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# Subspace Identification of a Class of Large-Scale Systems ⋆

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**Abstract:** This article concerns the system identification of a class of large scale systems called "circulant systems". Circulant systems have a special property that allows them to be decomposed into simpler subsystems through a state transformation. This property has been used in literature for control design, and here we show how it can be used for system identification. The approach that is proposed here will both reduce the complexity of the problem as well as provide models which have a circulant structure that can be exploited for control design.

Keywords: System Identification, Subspace Identification, Circulant Matrices, Fourier Matrix, Circulant Systems.

## 1. INTRODUCTION

Large scale systems have been object of interest in system and control theory since the late Seventies (Sandell et al. [1978]). The high dimensionality of these systems has led to the development of techniques which could reduce the complexity of the problem. A possible approach is to consider the large scale system as the result of the interconnection of many simpler subsystems, as in D'Andrea and Dullerud [2003].

In this paper we focus on a special class of large scale systems, which we call "circulant systems" (Denis and Looze [1999]). Circulant systems are the result of the periodic interconnection (D'Andrea and Dullerud [2003]) of a number of identical subsystems, as shown in Figure 1. Each subsystem has exactly two neighbors, and interacts with these neighbors in exactly the same way. Examples of circulant systems can be found in different fields, e.g. adaptive optics (Denis [1998]), paper machines (Laughlin et al. [1993]) and as result of the approximation of partial differential equations (Brockett and Willems [1974]).

Circulant systems have a remarkable property that allows their structure to be exploited by decomposing them into smaller systems; this property can be used for simplifying the complexity of the analysis (Lunze [1986]) and control design (Denis [1998], Denis and Looze [1999], Hovd and Skogestad [1994]). We show in this paper that the structural properties of circulant systems can also be exploited in order to simplify the identification of such systems from input-output data. We then develop an identification algorithm for models which have the circulant structure, allowing the exploitation of such structure for controller design.

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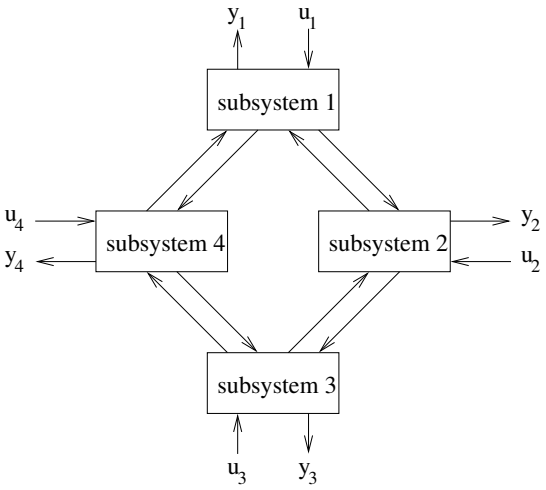


Fig. 1. Example of circulant system made of 4 identical subsystems.

The paper is organized as follows. In Section 2 the preliminary notions are presented; circulant systems are defined and their properties are explained. The focus is put on how circulant systems can be recognized *a priori* from physical insight, and then the decomposition property is presented together with its consequences for identification. Section 3 presents a novel general identification algorithm for circulant systems based on subspace identification methods, and Section 4 contains two simulated examples of the use of such algorithm in practice. The conclusions of the paper are in Section 5.

## 2. PRELIMINARIES

We start by showing the basic concepts that are needed for introducing the notion of a circulant system. These concepts include the definitions of some peculiar kind of matrices, like circulant and block circulant matrices, and

the Fourier matrix, that has some very special properties with respect to circulant matrices.

Let $j$ be the imaginary unit, and $I_n$ the identity matrix of order $n$; let $\otimes$ indicate the Kronecker product. For a generic matrix $A$, $A^T$ indicates its transpose while $A^H$ indicates its Hermitian (complex conjugate of the transpose); $\bar{b}$ indicates the complex conjugate of a matrix or scalar $b$.

*Definition 1.* (Permutation matrix). The permutation matrix of order $n$ is defined as:

$$\Pi_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline 1 & 0 \end{array}\right]$$

Notice that $\Pi_n$ is orthogonal:

$$\Pi_n^{-1} = \Pi_n^T$$

Right-multiplying an $n \times n$ matrix by $\Pi_n$ is equivalent to cyclically shifting all its columns of one position to the right. Left-multiplication instead cyclically shifts the rows up.

*Definition 2.* (Circulant matrix). A square matrix $E$ of size $n \times n$ is called "circulant" if it has the following structure:

$$E = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & \cdots & e_n \\ e_n & e_1 & e_2 & e_3 & \cdots & e_{n-1} \\ e_{n-1} & e_n & e_1 & e_2 & \cdots & e_{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ e_2 & e_3 & e_4 & e_5 & \cdots & e_1 \end{bmatrix}$$

where $e_i \in \mathbb{R}$ or $e_i \in \mathbb{C}$. This is the same as saying, a square matrix is circulant if and only if each row is obtained from the preceding one by a cyclic shift of one position to the right.

This definition is equivalent to saying that a circulant matrix is invariant to a similarity transformation with respect to $\Pi_n$:

$$E = \Pi_n^{-1} E \Pi_n = \Pi_n^T E \Pi_n$$

*Definition 3.* (Block circulant matrix). A block circulant matrix $E$ of order $n$ is a (non necessarily square) matrix with the following block structure:

$$E = \begin{bmatrix} E_1 & E_2 & E_3 & E_4 & \cdots & E_n \\ E_n & E_1 & E_2 & E_3 & \cdots & E_{n-1} \\ E_{n-1} & E_n & E_1 & E_2 & \cdots & E_{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ E_2 & E_3 & E_4 & E_5 & \cdots & E_1 \end{bmatrix}$$

where $E_i \in \mathbb{R}^{p \times q}$ or $E_i \in \mathbb{C}^{p \times q}$, with $p$, $q$ positive integers.

Let us now introduce some new notation. We will denote the set of block circulant matrices of order $n$, with blocks of size $p \times q$, as $\mathscr{C}_{n,p,q}$; we will use the symbol $\mathscr{C}^{\mathbb{R}}_{n,p,q}$ or $\mathscr{C}^{\mathbb{C}}_{n,p,q}$ if we want to specify that the values of such matrices are respectively real or complex. Let $\mathscr{D}_{n,p,q}$ instead denote the set of block diagonal matrices with $n$ block rows and block columns, and blocks of size $p \times q$. Again, we will use either $\mathscr{D}^{\mathbb{R}}_{n,p,q}$ or $\mathscr{D}^{\mathbb{C}}_{n,p,q}$ if we want to emphasize the nature of the values of such matrices. For a matrix $E \in \mathscr{D}_{n,p,q}$, $E_i$ will indicate the $i^{\text{th}}$ block on the diagonal; for a matrix $E \in \mathscr{C}_{n,p,q}$, $E_i$ indicates the $i^{\text{th}}$ block in the first row (as shown in Definition 3).

*Remark 4.* The sums and products of block circulant matrices of the same order are still block circulant. The inverse of a square invertible block circulant matrix is block circulant (Davis [1979]).

*Lemma 5.* (Block-permutation). A block circulant matrix $E \in \mathscr{C}_{n,p,q}$ is invariant to a block-permutation transformation, that means:

$$\left(\Pi_n \otimes I_p\right)^{-1} E \left(\Pi_n \otimes I_q\right) = \left(\Pi_n^{-1} \otimes I_p\right) E \left(\Pi_n \otimes I_q\right) = E$$

*Definition 6.* (Fourier matrix). We define the Fourier matrix of order $n$ as:

$$F_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n & w_n^2 & \cdots & w_n^{(n-1)} \\ 1 & w_n^2 & w_n^4 & \cdots & w_n^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w_n^{(n-1)} & w_n^{2(n-1)} & \cdots & w_n^{(n-1)(n-1)} \end{bmatrix}$$

with $w_n = \mathrm{e}^{-\frac{2\pi j}{n}} = \cos\frac{2\pi}{n} - j \sin\frac{2\pi}{n}$.

The matrix $F_n$ is unitary and symmetric:

$$F_n^H F_n = F_n F_n^H = I_n, \quad F_n^T = F_n$$

We call $f_i$ the $i^{\text{th}}$ row of $F_n$. We will now show that all the rows but the first of the Fourier matrix are complex conjugate between each other; if $n$ is even, then $f_1$ and $f_{n/2}$ are real, while the other rows form complex conjugate pairs; if $n$ is odd, then $f_1$ alone is real with the other rows forming complex conjugate pairs.

*Lemma 7.* The rows of $F_n$ are either real or in complex conjugate pairs according to the relation:

$$f_{n+2-i} = \bar{f}_i \quad \text{for } i = \{2, \ldots, n\}$$

Fourier matrices have the remarkable property of diagonalizing any circulant matrix. This property is crucial because it will allow the decomposition of large scale circulant systems to smaller independent ones, thus reducing the complexity of the identification problem. The property is stated in the Theorem that follows, and then generalized to block circulant matrices.

*Theorem 8.* (Diagonalization property). For a matrix $E \in \mathbb{C}^{n \times n}$, it holds that $F_n E F_n^H$ is a diagonal matrix if and only if $E$ is circulant.

**Proof.** The proof can be found in Davis [1979].

*Corollary 9.* Consider a matrix $E \in \mathbb{C}^{np \times nq}$. Then we have that $\mathbf{E} = \left(F_n \otimes I_p\right) E \left(F_n \otimes I_q\right)^H \in \mathscr{D}^{\mathbb{C}}_{n,p,q}$ if and only if $E \in \mathscr{C}^{\mathbb{C}}_{n,p,q}$.

It is possible to show that the complex block diagonal matrices obtained through the transformation via Fourier matrices from real block circulant matrices have some special features; and all of the block diagonal matrices of such kind can be transformed into real block circulant ones with the inverse transformation.

*Corollary 10.* For a matrix $E \in \mathscr{C}^{\mathbb{R}}_{n,p,q}$, then for $\mathbf{E} = (F_n \otimes I_p)\, E\, (F_n \otimes I_q)^H \in \mathscr{D}^{\mathbb{C}}_{n,p,q}$ it holds that $\mathbf{E}_1 \in \mathbb{R}^{p\times q}$ and $\mathbf{E}_{n+2-i} = \bar{\mathbf{E}}_i$ for $i = \{2,\dots,n\}$.

Conversely, for a matrix $G \in \mathscr{D}^{\mathbb{C}}_{n,p,q}$ for which $G_1 \in \mathbb{R}^{p\times q}$ and $G_{n+2-i} = \bar{G}_i$ for $i = \{2,\dots,n\}$, we have that $(F_n \otimes I_p)^H\, G\, (F_n \otimes I_q) \in \mathscr{C}^{\mathbb{R}}_{n,p,q}$.

We are now ready to introduce the notion of a circulant system and show its key features. After the definition, we will first state a property that characterizes such kind of systems, and then we will show how they can be decomposed into smaller independent systems, thus enabling efficient solutions to the identification problem.

*Definition 11.* (Circulant systems). Consider a discrete-time MIMO system with $nm$ inputs and $nr$ outputs, with state-space equations of the kind:

$$\begin{cases} x(k+1) = \mathcal{A}x(k) + \mathcal{B}u(k) \\ y(k) = \mathcal{C}x(k) + \mathcal{D}u(k) \end{cases} \tag{1}$$

with $\mathcal{A} \in \mathbb{R}^{nl\times nl}$, $\mathcal{B} \in \mathbb{R}^{nl\times nm}$, $\mathcal{C} \in \mathbb{R}^{nr\times nl}$, $\mathcal{D} \in \mathbb{R}^{nr\times nm}$. The vector $x \in \mathbb{R}^{nl\times 1}$ is the state, $u \in \mathbb{R}^{nm\times 1}$ is the input signal and $y \in \mathbb{R}^{nr\times 1}$ is the output signal. We call the system "circulant" (or block circulant) if and only if $\mathcal{A} \in \mathscr{C}^{\mathbb{R}}_{n,l,l}$, $\mathcal{B} \in \mathscr{C}^{\mathbb{R}}_{n,l,m}$, $\mathcal{C} \in \mathscr{C}^{\mathbb{R}}_{n,r,l}$ and $\mathcal{D} \in \mathscr{C}^{\mathbb{R}}_{n,r,m}$. We consider also the input $u$ to be made of $n$ blocks of size $m \times 1$, which we denote as $u_i$, and the output $y$ to be made of $n$ blocks of size $r \times 1$, which we denote as $y_i$ $(i = 1,\dots,n)$. We call these blocks "local inputs" and "local outputs".

An important property of circulant systems is the invariance with respect to shifts in the inputs and outputs. If a certain input signal $u$ generates an output signal $y$, then a permuted version of the same input $((\Pi_n \otimes I_m)\, u)$ will generate a permuted version of the same output $((\Pi_n \otimes I_r)\, y)$. This is better explained in the following Lemma.

*Lemma 12.* (Invariance to input/output shift). Let the signal $y(k)$ be the output of the circulant system of (1) when excited by the input signal $u(k)$. Then also $\tilde{u}(k) = (\Pi_n \otimes I_m)\, u(k)$ and $\tilde{y}(k) = (\Pi_n \otimes I_r)\, y(k)$ are a valid input/output pair for the same system.

**Proof.** From Lemma 5, we can rewrite (1) as:

$$\begin{cases} x(k+1) = (\Pi_n^{-1} \otimes I_l)\mathcal{A}(\Pi_n \otimes I_l)x(k) + \\ \qquad + (\Pi_n^{-1} \otimes I_l)\mathcal{B}(\Pi_n \otimes I_m)u(k) \\ y(k) = (\Pi_n^{-1} \otimes I_r)\mathcal{C}(\Pi_n \otimes I_l)x(k) + \\ \qquad + (\Pi_n^{-1} \otimes I_r)\mathcal{D}(\Pi_n \otimes I_m)u(k) \end{cases}$$

If we perform the state transformation: $\tilde{x}(k) = (\Pi_n \otimes I_l)x(k)$, then the system becomes:

$$\begin{cases} \tilde{x}(k+1) = \mathcal{A}\tilde{x}(k) + \mathcal{B}\underbrace{(\Pi_n \otimes I_m)u(k)}_{\tilde{u}(k)} \\ \underbrace{(\Pi_n \otimes I_r)y(k)}_{\tilde{y}(k)} = \mathcal{C}\tilde{x}(k) + \mathcal{D}\underbrace{(\Pi_n \otimes I_m)u(k)}_{\tilde{u}(k)} \end{cases}$$

So we see that the dynamic equations for the input/output pair $\tilde{u}(k)$ and $\tilde{y}(k)$ are the same as for $u(k)$ and $y(k)$. So if $y(k)$ is a valid output for $u(k)$, then $\tilde{y}(k)$ is a valid output for $\tilde{u}(k)$. □

The property shown in this Lemma 12 is of fundamental importance, because it makes it possible to recognize a system as circulant *a priori*, from physical insight, without knowing its dynamic equations. If a system possesses certain symmetries such that it is possible to know that a shift in the input signals will generate a shift in the output signals, then it is possible to assume a circulant structure in the identification process. This circulant structure can be exploited to derive a specific subspace identification algorithm that assumes such structure. The following Theorem is key to the development of such an algorithm.

*Theorem 13.* (Decomposition property). A circulant system of order $nl$ as described in Definition 11 is equivalent to $n$ independent systems of order $l$ in the complex domain. Each of these subsystem has only $m$ inputs and $r$ outputs.

**Proof.** According to Corollary 9, it holds that:

$$\begin{aligned} \mathcal{A} &= (F_n \otimes I_l)^H \mathbf{A}(F_n \otimes I_l) \\ \mathcal{B} &= (F_n \otimes I_l)^H \mathbf{B}(F_n \otimes I_m) \\ \mathcal{C} &= (F_n \otimes I_r)^H \mathbf{C}(F_n \otimes I_l) \\ \mathcal{D} &= (F_n \otimes I_r)^H \mathbf{D}(F_n \otimes I_m) \end{aligned} \tag{2}$$

with $\mathbf{A} \in \mathscr{D}^{\mathbb{C}}_{n,l,l}$, $\mathbf{B} \in \mathscr{D}^{\mathbb{C}}_{n,l,m}$, $\mathbf{C} \in \mathscr{D}^{\mathbb{C}}_{n,r,l}$, $\mathbf{D} \in \mathscr{D}^{\mathbb{C}}_{n,r,m}$. So we can rewrite (1) as:

$$\begin{cases} x(k+1) = (F_n \otimes I_l)^H \mathbf{A}(F_n \otimes I_l)x(k) + \\ \qquad + (F_n \otimes I_l)^H \mathbf{B}(F_n \otimes I_m)u(k) \\ y(k) = (F_n \otimes I_r)^H \mathbf{C}(F_n \otimes I_l)x(k) + \\ \qquad + (F_n \otimes I_r)^H \mathbf{D}(F_n \otimes I_m)u(k) \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} (F_n \otimes I_l)x(k+1) = \mathbf{A}(F_n \otimes I_l)x(k) + \mathbf{B}(F_n \otimes I_m)u(k) \\ (F_n \otimes I_r)y(k) = \mathbf{C}(F_n \otimes I_l)x(k) + \mathbf{D}(F_n \otimes I_m)u(k) \end{cases}$$

If we apply the following invertible transformations for state, input and output:

$$\begin{aligned} \hat{x}(k) &= (F_n \otimes I_l)x(k) \\ \hat{u}(k) &= (F_n \otimes I_m)u(k) \\ \hat{y}(k) &= (F_n \otimes I_r)y(k) \end{aligned} \tag{3}$$

then the system turns into:

$$\begin{cases} \hat{x}(k+1) = \mathbf{A}\hat{x}(k) + \mathbf{B}\hat{u}(k) \\ \hat{y}(k) = \mathbf{C}\hat{x}(k) + \mathbf{D}\hat{u}(k) \end{cases} \tag{4}$$

All the matrices involved in this system are block diagonal, so this system is equivalent to the following $n$ independent $l^{\text{th}}$ order subsystems (of complex variables), each of them with $m$ inputs and $r$ outputs:

$$\begin{cases} \hat{x}_i(k+1) = \mathbf{A}_i\hat{x}_i(k) + \mathbf{B}_i\hat{u}_i(k) \\ \hat{y}_i(k) = \mathbf{C}_i\hat{x}_i(k) + \mathbf{D}_i\hat{u}_i(k) \end{cases} \quad \text{for } i = 1,\dots,n \tag{5}$$

where $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ and $\mathbf{D}_i$ are respectively the blocks in the diagonal of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, and $\hat{x}_i(k)$, $\hat{u}_i(k)$ and $\hat{y}_i(k)$ are the blocks of the column vectors $\hat{x}(k)$, $\hat{u}(k)$ and $\hat{y}(k)$; $\mathbf{A}_i \in \mathbb{C}^{l\times l}$, $\mathbf{B}_i \in \mathbb{C}^{l\times m}$, $\mathbf{C}_i \in \mathbb{C}^{r\times l}$, $\mathbf{D}_i \in \mathbb{C}^{r\times m}$, $\hat{x}_i(k) \in \mathbb{C}^{l\times 1}$, $\hat{u}_i(k) \in \mathbb{C}^{m\times 1}$ and $\hat{y}_i(k) \in \mathbb{C}^{r\times 1}$. □

Notice that the subsystems into which the global system is decomposed have nothing to do with the "physical" subsystems, like the ones shown in Figure 1. The state-space systems of complex variables found here can be seen as a kind of modal decomposition of the global system; in order to stress the difference between these and the "physical" subsystems, we will call the former "modal" subsystems.

*Remark 14.* The decomposition property (Theorem 13) can be interpreted under the formalism of systems over

spatial groups, as in Bamieh et al. [2002]. In this perspective, the circulant dynamic system is an operator with spatial coordinates ranging over $\mathbb{Z}_n$ (the finite group of integers modulo $n$) that has the property of "spatial invariance" (Lemma 12). A Fourier transform of the coordinates into its dual group ($\mathbb{Z}_n$ again in this case) is then able to block-diagonalize the system.

It is important also to point out that not all the $n$ modal subsystems of (5) are independent; actually, as a direct consequence of Corollary 10, the systems of index $n+2-i$ are the complex conjugate version of the systems of index $i$, for $i = \{2, \dots, n\}$. So there are only $n/2+1$ independent systems if $n$ is even and $(n+1)/2$ independent systems if $n$ is odd.

*Corollary 15.* (Properties of decomposition). With respect to (5), let $P_i$ indicate any among the following: $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$, $\hat{x}_i(k)$, $\hat{u}_i(k)$ and $\hat{y}_i(k)$. It holds that:

$$\begin{gathered} P_1 \text{ is real} \\ P_{n+2-i} = \bar{P}_i \quad \text{for } i = \{2, \dots, n\} \end{gathered}$$

## 3. IDENTIFICATION OF CIRCULANT SYSTEMS

### *3.1 Motivation and rationale*

As a consequence of Lemma 12, we have seen that there exist categories of systems which can be identified as circulant just from physical insight, as a result of the invariance of their input and output pairs to shifts. As an example, consider again the system shown in Figure 1: the global system is made of four smaller identical subsystems, each with its input and output, connected in a circular way. The interconnections between neighboring systems are all the same, and actually it is impossible to distinguish one system from the other.

In such a situation, putting Subsystem 1 in the place of Subsystem 2, Subsystem 2 in the place of Subsystem 3, Subsystem 3 in the place of Subsystem 4, and Subsystem 4 in the place of Subsystem 1 would still yield the same global system. Then we know that the invariance to shift of input/output pairs of Lemma 12 must hold, as it impossible to know if we are looking at the original system or its shifted (or "rotated") version.

So there might be the necessity of identifying such a kind of systems from data. Subspace methods (Van Overschee and De Moor [1994], Verhaegen [1994]) are the most common choice for MIMO systems, and they could be used in a situation as this to identify a discrete-time state-space model of the global system, from the set of all outputs and all inputs. The problem of this approach is in the fact that subspace methods return state-space matrices up to an arbitrary similarity transformation, that disrupts any structure the system may have. Moreover, we will also demonstrate that it can be useful to force the circulant structure to the model in order to improve the accuracy of the estimation, using the knowledge of the symmetries of the system as *a priori* information on the MIMO model.

We will shortly show that it is indeed possible to exploit the structure of circulant systems for identification; in fact, we will illustrate an identification algorithm that:

(1) allows using the prior knowledge of the system as circulant;
(2) reduces the computational complexity of the problem;
(3) preserves the circulant structure, that is, the identified model is again a circulant system.

The algorithm is a direct consequence of the diagonalization property of circulant systems (Theorem 13) and it can be outlined as follows. As the system can be transformed into $n$ independent subsystems, and as for each of these subsystem we can find *a priori* which are the inputs and outputs, then it is possible to identify each of these modal subsystems separately from each other. To this purpose, it is sufficient to transform the inputs and the outputs as in (3), and use them with any method (subspace identification, prediction error, etc., see Ljung [1987]) to identify the state-space matrices of the modal systems; the only additional care we will need to take is that we should extend the method to models with complex values. Actually not all the $n$ subsystems have to be identified, but only the independent ones, while the others are just the complex conjugates as explained in Corollary 15. Then, once these systems have been identified, the global model can be retrieved with the use of (2). Corollary 9 will grant that the global matrices obtained are block circulant, while Corollary 10 will grant that such matrices have real values.

We said in the previous paragraph that any method can be used for identifying the modal subsystems; actually, subspace methods seem to be the best choice at this point, as they are inherently fit to deal with state-space models (instead of transfer functions) and they can naturally be extended to the complex domain. The subspace identification process is a "numerical recipe" that yields four matrices as result of an input/output couple; all the algebraic operations used in subspace identification (matrix sum, matrix product, singular value decomposition or QR factorization) can be extended to complex numbers. Moreover, subspace methods will offer insight on the order of the subsystems ($l$), making it possible to choose a good value for it (although the different subsystems may yield different results, it is necessary to choose the same order $l$ for all of them). For these reasons, in the sequel of the paper we will use a subspace algorithm, specifically the MOESP (Multi-variable Output-Error State sPace) algorithm (Verhaegen [1994]). MOESP is fit for systems with white measurement noise only, and in the examples here we will restrict to them. But of course the idea of the algorithm can be extended to more sophisticated subspace methods that take into account different models of noise, like PI-MOESP, PO-MOESP (Verhaegen and Verdult [2007]) or N4SID (Van Overschee and De Moor [1994]).

Now we are ready to write the algorithm explicitly.

### *3.2 The novel algorithm*

*Algorithm 16.* (Circulant system identification). A set of $n$ input signals $u_i(k) \in \mathbb{R}^{m\times 1}$ and $n$ output signals $y_i(k) \in \mathbb{R}^{m\times 1}$ is given, for $i = \{1, \dots, n\}$ and $k = \{1, \dots, k_{\max}\}$. This set of data is associated with a dynamic system; we know, thanks to considerations stemming from Lemma 12, that this system has a circulant structure and that we can use a circulant system model according to Definition 11 to

describe it, where $n$, $m$ and $r$ are already known and $l$ is unknown.

Problem: identify an $ln^{\text{th}}$ order state-space circulant model from input-output data.

The problem is solved in the following steps:

(1) Compute the Fourier matrix $F_n$ of order $n$.
(2) Transform input and output signals, by computing:

$$\hat{u}(k) = (F_n \otimes I_m)u(k)$$
$$\hat{y}(k) = (F_n \otimes I_r)y(k)$$

(3) Verify that each signal $\hat{u}(k)$ is persistently exciting (Verhaegen and Verdult [2007]) of at least order $l$.
(4) Use MOESP to identify the independent state-space models of order $l$ from each $\hat{u}_i/\hat{y}_i$ pair:

$$\begin{cases} \hat{x}_i(k+1) = \hat{\mathbf{A}}_i\hat{x}_i(k) + \hat{\mathbf{B}}_i\hat{u}_i(k) \\ \hat{y}_i(k) = \hat{\mathbf{C}}_i\hat{x}_i(k) + \hat{\mathbf{D}}_i\hat{u}_i(k) \end{cases} \text{ for } i = 1, \ldots, n$$

If $n$ is even, then identify the systems for $i = \{1, \ldots, n/2\}$; if $n$ is odd instead, identify the systems for $i = \{1, \ldots, (n+1)/2\}$: the method will yield as results the identified (complex) matrices $\hat{\mathbf{A}}_i$, $\hat{\mathbf{B}}_i$, $\hat{\mathbf{C}}_i$ and $\hat{\mathbf{D}}_i$. Then use Corollary 15 to get the matrices of the other (dependent) systems:

$$\begin{matrix} \hat{\mathbf{A}}_i = \overline{\hat{\mathbf{A}}}_{n+2-i} \\ \hat{\mathbf{B}}_i = \overline{\hat{\mathbf{B}}}_{n+2-i} \\ \hat{\mathbf{C}}_i = \overline{\hat{\mathbf{C}}}_{n+2-i} \\ \hat{\mathbf{D}}_i = \overline{\hat{\mathbf{D}}}_{n+2-i} \end{matrix} \text{ for } i = \begin{cases} \{n/2+1, \ldots, n\} \text{ if } n \text{ even} \\ \{(n+1)/2, \ldots, n\} \text{ if } n \text{ odd} \end{cases}$$

(5) Construct the block diagonal matrices: $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ putting the identified blocks together.
(6) Retrieve the global system matrices with the following formulas:

$$\begin{aligned} \hat{\mathcal{A}} &= (F_n \otimes I_l)^H \hat{\mathbf{A}} (F_n \otimes I_l) \\ \hat{\mathcal{B}} &= (F_n \otimes I_l)^H \hat{\mathbf{B}} (F_n \otimes I_m) \\ \hat{\mathcal{C}} &= (F_n \otimes I_r)^H \hat{\mathbf{C}} (F_n \otimes I_l) \\ \hat{\mathcal{D}} &= (F_n \otimes I_r)^H \hat{\mathbf{D}} (F_n \otimes I_m) \end{aligned} \tag{6}$$

$\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ are real and block circulant, thanks to Corollaries 15, 9 and 10.

## 4. SOME SIMULATION RESULTS

### *4.1 Measurement noise*

For demonstrating the use of the algorithm, a stable circulant system of $12^{\text{th}}$ order, with $n = 4$, $l = 3$, $m = 1$ and $r = 1$ was randomly generated. The four input signals are made of 200 random samples each; white measurement noise has been added to all the four outputs.

In the test, we generated 250 different input/output pairs, and used them to identify the system. The algorithm shown in this paper (from now on, we will call it "circulant MOESP") was used and compared to a standard MOESP that assumes no structure at all for the system. In Figure 2 are shown the poles of the true system, together with the poles identified with the two different methods in 50 of the 250 runs; the poles identified with standard MOESP are indicated by a cross, while those which were found with the algorithm which assumes a circulant structure are indicated by a circle. At a glance it is possible to see that the circles are in general closer to the true poles if compared to the crosses (Figure 3 shows a magnification around one of the poles). Table 1 shows this observation in a more rigorous way, by comparing the mean square of the error in identifying each of the poles of the system.

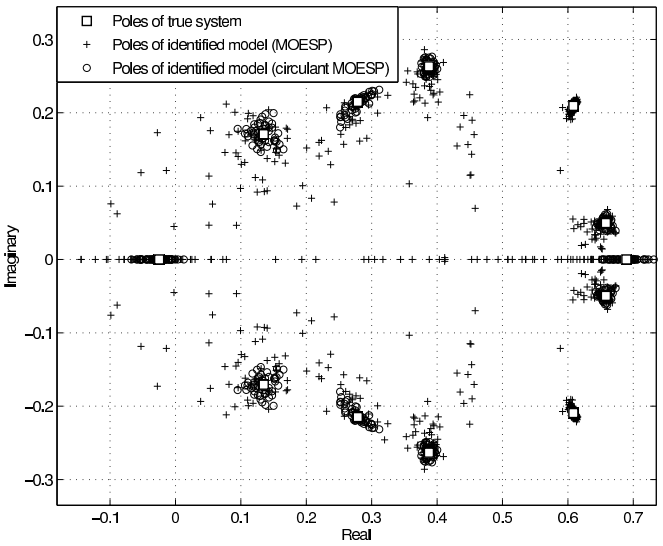


Fig. 2. Poles of the identified model in a set of 50 different experiments.

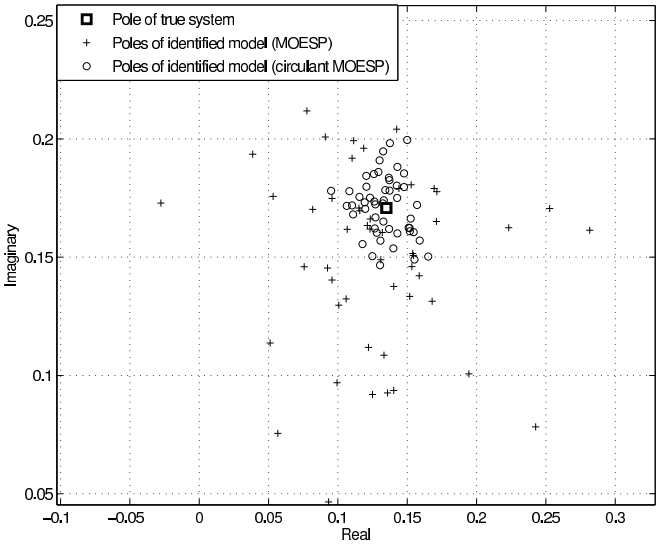


Fig. 3. Detail of Figure 2 around one of the poles.

| Pole | Root mean square error standard MOESP | circulant MOESP |
|---|---|---|
| $-0.02486$ | 0.04641 | 0.01958 |
| $0.13497 \pm 0.17077j$ | 0.06768 | 0.01879 |
| $0.27881 \pm 0.21487j$ | 0.08064 | 0.02038 |
| $0.38761 \pm 0.26329j$ | 0.02801 | 0.00877 |
| $0.60841 \pm 0.20941j$ | 0.00783 | 0.00394 |
| $0.65795 \pm 0.04966j$ | 0.02973 | 0.00870 |
| 0.68937 | 0.05027 | 0.01679 |

Table 1. Comparison of performances of the two different methods in identifying the poles with measurement noise.

So this example suggests that if we have a system with circulant structure, the novel method performs better than standard MOESP.

### *4.2 Non perfectly circulant systems*

Another test has been done adding "random perturbation" to the $\mathcal{A}$ matrix as well. This causes the system to be not perfectly circulant (that is most likely in real-life situations), but it has been verified that the method is still applicable; the idea is to show that small perturbations in the circulant structure do not cause completely wrong results. Again, we generated 250 different input/output pairs (with measurement noise), and used them to identify the system. For each pair, the $\mathcal{A}$ matrix has been perturbed with a different random matrix, each element of which was smaller than 1/1000 in modulus. For small perturbations such as these, there is still an advantage in the accuracy of the method with respect to standard MOESP, as shown in Figure 4 and in Table 2.

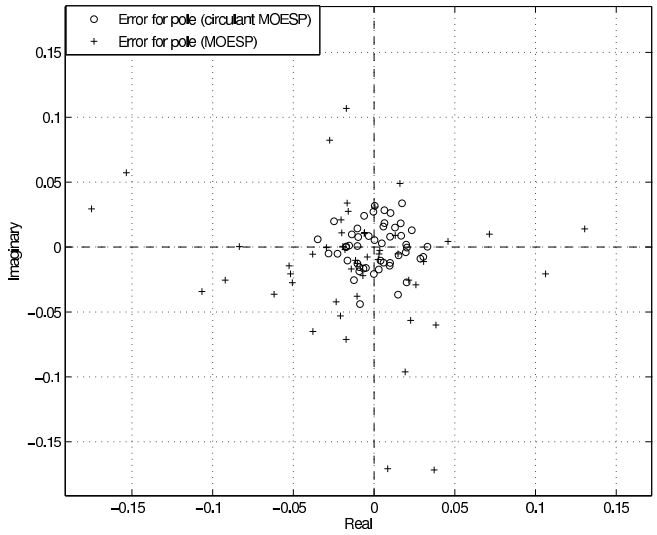


Fig. 4. Error in identifying one of the poles (the second in Table 2) in 50 experiments with perturbations on the $\mathcal{A}$ matrix.

| | Root mean square error | |
|---|---|---|
| Pole (if no noise) | standard MOESP | circulant MOESP |
| $-0.02486$ | 0.05287 | 0.02033 |
| $0.13497 \pm 0.17077j$ | 0.06198 | 0.02002 |
| $0.27881 \pm 0.21487j$ | 0.07897 | 0.02169 |
| $0.38761 \pm 0.26329j$ | 0.02744 | 0.00926 |
| $0.60841 \pm 0.20941j$ | 0.00771 | 0.00426 |
| $0.65795 \pm 0.04966j$ | 0.02874 | 0.00922 |
| 0.68937 | 0.05199 | 0.01857 |

Table 2. Comparison of performances of the different methods in identifying the poles, with measurement noise and perturbations on $\mathcal{A}$.

## 5. CONCLUSIONS

This paper has shown a new method for identifying a certain class of large scale systems possessing the property of circulant symmetry. This new method is based on a special property of circulant systems that allows them to be decomposed into a number of smaller "modal" subsystems of smaller order, allowing the independent identification of each one of them. The method can be used as a complement to any identification algorithm, but subspace methods are more appropriate, so in this paper the MOESP algorithm has been used and tested.

A complete algorithm that makes use of MOESP to identify circulant systems was developed. This algorithm allows maintaining the circulant structure in the final result, while subspace methods in general generate outputs up to an unpredictable similarity transformation. Moreover, the method uses the *a priori* information on the symmetries of the system to get better results, with a smaller computational effort.

The algorithm has been applied to an academic example, and the tests have verified the better ability of the algorithm in identifying circulant systems and the robustness to small perturbations of the circulant structure.

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