. The following control law is an uplimited realization of a discontinuous control in the form of nonlinear continually differentiated limited function.

4.3 Nonlinear stabilizing feedback

For system (11) let us form control law in the from

$$B_1(q_1)u_1 = -M\mathrm{arctg}(k_2e_2) - \varphi(.), \quad M = \mathrm{const} > 0 , \quad (28)$$

where $\mathrm{arctg}(k_2e_2) = \mathrm{col}(\mathrm{arctg}(k_{21}e_{21}),...,\mathrm{arctg}(k_{2m}e_{2m}))$, $|\mathrm{arctg}(k_{2i}e_{2i})| < \frac{\pi}{2}$, $k_2 = k_{2i} = \mathrm{const} > 0$,

$$\mathrm{arctg}(k_{2i}e_{2i}) \underset{k_{2i}\to\infty}{\to} \tfrac{\pi}{2}\mathrm{sign}e_{2i} , \; i = \overline{1,m} . \quad (29)$$

In closed system (11), (28), (13) $\dot e_2 = -M\mathrm{arctg}(k_2e_2) + e_3$ with conditions $0 < m_i \le m_{i\max}(U_{1i}) - F_i$ holding true, asymptotic convergence (8) is ensured, because $e_{2i}\dot e_{2i} < 0$ with $e_{2i} \ne 0$ and $\mathrm{arctg}(k_{2i}e_{2i}) \sim k_{2i}e_{2i}$ with $e_{2i} \to 0$, $i = \overline{1,m}$. Control law (28) information support: measurements q_1 and state observer (18), (20), (21). Let us show that control (28) without the compensating component in the form

$$B_1(q_1)u_1 = -M\mathrm{arctg}(k_2e_2) \quad (30)$$

allows to ensure the given tracking precision. Precision that it is necessary to ensure in closed system (11), (30)

$$\dot e_2 = \varphi(.) + e_3 - M\mathrm{arctg}(k_2e_2) , \quad (31)$$

is defined by expression (26). Let us register the value of coefficient k_2 out of the following considerations:

$$\pi/2 - \mathrm{arctg}(k_2\delta_2) < \varsigma \Rightarrow k_2 = k_2^* > \mathrm{ctg}\varsigma/(k_1^*\delta_1) , \quad (32)$$

where ς is a small positive value.

For the derivative of the second summand of quadratic form (25), in view of (17), (31)–(32), beyond the neighborhood $\|e_2\| \le \delta_2$ the following estimate is valid: $\dot V_2 = e_2^T\dot e_2 = e_2^T \times$ $\times(\varphi(.) + e_3 - M\mathrm{arctg}(k_2^*e_2)) \le \|e_2\|(\bar F + \|e_3\| - M(\pi/2 - \varsigma))$, $\dot V_2 < 0$ if $M_1 > 2\bar F/(\pi - 2\varsigma)$, $M_2 > 2\|e_3\|/(\pi - 2\varsigma)$, $M =$ $= M_1 + M_2 \le M_{\max}(U_1)$. These inequalities are the estimates for the choice of coefficients in control law (30). To implement (30), measurements q_1 and a reduced observer (23), (20) are required.

5. MAIN RESULT

Basic algorithms (12) allow for diagonalization of a closed system and realization of autonomous control of output coordinates, but require calculation in real time of matrix $B_1(q_1)$ and its inverse. Let us attempt to avoid these calculations. First, let us consider a particular case when $B_1(q_1) = (b_{ij})$ is a matrix with a dominating diagonal in the area being considered, namely

$$\underline{b}_{ii} > \sum_{j=1,j\ne i}^{m} \bar b_{ij} , \; b_{ii} \ne 0 , \; |b_{ij}| \le \bar b_{ij} \;\; \forall q_1 \in \bar Q_1 , \; i,j = \overline{1,m} . \quad (33)$$

Considering (33) for system (11) the control law has the form

$$u_1 = -MS(b_{ii})\mathrm{arctg}(k_2e_2) , \quad (34)$$

where $S(b_{ii}) = \mathrm{diag}\{\mathrm{sign}b_{ii}\}$, which is different from (30) in that for its realization the precise matrix $B_1(q_1)$ coefficients are not required, but only the possible range of their change in the area being considered, and additionally, for nonzero diagonal elements, their signs are required whose determination may be considerably easier than calculation of the values.

Let us show that in closed system (11), (34) given precision (9) is ensured. To obtain the lower estimate for choice $M \le \frac{2}{\pi}U_1$, let us research the second summand of quadratic form (25) represented in coordinate-wise form

$$V_2 = \sum_{i=1}^{m} V_{2i} , V_{2i} = \tfrac{1}{2}(e_{2i})^2 . \quad (35)$$

For the derivative of the i-th ($i = \overline{1,m}$) summand of quadratic from (35) accounting for (11), (33), (34), (17) valid is the estimate

$$\dot V_{2i} = e_{2i}\dot e_{2i} = e_{2i}[\varphi_i(.) - M\sum_{j=1,j\ne i}^{m} b_{ij}\mathrm{sign}b_{jj}\mathrm{arctg}(k_2e_{2j}) - Mb_{ii} \times$$

$$\times\, \mathrm{sign}b_{ii}\mathrm{arctg}(k_2e_{2i})] \le |e_{2i}|[F_i - M(\underline{b}_{ii}(\tfrac{\pi}{2} - \varsigma) - \sum_{j=1,j\ne i}^{m}\bar b_{ij}\tfrac{\pi}{2})].$$

Outside of the neighborhood $|e_{2i}| \le \delta_2$ valid are inequalities

$$\forall \dot V_{2i} < 0 \text{ if } 2\bar F/(\pi\underline{B}) < M , \; \underline{B} = \min\{\underline{b}_{ii}(1 - \tfrac{2\varsigma}{\pi}) - \sum_{j=1,j\ne i}^{m}\bar b_{ij}\} .$$

In a general case when assumption (33) does not hold true, we use the control hierarchy method ideology developed for systems with discontinuous controls (Utkin, 1992). For uplimited realization (34) of discontinuous controls being considered, below is designed a step-by-step procedure for choosing coefficients $M = \mathrm{diag}\{m_i\}$, $m_i = \mathrm{const} > 0$, $k_{2i} > 0$, $i = \overline{1,m}$ based on inequalities, allowing to artificially obtain a matrix with a dominating diagonal (in a general case, with dominating elements from different columns) before a control, for an i-th component of which outside the neighborhood $|e_{2i}| \le \delta_{2i}$ with account of (32) valid are the estimates:

$$\overline{u}_{1i} < |u_{1i}| < \overline{\overline{u}}_{1i},\ \overline{u}_{1i} = m_i(\tfrac{\pi}{2} - \varsigma_i),\ \overline{\overline{u}}_{1i} = m_i \tfrac{\pi}{2},\ i = \overline{1,m}. \quad (36)$$

For simplicity, let us establish the hierarchy of vector e_2 components, which matches their order: $|e_{21}| \le \delta_{21}, \ldots, |e_{2m}| \le \delta_{2m}$. This sequence means that $|e_{2i}| \le \delta_{2i}$ will be ensured only after $|e_{21}| \le \delta_{21}, \ldots, |e_{2,i-1}| \le \delta_{2,i-1}$ hold true.

Step 1. In the first equation of system (11) let us choose a controlling coordinate $u_{11} = -m_1 \text{sign} b_{11} \text{arctg}(k_{21}e_{21})$, $b_{11}(q_1) \ne 0$. For the derivative of the first summand of quadratic from (35) with account of (36) outside the neighborhood $|e_{21}| \le \delta_{21}$ valid is the estimate

$$\dot{V}_{21} = e_{21}\dot{e}_{21} = e_{21}(\varphi_1 + \sum_{j=2}^{m} b_{1j}u_{1j} - b_{11}m_1 \text{sign} b_{11} \text{arctg}(k_{21}e_{21}) \le$$

$$\le |e_{21}|(F_1 + \sum_{j=2}^{m} \overline{b}_{1j}\overline{\overline{u}}_{1j} - \overline{b}_{11}m_1(\tfrac{\pi}{2} - \varsigma_1)) < 0 \Rightarrow$$

$$\Rightarrow (F_1 + \sum_{j=2}^{m} \overline{b}_{1j}\overline{\overline{u}}_{1j})/(\overline{b}_{11}(\tfrac{\pi}{2} - \varsigma_1)) < m_1 \le \tfrac{2}{\pi}U_{1i}.$$

For problem regularization, let us introduce a new control $\tilde{u}_{11}$ and consider an uplimited realization of the equivalent control method (Utkin, 1992). In asymptotic, with $k_{21} \to +\infty$ relations (29), $\dot{e}_{21} \to 0$ and $\tilde{u}_{11} \to u_{11\text{eq}}$ hold true. From a static equation $\dot{e}_{21} = \varphi_1 + \sum_{j=2}^{m} b_{1j}u_{1j} + b_{11}\tilde{u}_{11} = 0 \Rightarrow \tilde{u}_{11} = (-\varphi_1 - \sum_{j=2}^{m} b_{1j}u_{1j})/b_{11}$ which holds true with a precision of an infinitely small $\alpha_1(1/k_{21})$, we find a new control $\tilde{u}_{11}$ and substitute it in the rest of the equations of system (11) for u_{11} and obtain

$$\dot{e}_{2i} = \varphi_i^1 + \sum_{j=2}^{m} b_{ij}^1 u_{1j},\ i = \overline{2,m}, \quad (37)$$

where $\varphi_i^1 = \varphi_i - b_{i1}\varphi_1 / b_{11}$, $b_{ij}^1 = b_{ij} - b_{i1}b_{1j}/b_{11}$. On the second step in the second equation of system (37) control $u_{12} = -m_2 \text{sign} b_{22}^1 \text{arctg}(k_{22}e_{22})$, $b_{22}^1(q_1) \ne 0$ is formed, and the regularization of the problem occurs in the said manner, etc.

Step μ. As a result of the previous transformations, the last $m - \mu + 1$ equations of system (37) take form

$$\dot{e}_{2i} = \varphi_i^{\mu-1} + \sum_{j=\mu}^{m} b_{ij}^{\mu-1} u_{1j},\ i = \overline{\mu,m}, \quad (38)$$

where $b_{ij}^{\mu-1} = b_{ij}^{\mu-2} - b_{i,\mu-1}^{\mu-2} b_{\mu-1,j}^{\mu-2} / b_{\mu-1,\mu-1}^{\mu-2}$, $|b_{ij}^{\mu-1}| \le \overline{b}_{ij}^{\mu-1}$, $\varphi_i^{\mu-1} = \varphi_i^{\mu-2} - b_{i,\mu-1}^{\mu-2}\varphi_{\mu-1}^{\mu-2} / b_{\mu-1,\mu-1}^{\mu-2}$, $|\varphi_i^{\mu-1}| \le F_i^{\mu-1}$. Let us choose in μ-th equation of system (38) control

$$u_{1\mu} = -m_\mu \text{sign} b_{\mu\mu}^{\mu-1} \text{arctg}(k_{2\mu}e_{2\mu}),\ b_{\mu\mu}^{\mu-1}(q_1) \ne 0.$$

Then outside of neighborhood $|e_{2\mu}| \le \delta_{2\mu}$ we have

$$\dot{V}_{2\mu} = e_{2\mu}(\varphi_\mu^{\mu-1} + \sum_{j=\mu+1}^{m} b_{\mu j}^{\mu-1}u_{1j} - b_{\mu\mu}^{\mu-1}m_\mu \text{sign} b_{\mu\mu}^{\mu-1} \text{arctg}(k_{2\mu}e_{2\mu}) \le$$

$$\le |e_{2\mu}|(F_\mu^{\mu-1} + \sum_{j=\mu+1}^{m} \overline{b}_{\mu j}^{\mu-1}\overline{\overline{u}}_{1j} - \overline{b}_{\mu\mu}^{\mu-1}m_\mu(\tfrac{\pi}{2} - \varsigma_\mu)) < 0 \Rightarrow$$

$$\Rightarrow (F_\mu^{\mu-1} + \sum_{j=\mu+1}^{m} \overline{b}_{\mu j}^{\mu-1}\overline{\overline{u}}_{1j})/(\overline{b}_{\mu\mu}^{\mu-1}(\tfrac{\pi}{2} - \varsigma_\mu)) < m_\mu \le \tfrac{2}{\pi}U_{1\mu}. \quad (39)$$

With $k_{2\mu} \to +\infty$ relations (29), $\dot{e}_{2\mu} \to 0$ и $\tilde{u}_{1\mu} \to u_{1\mu\text{eq}}$ hold true. From a static equation which holds true with a precision of an infinitely small $\alpha_1(1/k_{21})$, we find $\tilde{u}_{1\mu}$,

$$\dot{e}_{\mu\mu} = \varphi_\mu^{\mu-1} + \sum_{j=\mu+1}^{m} b_{\mu j}^{\mu-1}u_{1j} + b_{\mu\mu}^{\mu-1}\tilde{u}_{1\mu} = 0 \Rightarrow \tilde{u}_{1\mu} = (-\varphi_\mu^{\mu-1} - \sum_{j=\mu+1}^{m} b_{\mu j}^{\mu-1}u_{1j})/b_{\mu\mu}^{\mu-1}$$

and substitute it in the rest of equations (38) for $u_{1\mu}$, etc.

As a result of this procedure, we obtain the last equation (38) $\dot{e}_{2m} = \varphi_m^{m-1} + b_{mm}^{m-1}u_{1m}$, where $\varphi_m^{m-1} = \varphi_m^{m-2} - b_{m,m-1}^{m-2}\varphi_{m-1}^{m-2}$: $: b_{m-1,m-1}^{m-2}$, $\times \varphi_{m-1}^{m-2} / b_{m-1,m-1}^{m-2}$, $b_{mm}^{m-1} = b_{mm}^{m-2} - b_{m,m-1}^{m-2}b_{m-1,m}^{m-2} / b_{m-1,m-1}^{m-2}$, $|b_{mm}^{m-1}| \le \overline{b}_{mm}^{m-1}$, $|\varphi_m^{m-1}| \le F_m^{m-1}$, $u_{1m} = -m_m \text{sign} b_{mm}^{m-1} \text{arctg}(k_{2m}e_{2m})$.

Outside of the neighborhood $|e_{2m}| \le \delta_{2m}$ we have:

$$\dot{V}_{mm} = e_{2m}(\varphi_m^{m-1} - b_{mm}^{m-1}m_m \text{sign} b_{mm}^{m-1} \text{arctg}(k_{2m}e_{2m}) \le$$

$$\le |e_{2m}|(F_m^{m-1} - \overline{b}_{mm}^{m-1}m_m(\tfrac{\pi}{2} - \varsigma_m)) < 0 \Rightarrow$$

$$\Rightarrow F_m^{m-1}/(\overline{b}_{mm}^{m-1}(\tfrac{\pi}{2} - \varsigma_m)) < m_m \le \tfrac{2}{\pi}U_{1m}.$$

Value $m_m = m_m^*$ chosen from the indicated range is substituted in the estimates of previous coefficients (39), from which, with $\mu = m - 1$, we determine the fixed value m_{m-1}^*, etc.:

$$(F_\mu^{\mu-1} + \sum_{j=\mu+1}^{m} \overline{b}_{\mu j}^{\mu-1}m_j^* \tfrac{\pi}{2})/(\overline{b}_{\mu\mu}^{\mu-1}(\tfrac{\pi}{2} - \varsigma_\mu)) < m_\mu^* \le \tfrac{2}{\pi}U_{1\mu}, \mu = \overline{m-1,1}.$$

It is right to consider the designed procedure of choosing feedback coefficient as hierarchic in asymptotic with $k_{21} >> k_{22} >> \ldots >> k_{2m} \to +\infty$. It is essential that the indicated constructs are conducted at the stage of researching the problems which allows to reduce the volume of calculations performed in real time. To realize the given algorithm, the immediate measurements of output variables y_1 and observer (23), (20) are sufficient. Requirements to additional measurements of coordinates q_1 are determined by the possibility of realizing the control law

$$u_1 = -MS(b_{ii}^{i-1}) \text{arctg}(k_2e_2),\ S(b_{ii}^{i-1}) = \text{diag}\{\text{sign} b_{11}^{i-1}\} \quad (40)$$

6. EXAMPLE

As an example, let us consider the dynamic model of the two-link planar manipulator with $n = m = 2$ degrees of freedom. Components of plant operator (1), where $q_1 = \text{col}(q_{11}, q_{12})$, $y_1 = \text{col}(y_{11}, y_{12})$, have the following form:

$$H_{11} = m_1 l_{c1}^2 + I_1 + m_2(l_1^2 + l_{c1}^2 + 2l_1 l_{c2}\cos q_{12}) + I_2, \quad (41)$$

$$H_{22} = m_2 l_{c2}^2 + I_2,\ H_{12} = H_{21} = m_2 l_1 l_{c2}\cos q_{12} + m_2 l_{c2}^2 + I_2,$$

$$C_{11} = -m_2 l_1 l_{c2} \sin q_{12} q_{22},\ C_{21} = m_2 l_1 l_{c2} \sin q_{12} q_{21},\ C_{22} = 0,$$

$$C_{12} = -m_2 l_1 l_{c2} \sin q_{12} q_{22} - m_2 l_1 l_{c2} \sin q_{12} q_{21},$$

$$G_1 = m_1 l_{c1}^2 g \cos q_{11} + m_2 g[l_{c2}\cos(q_{11} + q_{12}) + l_1 \cos q_{11}],$$

$G_2 = m_2 l_{c2} g \cos(q_{11} + q_{12})$,
$y_{11} = h_1(q_1) = l_1 \cos q_{11} + l_2 \cos(q_{11} + q_{12})$,
$y_{12} = h_2(q_2) = l_1 \sin q_{11} + l_2 \sin(q_{11} + q_{12})$,

where $g = 9{,}8\,[H/kg]$, $I_1 = 0{,}4$, $I_2 = 0{,}25$ $[kg \cdot m^2]$ are applied moments of inertia, $m_1 = 4$, $m_2 = 3$ $[kg]$ are mass of links, $l_1 = 0{,}5$, $l_2 = 0{,}5$ $[m]$ are lengths of links, $l_{c1} = 0{,}3$, $l_{c1} = 0{,}25$ $[m]$ are distances to the center of gravity of links. The program trajectory has the following form (fig. 1)

$$y_{11d} = g_{10} + R\sin t,\ y_{12d} = g_{20} + R\cos t. \tag{42}$$

In this case, two manipulator configurations (q_{11}, q_{12}), $(\tilde{q}_{11}, \tilde{q}_{12})$ may correspond to a particular endpoint position (fig. 2). Let us note that mutually single-valued $y_1 \leftrightarrow q_1$ correspondence between endpoint position and angular positions of the manipulator may be established with the help of information about the sign of q_{12}.

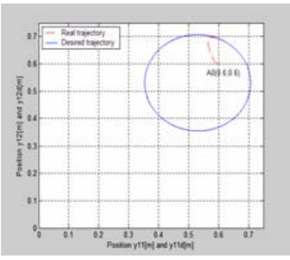

Fig. 1.

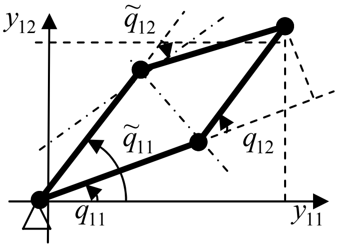

Fig. 2.

There is $\det J = l_1 l_2 \sin q_{12} \neq 0$ $\forall q_{12} \in \overline{Q}_1$ in output image system (1), (41) to BCCFO (4) with the exception of particular points $q_{12}^* = \pm k\pi$, $k = 0,1,\ldots$, which corresponded to the extended or folded manipulator arm.

Let us transform system (1), (41) into form (7), (11), namely

$$\dot{e}_{1i} = -k_{1i} e_{1i} + e_{2i}, \dot{e}_{2i} = \varphi_i(.) + b_{i1}(.)u_1 + b_{i2}(.)u_2,\ i = 1,2, \tag{43}$$

where $\varphi_1 = -[y_{22} q_{21} + (y_{22} - l_1 \cos q_{11} q_{21}) q_{22}] + \frac{1}{\Delta}[y_{12} H_{22} G_1^* -$
$- y_{12} H_{12} G_2^* - (y_{12} - l_1 \sin q_{11}) H_{21} G_1^* + (y_{12} - l_1 \sin q_{11}) H_{11} G_2^*] +$
$+ k_{11}(-k_{11} e_{11} + e_{21}) - \ddot{y}_{11d}, \varphi_2 = [y_{21} q_{21} + (y_{21} + l_1 \sin q_{11} q_{21}) q_{22}] -$
$- \frac{1}{\Delta}[y_{11} H_{22} G_1^* + y_{11} H_{12} G_2^* + (y_{11} - l_1 \cos q_{11}) H_{21} G_1^* - (y_{11} -$
$- l_1 \cos q_{11}) H_{11} G_2^*] + k_{12}(-k_{12} e_{12} + e_{22}) - \ddot{y}_{12d}$,
$G_1^* = (c_{11} q_{21} + c_{12} q_{22}) + G_1(q_1)$, $G_2^* = (c_{21} q_{21} + c_{22} q_{22}) + G_2(q_1)$
$\Delta = H_{11} H_{22} - H_{21} H_{12}$, $b_{11} = \frac{1}{\Delta}[(y_{12} - l_1 \sin q_{11}) H_{21} - y_{12} H_{22}]$,
$b_{12} = \frac{1}{\Delta}[y_{12} H_{12} - (y_{12} - l_1 \sin q_{11}) H_{11}]$,
$b_{21} = \frac{1}{\Delta}[y_{11} H_{22} - (y_{11} - l_1 \cos q_{11}) H_{21}]$,
$b_{22} = \frac{1}{\Delta}[(y_{11} - l_1 \cos q_{11}) H_{11} - y_{11} H_{12}]$.

Mismatches $e_{11}(t) = y_{11} - y_{11d}$, $e_{12}(t) = y_{12} - y_{12d}$ of assignment (42) in system (43) under linear stabilizing feedback (22) (fig. 3-4), under discontinuous stabilizing feedback (27) (fig. 5-6) are shown. Nonlinear controls (40) $u_1(t), u_2(t)$ (fig. 7-8) and corresponding mismatches $e_{11}(t)$, $e_{12}(t)$ (fig. 9-10.) are shown.

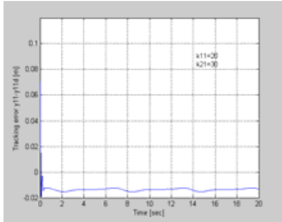

Fig. 3. $|e_{11}| \leq 0{,}005$

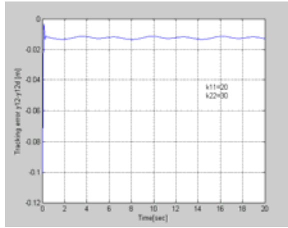

Fig. *4.* $|e_{12}| \leq 0{,}0045$

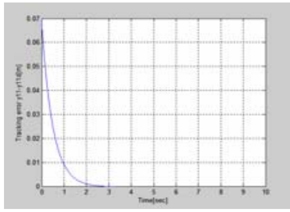

Fig. 5. $e_{11}(t) = 0$, $t > 3\,\text{sec}$

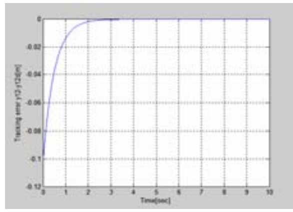

Fig. *6.* $e_{12} = 0$, $t > 3\,\text{sec}$

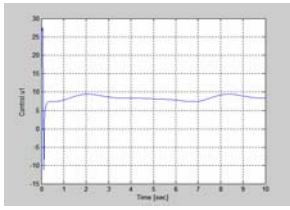

Fig. 7. $u_1(t)$

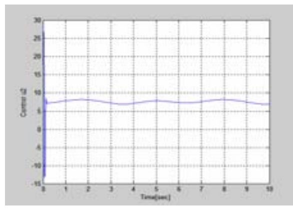

Fig. *8.* $u_2(t)$

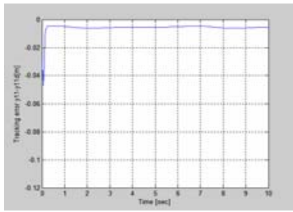

Fig. 9. $|e_{11}| \leq 0{,}001$

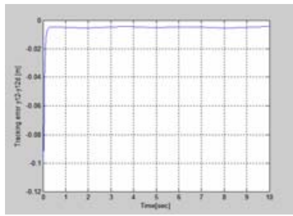

Fig. *10.* $|e_{12}| \leq 0{,}001$

REFERENCES

Drakunov, S.V., D.B. Izosimov, A.G. Loukianov, V.A. Utkin and V.I. Utkin (1990). Block control principle. *Automation and Remote Control,* **Vol. 51, No. 5,** pp. 601-609.

Krasnova, S.A. (2001). Cascade design of a manipulator control system with consideration for dynamics of electric drives. *Automation and Remote Control,* **Vol. 62, No.11,** pp. 1803-1825.

Krasnova, S.A., V.A. Utkin and Yu.V. Mikheev (2001). Cascade design of nonlinear state observers. *Automation and Remote Control,* **Vol. 62, No. 2,** pp. 207-226.

Utkin, Anton V. (2007). Method of state space expansion in autonomous control problem. *Automation and Remote Control,* **Vol. 68, No. 6,** pp. 81-98.

Utkin, Vadim I. (1992). *Sliding modes in control and optimizations.* Springer-Verlag, Berlin.