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Boundary Output Feedback Stabilization of A One-Dimensional Wave Equation System With Time Delay ⋆

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Abstract: The stabilization with time delay in observation and control represents extremely difficult mathematical challenges in distributed parameter systems control. It is well-known that the closed-loop stability achieved by some stabilizing output feedback laws is not robust for any small time delay. When this happens, it necessary to reconsider the stabilizing feedback control laws. We are concerned with a particularly interesting case: boundary output feedback stabilization of one-dimensional wave equation system for which the boundary observation suffers a time delay. This stabilization problem has been unsolved for over two decades. We construct an infinite-dimensional observer such that the estimation error converges exponentially to zero as time goes to infinity and we design a stabilizing state feedback law. Using the separation principle as in the finite dimensional cases, we show that the delay system is exponentially stabilized by the feedback law based on the estimated state.

1. INTRODUCTION

In a practical control system, there is often a time delay between the sent control and the information via the observation of the system. A general case occurs in sampled-data control systems. Time delay may destroy the stability or cause periodic oscillations for a control system Gumowski and Mira [1968]. For distributed parameter control systems, the stabilization with time delay in observation and control represents difficult mathematical challenges [Fleming, 1988, p.69]. The first example of one-dimensional wave equation with boundary feedback in Datko et al. [1986], Datko [1988] illustrates that every small time delay in the known stabilizing boundary output feedback schemes could destabilize the system. This interesting case was particularly mentioned in (Fleming [1988], p.69). Afterwards the time delay problem was raised over and over again: whether a small time delay causes catastrophic behavior in actual systems (see Datko [1995], Datko and You [1991], Datko [1991, 1997]). If it does, the current control theory for elastic systems is invalid. The difficulty to overcome time delay problem for elastic systems is that there are an infinite number of eigenvalues on the imaginary axis, which is in sharp contrast to parabolic systems. A general result of Logemann et al. [1996] shows that if there is a time delay in the output of an infinite-dimensional control system, the stabilization by the output PI feedbacks is not robust to time delay.

Recently inspired by the works of Deguenon et al. [2006], Smyshlyaev and Krstic [2005], we solved successfully the stabilization problem of one dimensional wave equation system with boundary control and non-collocated observation Guo and Xu [2007]. The idea behind is to use the separation principle which is valid for finite-dimensional linear systems [Callier and Desoer, 1991, p.329] and also for some nonlinear systems Gauthier and Kupka [1992]. For infinite-dimensional systems, applying the principle is generally more complex, since different stabilities like weak stability, strong stability and exponential stability are not equivalent (see also Nguyen and Egeland [2006]). However we show that the principle still works for the stabilization of time-delayed distributed parameter systems.

In this paper, we are concerned with the following one-dimensional wave equation under boundary control and collocated observation with time delay:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0,\ 0<x<1, t>0, \\ w(0,t)=0, w_x(1,t)=u(t),\ t>0, \\ w(x,0)=w_0(x), w_t(x,0)=w_1(x),\ 0<x<1, \\ y(t)=w_t(1,t-\tau),\ t>0, \end{cases} \tag{1}$$

where u is the control (input), $\tau>0$ is a constant time delay and y is the output observation which suffers a delay time τ. This is to say that the information concerning the initial condition is available only after time τ. Of course the delay is a dynamical system which represents somehow a memory. We have to precise the initial condition for the memory such that the trajectory of the system (1) is determined. Without lost of generality we assume that $w_t(1,s)=0\ \forall\, s\in[-\tau,0]$.

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It is well-known that if the system (1) is time delay free or $\tau = 0$, the proportional output feedback control $u(t) = -ky(t), k > 0$ exponentially stabilizes the system. However, if there is a sufficiently small time delay $\tau > 0$, then the closed-loop system by the output feedback control $u(t) = -ky(t)$, $k > 0$, has at least one eigenvalue with positive real part (see Datko et al. [1986], Datko [1988]), and so unstability appears. In other words the stabilizing output feedback control law can not tolerate any small delay in the control feedback law. The drawback of the feedback control law makes it almost useless or extremely dangerous to applications, since usually there is some small delay from the received information to the sent control. This problem has been pointed out in [Fleming, 1988, p.69] and the stabilization of (1) with $\tau > 0$ is the particularly interesting case discussed there.

For the system (1) the energy state space is the Hilbert space $H = H_L^1(0,1) \times L^2(0,1), H_L^1(0,1) = \{f | \ f \in H^1(0,1), f(0) = 0\}$ with state variable $(w(\cdot,t), w_t(\cdot,t))$. The input space and the output space are the same $U = Y = \mathbb{C}$. The norm of $(w(\cdot,t), w_t(\cdot,t))$ in H is defined by the energy

$$E(t) = \frac{1}{2}\|(w(\cdot,t), w_t(\cdot,t))\|^2 = \frac{1}{2}\int_0^1 [w_x^2(x,t)+w_t^2(x,t)]dx.$$

Our idea of solving the stabilization of system (1) is to use the separation principle. We design an infinite-dimensional observer for the system (1) such that the estimation error converges exponentially to zero as time goes to infinity. To stabilize the system we apply a stabilizing state feedback law based on the estimated state through the observer.

In the next section, Section 2, we show that the system is well-posed in the sense of D. Salamon (cf. Curtain [1997]). This seems necessary for the design of the observer. The exact observability and controllability are illustrated in Section 3. Section 4 is devoted to the design of the observer. Finally, in Section 5, we design a stabilizing feedback control law and show that the closed-loop system is exponentially stable.

2. WELL-POSEDNESS

We begin by considering the dynamical system (1) without delay. The first question is its well-posedness in the sense of D. Salamon. The question is to know if the following mapping is continuous: the mapping which, to each pair of initial condition and control input signal, associates the state and the output observation signal.

It is well known that the following system

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, \ 0 < x < 1, t > 0, \\ w(0,t) = 0, \ t \geq 0, \\ w_x(1,t) = u(t), \ t \geq 0, \\ w(x,0) = w_0(x), \ \ w_t(x,0) = w_1(x), \ 0 < x < 1, \\ y_w(t) = w_t(1,t), \ t \geq 0, \end{cases} \tag{2}$$

can be written as a second-order system in H studied in Guo and Luo [2002]:

$$\begin{cases} w_{tt}(\cdot,t) + Aw(\cdot,t) + Bu(t) = 0, \\ y_w(t) = -B^* w_t(\cdot,t), \end{cases} \tag{3}$$

where

$$\begin{aligned} &Af = -f'', \forall \ f \in D(A) \\ &= \{f \in H^2(0,1) \cap H_L^1(0,1) | f'(1) = 0\} \\ &B = -\delta(x-1) \end{aligned} \tag{4}$$

and $\delta(\cdot)$ denotes the Dirac distribution. The operator A has eigen-pairs $\{\lambda_n = (n - 1/2\pi)i, \varphi_n = \sin(n - 1/2)\pi x\}_{n\in\mathbb{Z}}$. Since $|b_n| = |\langle b, \sin(n-1/2)\pi x\rangle| = 1$ for any $n \in \mathbb{Z}$, it follows from Proposition 2 of Guo and Luo [2002] that B is an admissible input operator. A direct computation shows that the transfer function of (2) is

$$H(s) = \frac{e^s - e^{-s}}{e^s + e^{-s}}, \ \forall \ \ \text{Re}s > 0,$$

which is bounded on the open right half complex plane. Hence the system (2) is well-posed in the sense of D.Salamon (see Curtain [1997]), that is to say, for any $u \in L_{loc}^2(0,\infty;\mathbb{R})$ and $(w_0, w_1) \in H$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t)) \in C(0,\infty; H_L^1(0,1) \times L^2(0,1))$ to (2), and moreover, for any $T > 0$, there exists a constant D_T such that

$$\begin{aligned} &\|(w(\cdot,T), w_t(\cdot,T))\|^2 + \int_0^T |y_w(s)|^2 ds \\ &\leq D_T \left[\|(w_0, w_1)\|^2 + \int_0^T |u(s)|^2 ds\right]. \end{aligned} \tag{5}$$

Now let us return to our dynamical system (1). Since there is a time delay in the output observation, the state space for the whole system is the Hilbert space $\mathcal{H} = H \times L^2(0,\tau)$. As we have said in the Introduction, the time delay is a dynamical system of memory and described by the state equation

$$\begin{aligned} &\xi(s,t) = w_t(1, t-s), \quad 0 < s < \tau, \quad t > 0, \\ &\xi(s,0) = \xi_0(s), \quad 0 < s < \tau, \end{aligned} \tag{6}$$

where $\xi_0(s)$ is a known function in $L^2(0,\tau)$. The norm on the state space $\mathcal{H}$ is given by

$$\|(w_1, w_2, \xi)'\|_{\mathcal{H}}^2 = \|(w_1, w_2)'\|_H^2 + \int_0^\tau \xi^2(s)ds.$$

Then the well-posedness of (1) follows immediately from (5):

$$\begin{aligned} &\left\| \begin{bmatrix} w(\cdot,T) \\ w_t(\cdot,T) \\ \xi(\cdot,T) \end{bmatrix} \right\|_{\mathcal{H}}^2 + \int_0^T |y(s)|^2 ds \\ &\leq \tilde{D}_T \left[\left\| \begin{bmatrix} w_0 \\ w_1 \\ \xi_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 + \int_0^T |u(s)|^2 ds \right] \end{aligned} \tag{7}$$

where $\tilde{D}_T = \max(1, D_T)$. Hence the system (1) with delay is also well-posed.

So we have proved the following Theorem 1.

Theorem 1. The system (1) is well-posed: For any $(w_0, w_1, \xi_0) \in \mathcal{H}$ and $u \in L_{loc}^2(0,\infty;\mathbb{R})$ there exists a unique solution to (1)+(6) such that $(w(\cdot,t), w_t(\cdot,t), \xi(\cdot,t)) \in C(0,\infty;\mathcal{H})$, and moreover, for any $T > 0$ there exists a constant D_T such that

$$\begin{aligned} &\|(w(\cdot,T), w_t(\cdot,T), \xi(\cdot,T))\|^2 + \int_0^T |y(t)|^2 dt \\ &\leq D_T \left[\|(w_0, w_1, \xi_0)\|^2 + \int_0^T |u(t)|^2 dt\right]. \end{aligned} \tag{8}$$

3. EXACT OBSERVABILITY AND EXACT CONTROLLABILITY

In this section we consider the exact observability of the dynamical system with delay:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, \ 0 < x < 1, t > 0, \\ w(0,t) = w_x(1,t) = 0, \ t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \ 0 < x < 1, \\ y(t) = w_t(1, t-\tau), \ t > 0. \end{cases} \tag{9}$$

To complete the description of (9) we have to add to it the state equation with an initial condition

$$\begin{aligned} &\xi(s,t) = w_t(1, t-s), \quad 0 < s < \tau, \quad t > 0, \\ &\xi(s,0) = \xi_0(s), \quad 0 < s < \tau. \end{aligned} \tag{10}$$

We will study the exact observability and the exact controllability of (9)+(10) on the state space $\mathcal{H}$. Let us begin by the exact observability.

3.1 Exact observability

Define the energy function for the system (9) by

$$E(t) = \frac{1}{2}\int_0^1 [|w_x(x,t)|^2 + |w_t(x,t)|^2]dx.$$

Then $E(t) = E(0)$ for any $t \geq 0$. On the other hand, define

$$\rho(t) = \int_0^1 x w_x(x,t) w_t(x,t) dx.$$

Then $|\rho(t)| \leq E(t)$ for any $t \geq 0$. Notice that

$$\dot{\rho}(t) = \frac{1}{2}|w_t(1,t)|^2 - E(t).$$

We have

$$(T-2)E(0) \leq \frac{1}{2}\int_0^T |w_t(1,t)|^2 dt \leq (T+2)E(0). \tag{11}$$

As a consequence the system (9) without delay is exactly observable on each interval [0,T] with $T > 2$. $T = 2$ is the minimal time needed for the propagation of all the wave to be observed on the only boundary $x = 1$. The question is to know if the dynamical system (9) with delay is exactly observable. It is easy to see that the response is positive.

Indeed, from (11) we write for each $T > 2 + \tau$

$$2(T-\tau-2)E(0) \leq \int_0^{T-\tau} w_t^2(1,t)\, dt \leq 2(T-\tau+2)E(0). \tag{12}$$

Since

$$\int_0^T y^2(t)dt = \int_{-\tau}^0 w_t^2(1,s)ds + \int_0^{T-\tau} w_t^2(1,s)ds,$$

by (12) we get easily

$$c_1\tilde{E}(0) \leq \int_0^T y^2(t)\, dt \leq c_2\tilde{E}(0),$$

where

$$\begin{aligned} &c_1 = \min\{2(T-\tau-2), 2\} \\ &c_2 = \max\{2(T-\tau+2), 2\} \\ &2\tilde{E}(0) = \int_0^1 (w_0^2(x) + w_1^2(x))dx + \int_{-\tau}^0 \xi_0^2(s)ds. \end{aligned}$$

We have proved the following result.

Theorem 2. The system (9) is exactly observable on each interval $[0,T]$ with $T > \tau + 2$.

3.2 Exact controllability

We consider the system on $\mathcal{H}$ with control

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, \ 0 < x < 1, t > 0, \\ w(0,t) = 0, \quad w_x(1,t) = u(t), \ t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \ 0 < x < 1, \\ \xi(s,t) = w_t(1, t-s), \quad 0 < s < \tau, \quad t > 0, \\ \xi(s,0) = \xi_0(s), \quad 0 < s < \tau. \end{cases} \tag{13}$$

We will say that the system (13) is exactly controllable on $\mathcal{H}$ if there exists a $T > 0$ such that for each couple of states $(w_0, w_1, \xi_0) \in \mathcal{H}$ and $(w_0^T, w_1^T, \xi_0^T) \in \mathcal{H}$, a control $u \in L^2(0,T)$ can be found so that the solution of the system satisfies

$$\begin{cases} w(\cdot,0) = w_0 \\ w_t(\cdot,0) = w_1 \\ \xi(\cdot,0) = \xi_0 \end{cases} \text{and} \begin{cases} w(\cdot,T) = w_0^T \\ w_t(\cdot,T) = w_1^T \\ \xi(\cdot,T) = \xi_0^T. \end{cases} \tag{14}$$

It is well-known that the system (13) is exactly controllable on H. For each couple $(w_0, w_1), (w_0^T, w_1^T) \in H$ we find some $u \in L^2(0,T)$ such that the conditions (14) are satisfied (except the last one on ξ), as soon as $T > 2$. Then we have the exact controllability on $\mathcal{H}$ for $T > 2+\tau$.

Theorem 3. The system (13) is exactly controllable on each interval $[0,T]$ with $T > \tau + 2$.

Proof. Let $T > \tau + 2$. We consider the backward evolution of the following system

$$\begin{cases} z_{tt}(x,t) - z_{xx}(x,t) = 0, \ 0 < x < 1, \ \ T-\tau < t < T, \\ z(0,t) = 0, \\ z_t(1,t) = \xi_0^T(T-t), \\ z(x,T) = w_0^T(x), \ \ z_t(x,T) = w_1^T(x). \end{cases} \tag{15}$$

This way the system (14) is drived to the state $(z(\cdot, T-\tau), z_t(\cdot, T-\tau)) \in H$ and we get a control signal $u \in L^2(T-\tau, T)$ by defining $u_2(t) = z_x(1,t) \ \forall\ t \in (T-\tau, T)$, as the system is well-posed as stated in Section 2. With this signal u_2 as control and $(w(\cdot, T-\tau), w_t(\cdot, T-\tau)) = (z(\cdot, T-\tau), z_t(\cdot, T-\tau))$ as initial condition, we consider the forward evolution of the same system but with a different boundary condition

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0, \ 0 < x < 1, \ \ T-\tau < t < T, \\ w(0,t) = 0, \\ w_x(1,t) = u_2(t). \end{cases} \tag{16}$$

Since the solution is unique, it has the same solution as (15) and so $\xi(s,T) = w_t(1, T-s) = \xi_0^T(s) \ \ \forall \ \ s \in (T-\tau, T)$. Il rests only to find a control $u_1 \in L^2(0, T-\tau)$ such that the solution of (13) verifies

$$\begin{cases} w(\cdot,0) = w_0 \\ w_t(\cdot,0) = w_1 \end{cases} \text{and} \begin{cases} w(\cdot, T-\tau) = z(\cdot, T-\tau) \\ w_t(\cdot, T-\tau) = z_t(\cdot, T-\tau). \end{cases} \tag{17}$$

This is possible because the system is exactly controllable on H for any $T > 2$. Then the concatenation $u(t)$ of the two controls does the work, this is, drives the system (13) from the initial condition to the final condition:

$$u(t) = \begin{cases} u_1(t), \ t \in (0, T-\tau); \\ u_2(t), \ t \in (T-\tau, T). \end{cases}$$

∎

4. OBSERVER DESIGN

From the discussions in previous sections, we know that the system (1) is well-posed and exactly observable. These imply that the output belongs to $L^2_{loc}(0,\infty)$ and could be used to recover the state. Let us design an observer for the system (1) (see Demetriou [2004] for other observers).

Now we proceed step by step. First, we construct the observer for $w(x,t-\tau)$. Since $z(x,t)=w(x,t-\tau)$ satisfies

$$\begin{cases} z_{tt}(x,t)-z_{xx}(x,t)=0,\ 0<x<1, t>\tau,\\ z(0,t)=0,\\ z_x(1,t)=u(t-\tau),\\ z(x,\tau)=w_0(x),\ z_t(x,\tau)=w_1(x),\ 0<x<1,\\ y(t)=z(1,t),\ t>\tau, \end{cases} \tag{18}$$

we construct naturally an observer for (18) by the principle of "copy of the plant and injection of the output":

$$\begin{cases} \widehat{w}_{tt}(x,t)-\widehat{w}_{xx}(x,t)=0,\ 0<x<1,\ t>\tau,\\ \widehat{w}(0,t)=0,\\ \widehat{w}_x(1,t)=u(t-\tau)-k_1\left[\widehat{w}_t(1,t)-y(t)\right],\ \ k_1>0,\\ \widehat{w}(x,\tau)=\widehat{w}_0(x),\ \widehat{w}_t(x,\tau)=\widehat{w}_1(x),\ 0<x<1. \end{cases} \tag{19}$$

where $(\widehat{w}_0,\widehat{w}_1)$ is an arbitrary initial state of the observer. Let

$$\varepsilon(x,t)=\widehat{w}(x,t)-z(x,t). \tag{20}$$

Then ε satisfies

$$\begin{cases} \varepsilon_{tt}(x,t)-\varepsilon_{xx}(x,t)=0,\ 0<x<1,\ t\ge\tau,\\ \varepsilon(0,t)=0,\\ \varepsilon_x(1,t)=-k_1\varepsilon_t(1,t),\\ \varepsilon(x,\tau)=\widehat{w}_0(x)-w_0(x),\\ \varepsilon_t(x,\tau)=\widehat{w}_1(x)-w_1(x). \end{cases} \tag{21}$$

It is well-known that for any $k_1>0$, the system (21) is exponentially stable in H:

$$\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|\le Me^{-\omega(t-\tau)}\|(\varepsilon(\cdot,\tau),\varepsilon_t(\cdot,\tau))\| \tag{22}$$

for some positive constants M and ω and for every $t\ge\tau$.

In order to show that (19) is indeed an observer of the system (18), we need to solve (19). This is a direct consequence of Corollary 1 of Guo and Luo [2002] since (19) can be written as

$$\widehat{w}_{tt}(\cdot,t)+A\widehat{w}(\cdot,t)+k_1bb^*\widehat{w}_t(\cdot,t)+b[u(t-\tau)+k_1y(t)]=0 \tag{23}$$

where A,b are defined in (4).

Proposition 4. The system (19) is well-posed: for any $(\widehat{w}_0,\widehat{w}_1)\in H$, $u\in L^2_{loc}(0,\infty)$, there exists a unique solution to (19) such that $(\widehat{w}(\cdot,t),\widehat{w}_t(\cdot,t))\in C(\tau,\infty;H)$. Moreover, for any $T>\tau$, there exists a constant C_T such that

$$\|(\widehat{w}(\cdot,T),\widehat{w}_t(\cdot,T))\|^2 \le C_T\left[\|(\widehat{w}_0,\widehat{w}_1)\|^2+\int_\tau^T|y(t)|^2dt+\int_0^{T-\tau}|u(t)|^2dt\right].$$

Keep in mind that the closed-loop system (1) is exponentially stable if $u(t)=-k_2w_t(1,t), k_2>0$. However, the variable $w_t(1,t)$ is not directly measurable, since there is a delay τ in the output $y(t)=w_t(1,t-\tau)$. Now the observer (19) gives us the estimated state $w(x,t-\tau)$. In order to get $w_t(1,t)$, we need to predict the values of w on $[t-\tau,t]$. To do this, we consider $\widehat{w}(x,s,t)=w(x,t-\tau+s)$ with $t\ge\tau$ and $s\in[0,\tau]$.

By (1), $\widehat{w}(x,s,t)$ satisfies

$$\begin{cases} \widehat{w}_{ss}(x,s,t)-\widehat{w}_{xx}(x,s,t)=0,\ 0<x<1,\\ \widehat{w}(0,s,t)=0,\ 0\le s\le\tau,\ t\ge\tau,\\ \widehat{w}_x(1,s,t)=u(t-\tau+s),\\ \widehat{w}(x,0,t)=\widehat{w}(x,t),\ \widehat{w}_s(x,0,t)=\widehat{w}_t(x,t), \end{cases} \tag{24}$$

where $(\widehat{w}(x,t),\widehat{w}_t(x,t))$ are determined by (19). Since (24) can be written as

$$\widehat{w}_{ss}(\cdot,s,t)+A\widehat{w}_{xx}(\cdot,s,t)+Bu(t-\tau+s)=0, \tag{25}$$

As the same as (18) the system (24) is well-posed.

Proposition 5. The system (24) is well-posed: for any $t\ge\tau$, $(w_0,w_1)\in H$ and $u\in L^2_{loc}(0,\infty)$, there exists a unique solution to (24) such that $\widehat{w}(\cdot,s,t),\widehat{w}_t(\cdot,s,t))\in C(0,\tau;H)$. Moreover there exists a constant C_t such that

$$\|\widehat{w}(\cdot,s,t),\widehat{w}_s(\cdot,s,t))\|^2\le C_t\left[\|(w_0,w_1)\|^2+\int_0^t|u(t)|^2dt\right].$$

We finally get the estimated state variable by

$$\widetilde{w}(x,t)=\widehat{w}(x,\tau,t),\ \widetilde{w}_t(x,t)=\widehat{w}_s(x,\tau,t),\ t\ge\tau. \tag{26}$$

And we have exponential convergence of the observer.

Theorem 6. For any $t\ge\tau$, we have

$$\left\|\begin{pmatrix} w(\cdot,t)-\widetilde{w}(\cdot,t)\\ w_t(\cdot,t)-\widetilde{w}_t(\cdot,t)\end{pmatrix}\right\|\le Me^{-\omega(t-\tau)}\left\|\begin{pmatrix}\widehat{w}_0-w_0\\ \widehat{w}_1-w_1\end{pmatrix}\right\| \tag{27}$$

where $(\widehat{w}_0,\widehat{w}_1)$ is the initial state of the observer (19), (w_0,w_1) is the initial state of the original system (1), and M,ω are constants in (22).

Proof. Let

$$\varepsilon(x,s,t)=\widehat{w}(x,s,t)-w(x,t-\tau+s). \tag{28}$$

Then $\varepsilon(x,s,t)$ satisfies

$$\begin{cases} \varepsilon_{ss}(x,s,t)-\varepsilon_{xx}(x,s,t)=0,\\ \varepsilon(0,s,t)=0,\ 0\le s\le\tau,\ t\ge\tau,\\ \varepsilon_x(1,s,t)=0,\\ \varepsilon(x,0,t)=\varepsilon(x,t)\varepsilon_s(x,0,t)=\varepsilon_t(x,t) \end{cases} \tag{29}$$

which is a conservative system

$$\|(\varepsilon(\cdot,\tau,t),\varepsilon_s(\cdot,\tau,t))\|=\|(\varepsilon(\cdot,t),\varepsilon_t(\cdot,t))\|. \tag{30}$$

This together with (21) and (22) gives (27). ∎

5. STABILIZATION BY THE ESTIMATED STATE FEEDBACK

Again the $u(t)=-k_2w_t(1,t)$ stabilizes exponentially the system (1). Now, by (26) and Theorem 6, we have the estimated state $\widetilde{w}_t(1,t)$ of $w_t(1,t)$. Naturally, we design the estimated state feedback control law:

$$u^*(t)=\begin{cases} -k_2\widehat{w}_s(1,\tau,t), & t>\tau,\\ 0, & t\in[0,\tau], \end{cases} \tag{31}$$

under which, the closed-loop system becomes a system of partial differential equations (32)-(34):

$$\begin{cases} w_{tt}(x,t)-w_{xx}(x,t)=0, 0<x<1, t>0,\\ w(0,t)=0,\\ w_x(1,t)=u^*(t),\\ w(x,0)=w_0(x),\ w_t(x,0)=w_1(x), \end{cases} \tag{32}$$

$$\begin{cases} \widehat{w}_{tt}(x,t)-\widehat{w}_{xx}(x,t)=0,\ 0<x<1,\ t\ge\tau,\\ \widehat{w}(0,t)=0,\\ \widehat{w}_x(1,t)=-k_2u^*(t-\tau)-k_1\left[\widehat{w}_t(1,t)-y(t)\right],\\ \widehat{w}(x,\tau)=\widehat{w}_0(x),\ \widehat{w}_t(x,\tau)=\widehat{w}_1(x) \end{cases} \tag{33}$$

$$\begin{cases} \widehat{w}_{ss}(x,s,t) - \widehat{w}_{xx}(x,s,t) = 0,\ 0 < s < \tau,\ t \geq \tau, \\ \widehat{w}(0,s,t) = 0, \\ \widehat{w}_x(1,s,t) = -k_2 u^*(t-\tau+s), \\ \widehat{w}(x,0,t) = \widehat{w}(x,t),\ \widehat{w}_s(x,0,t) = \widehat{w}_t(x,t). \end{cases} \tag{34}$$

We consider the closed-loop system (32)-(34) in the state space $X = H \times H \times H$. The system (32)-(34) is equivalent to (35)-(37) for $t \geq \tau$:

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0,\ 0 < x < 1, t \geq \tau, \\ w(0,t) = 0, \\ w_x(1,t) = -k_2 w_t(1,t) - k_2 \varepsilon_s(1,\tau,t), \end{cases} \tag{35}$$

$$\begin{cases} \varepsilon_{tt}(x,t) - \varepsilon_{xx}(x,t) = 0,\ 0 < x < 1,\ t \geq \tau, \\ \varepsilon(0,t) = 0, \\ \varepsilon_x(1,t) = -k_1 \varepsilon_t(1,t), \\ \varepsilon(x,\tau) = \widehat{w}_0(x) - w_0(x), \\ \varepsilon_t(x,\tau) = \widehat{w}_1(x) - w_1(x) \end{cases} \tag{36}$$

$$\begin{cases} \varepsilon_{ss}(x,s,t) - \varepsilon_{xx}(x,s,t) = 0,\ 0 < s < \tau,\ t \geq \tau, \\ \varepsilon(0,s,t) = 0, \\ \varepsilon_x(1,s,t) = 0, \\ \varepsilon(x,0,t) = \varepsilon(x,t),\ \varepsilon_s(x,0,t) = \varepsilon_t(x,t). \end{cases} \tag{37}$$

Theorem 7. Let $k_1 >, k_2 > 0$ and $t \geq \tau$. Then for any $(w_0, w_1) \in H, (\widehat{w}_0, \widehat{w}_1) \in H$, there exists a unique solution to system (35)-(37) such that $(w(\cdot,t), w_t(\cdot,t)) \in C(\tau,\infty;H)$, $(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t)) \in C(\tau,\infty;H)$, $(\varepsilon(\cdot,s,t), \varepsilon_s(\cdot,s,t)) \in C([0,\tau] \times (\tau,\infty);H)$ and for any $t \geq \tau$, there exists a constant $C_t > 0$ such that

$$\left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varepsilon(\cdot,t) \\ \varepsilon_t(\cdot,t) \end{pmatrix} \right\| + \left\| \begin{pmatrix} \varepsilon(\cdot,s,t) \\ \varepsilon_s(\cdot,s,t) \end{pmatrix} \right\| \leq C_t \left[\left\| \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\| + \left\| \begin{pmatrix} \widehat{w}_0 \\ \widehat{w}_1) \end{pmatrix} \right\| \right].$$

Proof. Only some essential ideas are given to outline the proof. First, it is well-known that for any $(w_0, w_1) \in H, (\widehat{w}_0, \widehat{w}_1) \in H$, there is a unique C_0-semigroup solution $(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t)) \in C(\tau,\infty;H)$ to (36) such that (22) holds true. Next, for any given time $t > \tau$, since (37) is a classical one-dimensional wave equation in (x,s), for any $(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t)) \in H$ that is guaranteed by (36), there exists a unique solution to (37) such that

$$\|(\varepsilon(\cdot,s,t), \varepsilon_s(\cdot,s,t))\| = \|(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t))\|,\ \forall\ s \in [0,\tau]. \tag{38}$$

Hence $(\varepsilon(\cdot,s,t), \varepsilon_s(\cdot,s,t)) \in C([0,\tau] \times (\tau,\infty);H)$. Let us now consider the following wave equation:

$$\begin{cases} p_{\xi\xi}(x,\xi) - p_{xx}(x,\xi) = 0,\ 0 < x < 1,\ \xi > t-\tau, \\ p(0,\xi) = 0,\ \xi \geq t-\tau, \\ p_x(1,\xi) = 0,\ \xi \geq t-\tau, \\ p(x,t-\tau) = \varepsilon(x,t),\ p_\xi(x,t-\tau) = \varepsilon_t(x,t). \end{cases} \tag{39}$$

Then we have

$$\varepsilon(x,s,t) = p(x,t-\tau+s),\ x \in [0,1], s \in [0,\tau], t \geq \tau. \tag{40}$$

Hence

$$\varepsilon_s(1,\tau,\xi) = p_\xi(1,\xi),\ \forall\ \xi \geq t-\tau. \tag{41}$$

On the other hand, it follows from the right hand side inequality of (11) that for any $T > t-\tau$,

$$\int_{t-\tau}^T |\varepsilon_s(1,\tau,\xi)|^2 d\xi = \int_{t-\tau}^T |p_\xi(1,\xi)|^2 d\xi \tag{42}$$

$$\leq 2(T-t+\tau+2)\|(\varepsilon(\cdot,t), \varepsilon_t(\cdot,t))\|^2. \tag{43}$$

In particular, setting $t = 2\tau$ leads to

$$\int_\tau^T |\varepsilon_s(1,\tau,t)|^2 dt \leq (T-\tau+2)\|(\varepsilon(\cdot,2\tau), \varepsilon_t(\cdot,2\tau))\|^2. \tag{44}$$

Thirdly, since (35) can be written as

$$w_{tt}(\cdot,t) + Aw(\cdot,t) + k_2 bb^* w_t(\cdot,t) + k_2 b \varepsilon_s(1,\tau,t) = 0 \tag{45}$$

which is well-posed as mentioned in previous section. Hence for any $t > \tau$, there exists a constants $\tilde{C}_t > 0$ such that

$$\|(w(\cdot,t), w_t(\cdot,t)\|^2 \leq \tilde{C}_t \left[\|(w(\cdot,\tau), w_t(\cdot,\tau))\| \right. \tag{46}$$

$$\left. + \int_\tau^t |\varepsilon_s(1,\tau,\xi)|^2 d\xi \right] \tag{47}$$

$$= \tilde{C}_t \left[\|(w_0, w_1)\| + \int_\tau^t |\varepsilon_s(1,\tau,\xi)|^2 d\xi \right]. \tag{48}$$

This together with (38), (44) and (22) gives the required result. ∎

The wave equation system with time delay is exponentially stabilized by the boundary output feedback law which is constituted of a stabilizing static state feedback and a converging observer.

Theorem 8. Let $k_1 > 0$ and $k_2 > 0$. Then there are some positive constants $M > 0$ and $\omega > 0$ such that, for every initial conditions $(w_0, w_1) \in H$ and $(\hat{z}_0, \hat{z}_1) \in H$, the solution of (32)-(34) for the closed-loop system decays exponentially to zero in function of time t, with $t > \tau$ and $0 < s < \tau$:

$$\left\| \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \right\|_H + \left\| \begin{pmatrix} \widehat{w}(\cdot,t) \\ \widehat{w}_t(\cdot,t) \end{pmatrix} \right\|_H + \left\| \begin{pmatrix} \widehat{w}(\cdot,s,t) \\ \widehat{w}_s(\cdot,s,t) \end{pmatrix} \right\|_H$$

$$\leq Me^{-\omega(t-\tau)} \left[\left\| \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \right\|_H + \left\| \begin{pmatrix} \widehat{w}_0 \\ \widehat{w}_1 \end{pmatrix} \right\|_H \right]. \tag{49}$$

Proof. We give only some essential ideas. We prove first exponential convergence of the error $(\hat{z}(x,t) - w(x,t-\tau), \hat{z}_t(x,t) - w_t(x,t-\tau))$ to zero in H. It is obvious from (36). Let the initial condition be sufficiently smooth. We prove, by some elaborated computations, that the time function $\varepsilon_s(1,\tau,t) = (\hat{w}_\xi(1,\tau,t) - w_t(1,t))$ is square summable. From (35) it implies that the state variable $(w(.,t), w_t(.,t))$ is square-summable in H, or it is in $L^2(R^+, H)$. Actually it decays exponentially to zero as function of time t. We have used several facts. In particular we have exponential stability of the underlying semigroups; we have the concatenation properties of the semigroups and the observation function $\varepsilon_s(1,\tau,t)$ with respect to the initial conditions. Due to the page limitation, we omit the detailed computation. ∎

We could modify $K_1 > 0$ and $K_2 > 0$ to adjust the decay rate of the solution for the closed-loop system.

In our future work the design of observers and stabilizing output feedback laws will be carried out for the wave equation with time delay via the approach of transfer functions and compared with the results of the present paper. Another interesting question is to what extent the stabilization feedback proposed in the paper is robust with respect to variations in the time-delay.

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