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Periodic use of time-varying state feedbacks for the Receding Horizon Control of bilinear systems

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Abstract: This paper provides a receding horizon control method for bilinear systems in the presence input constraints. Periodically-invariant sets are derived for a bilinear system with respect to a series of time-varying state feedback gains. The dual-mode control strategy is adopted and the periodically-invariant sets are used as target invariant sets. The state feedback gains used to define the target invariant sets are also used to render degrees of freedom to steer the current state into the target set. The region of attraction for the proposed algorithm is enlarged significantly with an extension of the horizon of periodicity while the on-line computation remains easy to handle.

1. INTRODUCTION

The 'dual-mode paradigm' is known to be an effective way to handle physical constraints in actuators.[3-4][6-8] The basic idea of the dual-mode paradigm is to use feasible control moves to steer the current state into a feasible and invariant terminal set in finite time steps. This dual-mode strategy has been adopted in many constrained MPC methods. The size of stabilizable regions of states of MPC methods depends on the size of underlying feasible and positively invariant terminal set and number of control moves. These results, however, could be conservative because the definition of positive invariance does not allow the state to leave the set, even temporarily. In the recent works of [2] and [5], a parameter dependent Lyapunov function has been used to reduce the conservativeness of the dual mode approach. But these works still assume the use of a single feedback gain and require strict invariance in the definition of a positively invariant set.

Motivated by these considerations, the concept of periodic invariance [7-8] was introduced for systems with polyhedral model uncertainties, in which states are allowed to leave a set temporarily but return into the set in finite time steps. Moreover, the periodic invariance involves the use of more than one state feedback gain and several ellipsoidal sets. These facts make it possible for the periodically invariant sets to provide considerably larger target sets in the dual-mode paradigm. In the work of [7] a single degrees of freedom was considered to steer the current state into the target set.

In this paper, we apply the concept of periodic invariance to the receding horizon control of bilinear systems. IOFL(Input Output Feedback Linearization) based methods[1][10-12] have been proposed for the control of SISO bilinear systems. The proposed method in this paper whereas can be applied to MIMO bilinear systems. First, periodically-invariant sets for the bilinear systems are derived. Based on these periodically-invariant sets, a sub-optimal dual-mode strategy is proposed. Unlike [7], a use of multiple steps of time-varying state feedback control is made to steer the current state into the periodically-invariant target sets. The on-line computation load and optimality of the proposed method are compared to those of nonlinear model predictive control methods based on non-convex optimisation and are shown to offer significant advantages.

2. SYSTEMS AND PERIODIC INVARIANCE

Consider an input affine nonlinear system with state space representation:

$$\begin{aligned} x_{t+1} &= f(x_t) + g(x_t)u_t \\ y_t &= Hx_t \end{aligned} \tag{1}$$

with input constraint $|u_t| \le \bar{u}$, where $x_t \in R^n$, $u_t \in R^m$ and

$$f(x_t) = Ax_t, \quad g(x_t) = Fx_t + B. \tag{2}$$

Consider the following ellipsoidal sets:

$$E_j := \{x : x^T P_j x \le 1 (P_j > 0)\}, j = 0, 1, \cdots, \nu - 1 \tag{3}$$

and the periodic control strategy given as:

$$u_t = -K_{\text{mod}(t/\nu)} x_t, \tag{4}$$

where the notation $\text{mod}(t/\nu)$ is defined as:

$$\begin{aligned}&\mathrm{mod}(0/v)=0,\mathrm{mod}(1/v)=1,\ldots\mathrm{mod}(v-1/v)=v-1,\\&\mathrm{mod}(v/v)=0,\mathrm{mod}(v+1/v)=1\ldots\end{aligned} \tag{5}$$

The set E_0 is defined to be periodically invariant with respect to the state feedback gains of (4), if $\forall x_0 \in E_0$ is steered through E_j, $j=1,2,\cdots,v-1$ and back into E_0 by the use of state feedback law of (4) for $t=0,1,\cdots,v-1$. The condition of this periodic invariance can be obtained as follows.
Substituting (4) into (1), we have:

$$\begin{aligned}x_{t+1} &= Ax_t-(Fx_t+B)K_{\mathrm{mod}(t/v)}x_t\\&=\Psi_{t,v}(x_t)x_t\end{aligned} \tag{6}$$

where

$$\Psi_{t,v}(x_t) := \left(A-(Fx_t+B)K_{\mathrm{mod}(t/v)}\right).$$

Thus $x_t \in E_{\mathrm{mod}(t/v)}$ would imply $x_{t+1} \in E_{\mathrm{mod}(t+1/v)}$ provided that

$$\Psi_{t,v}(x_t)^T P_{\mathrm{mod}(t+1/v)} \Psi_{t,v}(x_t) < P_{\mathrm{mod}(t/v)},\ \forall x_t \in E_j. \tag{7}$$

Note that (7) can be rewritten as the following matrix inequality: (Boyd et al, 1994)

$$\begin{bmatrix} Q_{\mathrm{mod}(t/v)} & * \\ \left(AQ_{\mathrm{mod}(t/v)}-(Fx_t+B)Y_{\mathrm{mod}(t/v)}\right) & Q_{\mathrm{mod}(t+1/v)}\end{bmatrix}>0 \tag{8}$$

for $\forall x_t \in E_{\mathrm{mod}(t/v)}$, where $Q_j = P_j^{-1}$, and $Y_j = K_j Q_j$.

Now, we focus on how to guarantee that (8) is satisfied for $\forall x_t \in E_{\mathrm{mod}(t/v)}$. Consider a polyhedron defined as:

$$\mathsf{W} = \left\{x \mid \|Wx\|_\infty = 1\right\}, \tag{9}$$

where W is a non-singular matrix and $\|\cdot\|_\infty$ represents infinite norms. Assume that $\mathsf{W} \supset E_j$, for $j=1,2,\cdots,v-1$, then (8) would be satisfied for $\forall x_t \in E_{\mathrm{mod}(t/v)}$, if only (8) is satisfied for $\forall x_t \in \mathsf{W}$, which can be checked considering only the corners of W. Based on the above argument the condition for the periodic invariance of E_0 can be formulated as per the following theorem.

Theorem 1. Consider the bilinear system (1) and feedback gains (4). The ellipsoidal set E_0 is periodically-invariant and feasible with respect to the state feedback control of (4) provided that

$$\begin{bmatrix} Q_j & \left(AQ_j-(F\upsilon_i+B)Y_j\right)^T \\ \left(AQ_j-(F\upsilon_i+B)Y_j\right) & Q_{j+1}\end{bmatrix}>0 \tag{10}$$

$$W_l Q_j W_l^T \le 1 \tag{11}$$

$$\begin{bmatrix} X_j & Y_j \\ Y_j^T & Q_j\end{bmatrix}>0,\ X_{j,kk} \le \bar{u}_k^2 \tag{12}$$

for $j=0,1,\cdots,v-1$, $i=1,2,\cdots,2^n$, $l=1,2,\cdots,n$ and $k=1,2,\cdots,m$ with $Q_v = Q_0$, where $\upsilon_i, i=1,2,\cdots,2^n$ denote the corners of the polyhedron W of (9), W_l represent the l^{th} row vector of W, and $X_{j,kk}$, $\bar{u}_k$ represent the k^{th} (diagonal) element of X_j and $\bar{u}$, respectively.

Proof : From (11), we have[1]

$$\begin{aligned}(W_l x)^2 &= \left|W_l P_j^{-1/2} P_j^{1/2} x\right|^2\\&\le W_l P_j^{-1} W_l^T x^T P_j x\\&\le W_l P_j^{-1} W_l^T \le 1\end{aligned} \tag{13}$$

for $\forall x \in E_j$. From (13), we can conclude that $E_j \subset \mathsf{W}$. On the other hand, (10) guarantees that (8) is satisfied for $\forall x_t \in \mathsf{W}$ and hence for $\forall x_t \in E_j$. Thus (7) is satisfied and $x_t \in E_j$ would imply $x_{t+1} \in E_{j+1}$ for $j=0,1,\cdots,v-1$ with $E_v = E_0$. Condition (12) ensures that $|K_j x| \le \bar{u}$ for all $x \in E_j$. □

Motivated by the desire to maximize the volume of W, one could sensibly/conveniently perform an optimisation such as:

$$\max_{Q_j, Y_j}\ trace(Q_0) \tag{14}$$

subject to (10-12)

Remark 1. [7] Multiplying (10) by λ_i and summing up it from $j=0$ to $v-1$, we have:

$$\begin{bmatrix} \sum_{j=0}^{v-1}\lambda_j Q_j & * \\ \left(A\sum_{j=0}^{v-1}\lambda_j Q_j-(F\upsilon_i+B)\sum_{j=0}^{v-1}\lambda_j Y_j\right) & \sum_{j=0}^{v-1}\lambda_j Q_{j+1}\end{bmatrix}\ge 0 \tag{15}$$

From (15), we can see that the convex-hull of the ellipsoids (4) defined as:

$$E = \left\{ x \mid x^T \left(\sum_{j=1}^{v-1} \lambda_j Q_j \right)^{-1} x \le 1, \lambda_j \ge 0, \sum_{j=1}^{v-1} \lambda_j = 1 \right\} \quad (16)$$

is invariant with respect to a state-dependent state feedback law:

$$K = \left(\sum_{j=0}^{v-1} \lambda_j Y_j \right) \left(\sum_{j=0}^{v-1} \lambda_j Q_j \right)^{-1}. \quad (17)$$

□

3. DUAL-MODE STRATEGY FOR BILINEAR SYSTEMS

In this section, we will consider a dual-mode strategy using the periodically-invariant sets as targets. The basic idea is to steer the current state into one of periodically-invariant sets $E_j, j = 0,1,\cdots,v-1$ in v time steps using v numbers of feasible control moves. Here we assume that the series of feedback gains $\boldsymbol{K}_i, \boldsymbol{i} = 0,1,\cdots,\boldsymbol{v}$ are given by solving (14). We consider the following cost-index for the choice of v control moves:

$$J(x_t, j) = \sum_{i=1}^{v} \left\| x_{t+i|t} \right\|_{P_{\mathrm{mod}(t+i+j/v)}} \quad (18)$$

where $x_{t+i|t}$ is the prediction of the state for the time step $t+i$, $\|x\|_P := x^T P x$ and j can be chosen to be an integer from 0 to $v-1$. Based on this cost index, define an optimisation problem as:

$$\arg = \min_{j, u_{t+i} (i = 0,\cdots, v-1)} J(x_t, j) \quad (19)$$

subject to

$$\left| u_{t+i} \right| \le \bar{u} \text{ and } x_{t+v|t} \in E_j. \quad (20)$$

Because of the nonlinearities of (1-2), the predictions $x_{t+i|t}$ would be nonlinear in terms of u_{t+i} and the problem of (19-20) is a non-convex problem. Instead of solving this non-convex optimisation problem which may involve heavy computational burden, we propose a sub-optimal strategy. Consider the following control move:

$$u_t(x_t, j) = \begin{cases} -K_{\mathrm{mod}(t+j/v)} x_t, & if \ \left| -K_{\mathrm{mod}(t+j/v)} x_t \right| < \bar{u} \\ \bar{u}, & if \ -K_{\mathrm{mod}(t+j/v)} x_t \ge \bar{u} \\ -\bar{u}, & if \ -K_{\mathrm{mod}(t+j/v)} x_t \le -\bar{u} \end{cases} \quad (21)$$

and a set of these control moves defined as:

$$U(t, j) = \left\{ u_{t+i}(x_{t+i|t}, j), i = 0,1,\cdots, v-1 \right\}, \quad (22)$$

If the control law is determined as (21) for the given state x_t at time step t, then $x_{t+1|t}$ will be obtained from the bilinear dynamics (1-2). Then $x_{t+2|t}$ could be determined based on the control law (21) with the given $x_{t+1|t}$. Repeating this procedure, the set of control moves $U(t, j)$ can be determined for the given x_t and j . Based on this argument a sub-optimal problem can be determined as:

$$\arg = \min_{U(t,j)(j = 0,\cdots, v-1)} J(x_t, j) \quad (22)$$

subject to $\quad x_{t+v|t} \in E_j. \quad (23)$

If the problem (22-23) is feasible for the given current state x_t, we would get the optimal j i.e. j^* and corresponding sequence of control moves $U(t, j^*)$. The closed-stability of the system when the control moves $U(t+kv, j^*)(k = 0,1,\cdots)$ are applied periodically is summarised as per the following lemma:

Lemma 1. Assume that the problem (22-23) is feasible with the corresponding optimal control sequence $U(t, j^*)$. Then the use of periodic state feedback gains i.e. $U(t+kv, j^*)(k = 0,1,\cdots)$ will ensure the asymptotic stability of the closed-loop system.

Proof : From the constraint (23), $U(t, j^*)$ will steer x_t into E_{j^*} . Once the state is steered into E_{j^*} , from (7-8) and Theorem 1, it is guaranteed that

$$\left\| x_{t+i} \right\|_{P_{\mathrm{mod}(t+i+j^*)}} > \left\| x_{t+i+v} \right\|_{P_{\mathrm{mod}(t+i+j^*)}}. \quad (24)$$

From (24) and (18), we have:

$$J(x_{t+i}) > J(x_{t+i+1}). \quad (25)$$

Thus the cost index is monotonically decreasing and the state should converge to the origin.

□

Although the periodic application of the sequence of the state feedback gains would result in asymptotic stability, the performance can be further enhanced by following the receding horizon strategy. A receding horizon control strategy based on (22-23) can be described as follows:

Algorithm 1

(Off-Line)

Step 1 : Solve the problem (14) to obtain Y_j, Q_j s and corresponding K_j, P_j s for $j = 0,1,\cdots,v-1$.

(On-line)

Step 2 : For the given initial state x_0, solve the problem (22-23) to obtain the optimal set of control moves $U(0, j^*)$. Apply $u_i(x_{i|0}, j^*)(i = 0,1,\cdots,v-1)$ to the system until $x_i \in E_{\hat{j}}$ for some i such that $v \ge i \ge 1$.

Step 3 : At time steps $t \ge \hat{j}$, solve

$$\arg = \min_{U(t,j)(j=0,\cdots,v-1)} J(x_t, j) \tag{26}$$

subject to

$$x_t \in E_{\mathrm{mod}(t+j)}. \tag{27}$$

Denote the optimal j as j^o.

Step 4 : Apply $u_t(x_t, j^o)$ to the system. Repeat Step 3 and 4 for the next time steps.

The stability of the Algorithm 1 can be established as per the following theorem:

Theorem 2. Consider the bilinear system (1) and K_j, P_j s for $j = 0,1,\cdots,v-1$ obtained by solving (14). If the Algorithm 1 is feasible at first time step, then it remains feasible and the resulting closed loop-system is asymptotically stable.

Proof : Assume that the problem (22-23) is feasible at $t = 0$ and yields minimal cost index $J(x_0, j^*)$. Because of the constraint (23), applying $U(t, j^*)$ guarantees that $x_{t+v|t} \in E_{j^*}$. Once the state is steered into an ellipsoidal set i.e. $x_t \in E_{\hat{j}}$, the periodic application of $U(t, \hat{j})$ will ensure that the cost index will decrease as mentioned in Lemma 1. The optimisation (26-27) looks for further reduction in the cost. Thus the cost index of Algorithm 1 is monotonically decreasing and the state will converge to the origin asymptotically. □

Remark 2. The problem (22-23) and (26-27) are similar to integer programming since the decision variable j can take only v integer values. The computation for solving these problems consists of the determination of the state predictions for the circular applications of different state feedback gains and the comparison of cost indices.

4. SIMULATION STUDIES

Consider the bilinear system (1)-(2) with $\bar{u} = 5$ and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.1803 & -0.4397 & 1.6389 \end{bmatrix}; \quad B = \begin{bmatrix} 0.19 \\ -0.098609 \\ -0.0351533 \end{bmatrix} \quad F = \begin{bmatrix} -0.0082 & 0 & 0 \\ -0.017639 & 0 & 0 \\ -0.032703 & 0 & 0 \end{bmatrix}; \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{28}$$

The matrix A is unstable and the system is nonminimum-phase at the origin. The polyhedral (9) is defined with:

$$W = \begin{bmatrix} \sqrt{2.5} & 0 & 0 \\ 0 & \sqrt{0.1} & 0 \\ 0 & 0 & \sqrt{0.1} \end{bmatrix}. \tag{29}$$

We will show the efficacy of the proposed method by comparing it with NMPC method based on the nonlinear optimisation (19-20).

Figure 1 has the maximum feasible sets for NMPC with $v = 4$ and 8(dotted lines) superimposed on the regions of attraction for $v = 4$,8, and 15 (solid lines). The execution times (in milliseconds, averaged over 1000 trials) on PC (3.2 GHz Pentium 4) for NMPC and the periodic feedback algorithm are:

v	NMPC	Algorithm 1	State
8	13.3	1.96	[0;1.13;1.57]
4	3.41	0.54	[0;1.16;0.92]

and the optimal predicted costs for the two algorithms were

v	NMPC	Algorithm1	State
8	1539.6	1548.7	[0;1.13;1.57]
4	139.9	171.5	[0;1.16;0.92]

so the Periodic is about 6.5 times faster than NMPC (for same v and x), whereas the costs are respectively 6% and 18% suboptimal for Periodic for $v=8$ and $v=4$.

To compare optimality and computation directly consider a single initial condition: $x=[0;1.76;1.52]$ which is feasible for Algorithm 1Periodic with $v=8$ and for NMPC with $v=4$:

v	NMPC(cost,time)	Algorithm1 (cost,time)
8	466.1, 15.8	495.1, 1.88
4	480.9, 3.67	Infeasible

so the Algorithm 1 with $v=8$ is better in terms of both computation and cost than NMPC with $v=4$, for this initial condition.

5. CONSLUSIONS

In this paper, a receding horizon control method for bilinear systems was derived using periodic invariance. The v different state feedback gains used for the definition of periodic invariance are also exploited as degrees of freedom to steer the current state into the target invariant set in the dual-mode approach. The use of extra degrees of freedom enlarges the stabilizable region significantly while the computational burden remains tractable. The computational burden and performance of the proposed method was compared with the nonlinear MPC and the results shows that the proposed approach could be better in terms of both computation and cost.

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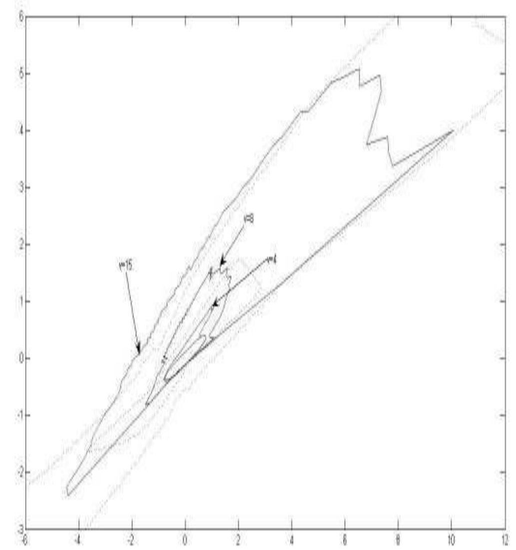

Fig. 1 Stabilization regions of Algorithm 1 and NMPC with $v=4,8,15$ and $v=4,8$.