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Minimum-time Control of a Two-wheeled Differentially Driven Vehicle in the Presence of Slip

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Abstract: In this paper we propose a solution for the minimum-time control of a two-wheeled differentially driven mobile robot in the presence of slip between the wheels and the ground. Starting from the Lagrangian equations of the system a Newton-Eulerian model of the robot is obtained by adding longitudinal and lateral forces between the tyres and the ground, expressed by means of the Pacejka equation. The travelling time of the robot from an initial point to an end point has to be minimised subject to actuator constraints. A Chebyshev series approach is used for the optimisation of the trajectory. Because this optimisation procedure requires a significant computational effort, several trajectories are calculated off line and then they are sampled and used for the training of a neural network controller. This controller is then employed on line in order to make the mobile robot follow the desired path.

1. INTRODUCTION

Time-optimal control is a desired task for mobile robots in many practical applications. Usually, the problem is addressed by first determining the time-optimal trajectory (subject to given kinematic and dynamic constraints) and then by implementing a (feedback) controller to track the trajectory.

Regarding the determination of the time-optimal trajectory, different solutions have been proposed in the literature, depending on the characteristics of the mobile robot. Regarding two-wheeled differentially driven vehicles, assuming an unobstructed environment, relevant works are (Reister and Pin (1994)) and (Renaud and Fourquet (1997)), where constraints on the maximum acceleration of the wheels are posed, and (Balkcom and Mason (2002)) where the maximum velocity is bounded. However, the presence of slip between the wheels and the ground is not considered.

In this paper we address the minimum-time control of a two-wheeled differentially driven mobile robot in the presence of slip. Constraints are posed on the maximum voltage and on the maximum torque applied to the motors. The trajectory optimisation is performed by means of a Chebyshev approach (Fox and Parker (1972); Vlassenbroeck (1988); Vlassenbroeck and Dooren (1988)). However, since this technique is time-consuming it cannot be used on line during the mobile robot motion. For this reason, several optimisations are performed off line. Then, the corresponding state and control variables are sampled and the samples are employed for the training of a neural network. The neural network is eventually used as the actual on-line (state-feedback) controller which drives the robot towards the desired end point in a minimum time.

The paper is organised as follows. In Section 2 the nonlinear model of the mobile robot is presented and the Pacejka formula that expresses the forces between the tyres and the ground is reviewed. The Chebyshev approach is described in Section 3, while the neural network controller is proposed in Section 4. Simulation results are given in Section 5 and conclusions are drawn in Section 6.

2. ROBOT MODEL

A sketch of the considered mobile robot is shown in Figure 1. Starting from the Lagrangian equations of the system, a Newton-Eulerian model of the robot is obtained. The holonomic coordinates are $\mathbf{q}' = [x_p, y_p, \Phi, \theta_l, \theta_r]$ where x_p and y_p are the coordinates of the intersection point P between the wheels axle and the symmetry axis of the robot, Φ is the robot yaw angle and θ_l and θ_r are the rotational angles of the left and right wheel respectively. Note that Φ does not depend on θ_l and θ_r because of the presence of slip. Since the potential energy is zero because the trajectory of the mobile robot is constrained in the horizontal plane, the Lagrangian $\mathcal{L}$ is equal to the total kinetic energy $\mathcal{K}$, i.e.,

$$\begin{aligned}\mathcal{K} = \; & \frac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2) + m_c L\dot{\Phi}(\dot{y}_p\cos(\Phi) - \dot{x}_p \\ & \cdot\sin(\Phi)) + \frac{1}{2}J_{tot}\dot{\Phi}^2 + \frac{1}{2}J_{wy}(\dot{\theta}_l^2 + \dot{\theta}_r^2)\end{aligned} \quad (1)$$

where J_{tot} is the total inertia referred to point P, J_{wy} is the inertia of the wheel with respect to the y axis, m is the total mass of the robot and m_c is the mass of the chassis. The Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{\mathbf{q}}'}\right) - \frac{\delta\mathcal{L}}{\delta\mathbf{q}'} = \mathbf{f_e} \quad (2)$$

where $\mathbf{f_e}$ is the vector of external forces. The solution is a system of five second-order differential equations:

$$\begin{cases} m\ddot{x}_p - m_c L(\ddot{\Phi}\sin(\Phi) + \dot{\Phi}^2\cos(\Phi)) = F_x \\ m\ddot{y}_p + m_c L(\ddot{\Phi}\cos(\Phi) - \dot{\Phi}^2\sin(\Phi)) = F_y \\ m_c L(\ddot{y}_p\cos(\Phi) - \ddot{x}_p\sin(\Phi)) + J_{tot}\ddot{\Phi} = F_\Phi \\ J_{wy}\ddot{\theta}_l = \bar{\tau}_l - F_{xl} r \\ J_{wy}\ddot{\theta}_r = \bar{\tau}_r - F_{xr} r \end{cases} \quad (3)$$

This system can be expanded into a system of ten first-order differential equations. In this model, $\bar{\tau}_l$ and $\bar{\tau}_r$ are the (constrained) torques applied to the left and right wheel respectively. Assuming that the two wheels are actuated by two voltage-piloted DC motors, we have two more differential equations that express the relationship between the torques and the voltages V_l and V_r applied to the motors, which are the actual (constrained) inputs of the system:

$$\begin{aligned} \dot{\bar{\tau}}_l(t) &= V_l(t)\frac{K}{L_a} - \bar{\tau}_l(t)\frac{R_a}{L_a} - \omega_l(t)\frac{K^2}{L_a} \\ \dot{\bar{\tau}}_r(t) &= V_r(t)\frac{K}{L_a} - \bar{\tau}_r(t)\frac{R_a}{L_a} - \omega_l(t)\frac{K^2}{L_a} \end{aligned} \quad (4)$$

where K is the DC motor constant and L_a and R_a are respectively the inductance and the resistance of the motor, which is of the same kind for the two wheels. By considering also equations (4), the overall model of the robot consists of a system of $n = 12$ first-order differential equations.

In (3), F_{xl} and F_{xr} are the longitudinal forces generated in the contact between the ground and the left and right tyre respectively. F_x, F_y and F_Φ are the generalised forces acting in the first three degrees of freedom and can be written as:

$$\begin{aligned} F_x &= (F_{xl} + F_{xr})\cos(\Phi) - (F_{yl} + F_{yr})\sin(\Phi) \\ F_y &= (F_{xl} + F_{xr})\sin(\Phi) + (F_{yl} + F_{yr})\cos(\Phi) \\ F_\Phi &= (F_{xr} - F_{xl})\, b \end{aligned} \quad (5)$$

where F_{yl} and F_{yr} are the lateral forces generated in the contact between the ground and the left and right tyre respectively.

In order to express the real behavior of the forces that a wheel exchanges with the ground, the Pacejka equation has been employed (Pacejka (2002)). These forces rise during the contact between the tyre and the surface. The longitudinal force F_x has the same direction as the forward speed of the wheel while the lateral force F_y is orthogonal to the previous one. By means of the Pacejka equation, the longitudinal forces can be expressed as a function of the slip ratio κ and lateral forces as a function of the slip angle ϑ. The slip ratio can be written as:

$$\kappa = -\frac{v_{Sx_w}}{v_{Hx}} \quad (6)$$

where $v_{w_{Sx}}$ is the forward speed of point S (the point of contact between tyre and surface) and v_{Hx} is the forward speed of point H (wheel center). Note that $v_{w_{Sx}}$ is obtained from the following equation of the rigid bodies:

$$v_{Sx_w} = v_{Hx} + \omega \times (S - P) \quad (7)$$

where $\times$ denotes the vectorial product and ω is the wheel angular velocity. The slip angle can be written as:

$$\vartheta = \arctan\left(-\frac{v_{Sy_w}}{v_{Hx}}\right) \quad (8)$$

where v_{Sy_w} is the lateral speed of point S.

The expression of the general Pacejka equation is:

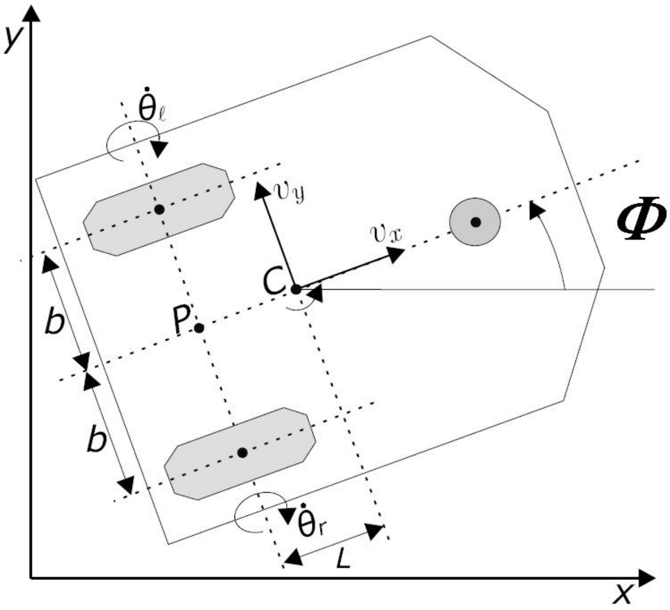

Fig. 1. Sketch of the two-wheeled differentially driven robot.

$$F(\xi) = c_1 \sin[c_2 \arctan(c_3\xi - c_4(c_3\xi - \arctan(c_3\xi)))] \quad (9)$$

where F is the longitudinal or lateral force of each wheel (F_{xl}, F_{xr}, F_{yl} or F_{yr}) and the argument ξ is the slip ratio κ or slip angle ϑ of each wheel. A general way to calculate this equation can be determined once coefficients c_i, $i = 1, \ldots, 4$ are determined after several tests on the tyre with certain initial conditions. The different shapes of Pacejka curves depend on the kind of ground. For example, asphalt, according to the weather condition, can be high grip, normal, rainy or icy.

3. CHEBYSHEV OPTIMISATION

3.1 Problem formulation

Consider the (nonlinear) model of the system in state-space form:

$$\frac{d\mathbf{x}(t)}{dt} = f(\mathbf{x}(t)\mathbf{u}(t)) \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{u} \in \mathbb{R}^m \quad (10)$$

where

$$\mathbf{x} = [x_p\ \dot{x}_p\ y_p\ \dot{y}_p\ \Phi\ \dot{\Phi}\ \theta_l\ \omega_l\ \theta_r\ \omega_r\ \bar{\tau}_l\ \bar{\tau}_r]^T \quad (11)$$

and

$$\mathbf{u} = [V_l\ V_r]^T \quad (12)$$

(thus, $n = 12$ and $m = 2$) and consider a transition from an initial state $\mathbf{x}(t_0) := \mathbf{x}_0$ to a final state $\mathbf{x}(t_f) := \mathbf{x}_f$. The time-optimal control problem consists in finding $\mathbf{u}(t) := [V_l(t)\ V_r(t)]$, $t \in [0, t_f]$ in order to minimise the transition time from the initial to the final equilibrium state subject to constraints on the maximum voltage and on the maximum torque to be applied to the motors of the wheels, namely, the posed optimal control problem can be rewritten compactly as

$$\min t_f \quad (13)$$

subject to

$$\frac{d\mathbf{x}(t)}{dt} = f(\mathbf{x}(t)\mathbf{u}(t)) \quad 0 \le t \le t_f, \quad (14)$$

$$\mathbf{x}(0) = \mathbf{x_0}, \quad \mathbf{x}(t_f) = \mathbf{x_f} \tag{15}$$

$$u_{min} \leq \mathbf{u}(t) \leq u_{max} \quad 0 \leq t \leq t_f \tag{16}$$

$$\bar{\tau}_{min} \leq [\bar{\tau}_l \ \bar{\tau}_r] \leq \bar{\tau}_{max} \quad 0 \leq t \leq t_f. \tag{17}$$

Note that not all the state elements have to be assigned at $t = t_f$. For example, the final yaw angle Φ can be left free (see Section 5).

This boundary value problem with differential-algebraic equations (BVP-DAE) can be solved with a numerical technique which consists in expanding the state and control variables in the Chebyshev series. This allows to convert the boundary conditions into algebraic equations in the unknown coefficients. In this way, the optimal control problem is replaced by a parameter optimisation problem which consists in the minimisation of the performance index subject to algebraic constraints (Vlassenbroeck and Dooren (1988)).

3.2 Chebyshev polynomials

The Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined as the solutions to the Chebyshev differential equation and denoted $T_i(\tau)$. They are normalised such that $T_i(1) = 1$, $i = 0, 1, \ldots$ and in their trigonometric form they are expressed as:

$$T_i(\tau) = \cos(i \cdot \arccos(\tau)) \qquad \tau \in [-1, 1]. \tag{18}$$

They can be also defined by the recurrence relation:

$$\begin{aligned} T_0(\tau) &= 1 \\ T_1(\tau) &= \tau \\ T_{i+1}(\tau) &= 2\tau T_i(\tau) - T_{i-1}(\tau) \quad i > 1. \end{aligned} \tag{19}$$

In order to use the Chebyshev polynomials of the first kind for the approximation of the system dynamics, the following time transformation is therefore necessary:

$$t = \frac{t_f}{2}(1 + \tau). \tag{20}$$

This transformation allows a change from the time domain $t \in [0, t_f]$ to the Chebyshev domain $\tau \in [-1, 1]$. The new system dynamics expressed in the Chebyshev domain is therefore:

$$\frac{d\mathbf{x}(\tau)}{d\tau} = f(\mathbf{x}(\tau; t_f)\mathbf{u}(\tau; t_f)) \qquad -1 \leq \tau \leq 1 \tag{21}$$

with initial and final conditions:

$$\mathbf{x}(-1) = \mathbf{x_0}, \qquad \mathbf{x}(1) = \mathbf{x_f}. \tag{22}$$

3.3 Approximation through Chebyshev series

Once the system dynamics has been rewritten in the Chebyshev domain, the next step is the expansion of both the state $\mathbf{x}$ and the input $\mathbf{u}$ through Chebyshev series of order h:

$$\mathbf{x_h}(\tau; t_f) = \frac{1}{2}\boldsymbol{\alpha_0} T_0(\tau; t_f) + \sum_{i=1}^{h} \boldsymbol{\alpha_i} T_i(\tau; t_f) \tag{23}$$

$$\mathbf{u_h}(\tau; t_f) = \frac{1}{2}\boldsymbol{\beta_0} T_0(\tau; t_f) + \sum_{i=1}^{h} \boldsymbol{\beta_i} T_i(\tau; t_f) \tag{24}$$

where $\tau \in [-1, 1]$ and $\bar{\boldsymbol{\alpha}} := [\boldsymbol{\alpha_0}, \boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_h}]$ (with $\boldsymbol{\alpha_i} = [\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}]^T$, $i = 0, \ldots, h$) and $\bar{\boldsymbol{\beta}} := [\boldsymbol{\beta_0}, \boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_h}]$ (with $\boldsymbol{\beta_i} = [\beta_{i1}, \beta_{i2}]^T$, $i = 0, \ldots, h$) are the unknown coefficients. The same order h has been assumed for both the state and the input for the sake of simplicity. The choice of h is related to the required accuracy. Actually, increasing its value yields to a better approximation (indeed, in principle, for h that tends to infinity, the approximation tends to the exact solution), but, from another point of view, increases also the complexity of the optimisation problem. The value of $h = 30$ has therefore been selected to provide a high accuracy with a reasonable computational effort.

3.4 Equality and inequality constraints

The approximation of the state and of the input variables allows to approximate the system dynamics as follows (Vlassenbroeck and Dooren (1988)):

$$\frac{d\mathbf{x_h}(\tau)}{d\tau} = f_h(\mathbf{x_h}(\tau; t_f), \mathbf{u_h}(\tau; t_f)) \qquad -1 \leq \tau \leq 1. \tag{25}$$

where

$$\begin{aligned} f_h(\mathbf{x_h}(\tau; t_f), \mathbf{u_h}(\tau; t_f)) = &\frac{1}{2} A_0(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}; t_f) T_0(\tau; t_f) \\ &+ \sum_{j=1}^{h-1} A_j(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}; t_f) T_j(\tau; t_f) \end{aligned} \tag{26}$$

where

$$\begin{aligned} &A_j(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}; t_f) = \\ &\frac{2}{K} \sum_{i=1}^{K} f(\mathbf{x_h}(\cos(\theta_i); t_f), \mathbf{u_h}(\cos(\theta_i); t_f)) \cos j\theta_i \end{aligned} \tag{27}$$

with $j = 0, \ldots, h-1$ and $K > h - 1$ and

$$\theta_i = \frac{2i - 1}{K} \frac{\pi}{2}. \tag{28}$$

Thus, we obtain that (26) is a system of n polynomials of order $h - 1$.

The left hand side of equation (25) can be derived by considering that the derivative of the series (23) with respect to τ is given by

$$\frac{1}{2}\boldsymbol{\alpha'_0} + \sum_{i=1}^{h-1} \boldsymbol{\alpha'_i} T_i(\tau; t_f) \tag{29}$$

where the coefficients $\bar{\boldsymbol{\alpha}}' := [\boldsymbol{\alpha'_0}, \boldsymbol{\alpha'_1}, \ldots, \boldsymbol{\alpha'_{h-1}}]$ can be expressed in terms of the coefficients $\boldsymbol{\alpha}$ by means of the following formula (Fox and Parker (1972)):

$$\boldsymbol{\alpha'_{r-1}} - \boldsymbol{\alpha'_{r+1}} - 2r\boldsymbol{\alpha_r} = 0, \quad r = 1, \ldots, h-1. \tag{30}$$

By equating the coefficients of same-order Chebyshev polynomials in (25) we obtain a system of $n \times h$ nonlinear equality constraints.

The substitution of $\mathbf{x_h}$ into the initial and final condition expression (22) yields at most to $2 \times n$ additional equality constraints, namely, (Vlassenbroeck and Dooren (1988))

$$\frac{1}{2}\boldsymbol{\alpha_0} + \sum_{i=1}^{h} (-1)^i \boldsymbol{\alpha_i} - \mathbf{x}(-1) = 0 \tag{31}$$

and

$$\frac{1}{2}\boldsymbol{\alpha_0} + \sum_{i=1}^{h} \boldsymbol{\alpha_i} - \mathbf{x}(1) = 0. \tag{32}$$

The inequality constraints (16) can be handled by rewriting them as

$$\begin{array}{l} u_{min} - u_{h1}(\tau; t_f) \leq 0 \\ u_{min} - u_{h2}(\tau; t_f) \leq 0 \\ u_{h1}(\tau) - u_{max} \leq 0 \\ u_{h2}(\tau) - u_{max} \leq 0 \end{array} \quad (33)$$

and by defining a vector $\mathbf{w_{14}}(\tau) = [w_1(\tau), \ldots, w_4(\tau)]^T$ of slack variables in order to rewrite them subsequently as

$$\begin{array}{l} u_{min} - u_{h,1}(\tau; t_f) = -w_1^2(\tau) \\ u_{min} - u_{h,2}(\tau; t_f) = -w_2^2(\tau) \\ u_{h,1}(\tau; t_f) - u_{max} = -w_3^2(\tau) \\ u_{h,2}(\tau; t_f) - u_{max} = -w_4^2(\tau) \end{array} . \quad (34)$$

Similarly, the inequality constraints (17) can be handled by rewriting them as

$$\begin{array}{l} \bar{\tau}_{min} - x_{h,11}(\tau; t_f) \leq 0 \\ \bar{\tau}_{min} - x_{h,12}(\tau; t_f) \leq 0 \\ x_{h,11}(\tau; t_f) - \bar{\tau}_{max} \leq 0 \\ x_{h,12}(\tau; t_f) - \bar{\tau}_{max} \leq 0 \end{array} \quad (35)$$

and by defining a vector $\mathbf{w_{58}}(\tau) = [w_5(\tau), \ldots, w_8(\tau)]^T$ of slack variables in order to rewrite them subsequently as

$$\begin{array}{l} \bar{\tau}_{min} - x_{h,11}(\tau; t_f) - w_5^2(\tau) \\ \bar{\tau}_{min} - x_{h,12}(\tau; t_f) - w_6^2(\tau) \\ x_{h,11}(\tau; t_f) - \bar{\tau}_{max} - w_7^2(\tau) \\ x_{h,12}(\tau; t_f) - \bar{\tau}_{max} - w_8^2(\tau) \end{array} . \quad (36)$$

At this point each $w_i(\tau)$, $i = 1, \ldots, 8$ can be expanded in a Chebyshev series with unknown coefficients $\bar{\boldsymbol{\gamma}}$ and by equating again the coefficients of same-order Chebyshev polynomials, a set of (nonlinear) equality constraint relations in $\bar{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}$ and $\bar{\boldsymbol{\gamma}}$ is obtained. In this way an optimisation problem with only (nonlinear) equality constraints is obtained.
Alternatively, the Chebyshev series in (33) and (35) can be evaluated at a number of points τ_i, $-1 = \tau_0 < \tau_1 < \cdots < \tau_k = 1$, so that a set of inequality constraint relations in $\bar{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ results. Although this approach is less rigorous, we preferred to use it because, overall, it requires a less computational effort.

3.5 Optimisation

By following the steps described before, the optimal control problem (13)-(17) is therefore transformed into a parameter optimisation problem which consists in finding the optimal values t_f^*, $\bar{\boldsymbol{\alpha}}^*$ and $\boldsymbol{\beta}^*$ in order to minimize the transition time t_f subject to the posed equality and inequality constraints. The minimum-time feedforward control law is then obtained by employing expression (24).
To solve this optimisation problem, a sequential quadratic programming (SQP) method, such as the one implemented in the function "`fmincon`" of Matlab can be used (Matlab (2006)).
In this context the starting values of the parameters t_f, $\bar{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$, denoted respectively as t_f^0, $\bar{\boldsymbol{\alpha}}^0$ and $\boldsymbol{\beta}^0$ can be selected by considering a control law $\mathbf{u}(t)$ determined without minimising any objective function and by determining a Chebyshev interpolation of the state variables evolution and of the input of the system (Quarteroni and Valli (1997)).
Note that the optimisation algorithm can be made faster by providing the explicit expression of the gradient of both the equality and inequality constraints with respect to t_f, $\bar{\boldsymbol{\alpha}}$ and $\bar{\boldsymbol{\beta}}$.

m	0.5	$[kg]$
m_c	0.47	$[kg]$
J_{tot}	$1.0 \cdot 10^{-3}$	$[kg \cdot m^2]$
J_m	$1.5 \cdot 10^{-5}$	$[kg \cdot m^2]$
J_{wy}	$1.5 \cdot 10^{-4}$	$[kg \cdot m^2]$
r	0.02	$[m]$
b	0.05	$[m]$
L	0.04	$[m]$
K	0.24	$[Nm/A]$
L_a	$1.8 \cdot 10^{-3}$	$[H]$
R_a	26.6	$[\Omega]$
u_{min}	-9	$[V]$
u_{max}	$+9$	$[V]$
$\bar{\tau}_{min}$	-0.08	$[Nm]$
$\bar{\tau}_{max}$	$+0.08$	$[Nm]$

Table 1. Parameters of the robot model used in the simulation.

	c_1	c_2	c_3	c_4
F_x	1.31	1.65	0.85	0.48
F_y	1.16	1.30	2.22	0.71

Table 2. Values of the coefficients of the Pacejka equation used in the simulation.

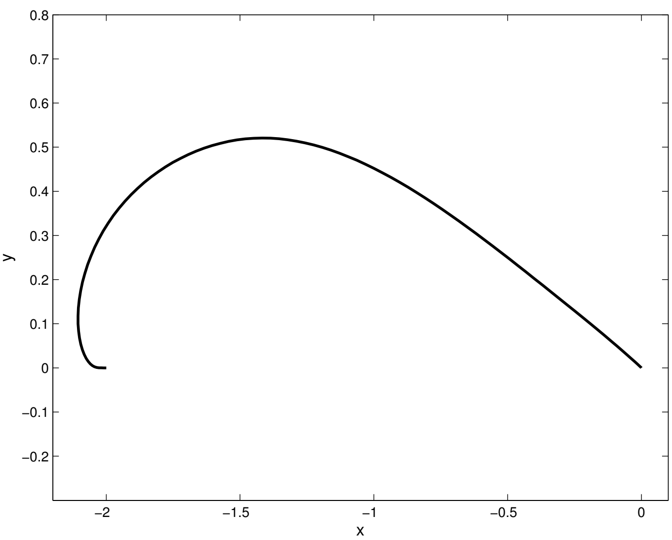

Fig. 2. Example of a trajectory of the robot after Chebyshev optimisation.

3.6 Illustrative Result

In order to show the effectiveness of the proposed approach, an illustrative result is shown. The values of the employed robot model parameters and of employed values of the constraints are shown in Table 1, while the coefficients of the Pacejka equation are shown in Table 2.
The robot initial position is $x_0 = -2$ and $y_0 = 0$, with a yaw angle $\Phi_0 = \pi$ and with an angular velocity of the wheels $\omega_{l0} = 0$, $\omega_{r0} = 0$. The target final position is $x_f = 0$ and $y_f = 0$ with null velocity. The determined trajectory of the robot and the determined torques and voltages applied to the two wheels are shown in Figures 2 and 3 respectively ($t_f^* = 4.65$ s). It can be seen that the robot is capable to reach the target position in a minimum time with high precision and without violating any posed constraint.

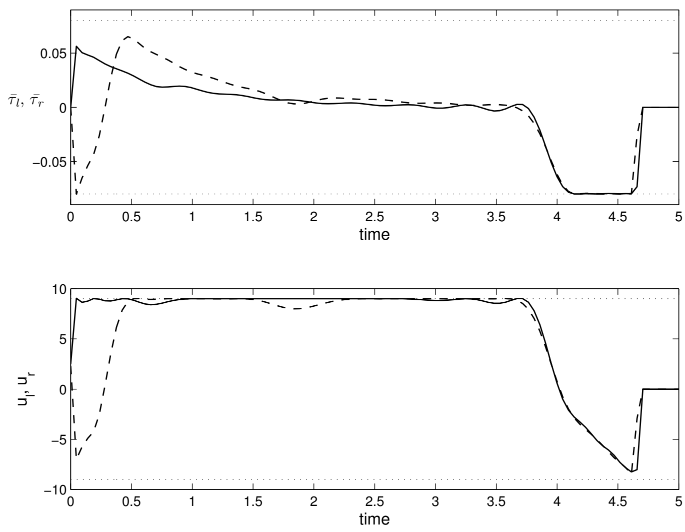

Fig. 3. Torques and voltages applied to the motor in the example of the Chebyshev optimisation. Solid line: left wheel; dashed line: right wheel.

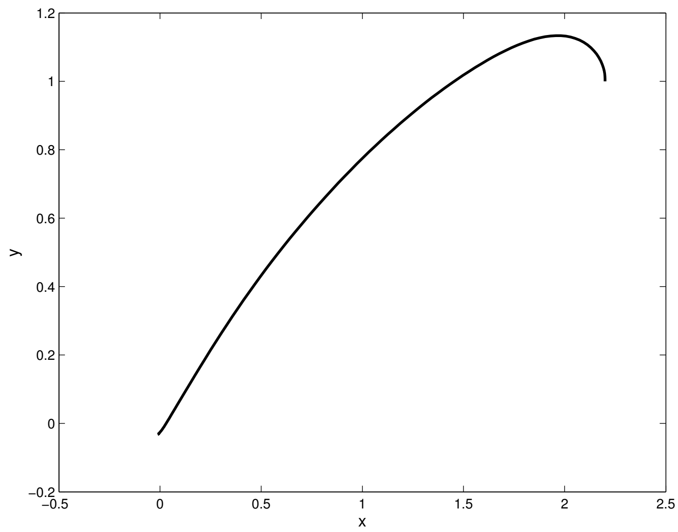

Fig. 4. Trajectory of the robot with the NN controller (first example).

4. NEURAL NETWORK CONTROLLER

It has been shown in Section 3 that the Chebyshev approach allows to find the minimum-time robot trajectory effectively. However, the computational effort required for the optimisation is significant and this prevents the use of the technique for on-line control.

To solve the problem of real-time control a standard feedforward neural network (NN) based on sigmoidal functions was implemented as a feedback controller. The neural network has four inputs $[x\ \Phi\ \omega_l\ \omega_r]$ and two outputs (one for each voltage to be applied to the two motors of the wheels). The hidden layer has sixteen neurons. Thus, for each state the neural network controller gives the voltage for a sampling interval.

The neural network is trained (by means of the backpropagation algorithm) with data that are collected from several Chebyshev optimisations, all with different initial state. To reduce the amount of training data and therefore the number of optimisations, symmetry can be exploited to eliminate redundant data. First, the trajectory can be translated so that the target position is in the origin, then by rotating and translating the initial state to the negative x-axis we only have to train the initial states with the initial position on the negative x-axis. In other words, we can always set $y = 0$ by accordingly modifying the value of Φ (note that y is not an input of the neural network). Furthermore, negative values of the robot yaw angle Φ can be mirrored about the x-axis. Finally, it appeared from the optimisations that the robot attains its maximum speed (which corresponds to the maximum voltages on both wheels) starting from point $x = -2$ and $y = 0$. Thus, trajectories starting more distant from $x = -2$ differ from that starting from $x = -2$ only for the length of the part of trajectory covered with maximum speed and therefore it is not necessary to consider them for the training of the neural network. Summarising, it is necessary to optimise and train the neural network for values with $0 < x_0 \leq 2, 0 \leq \Phi_0 \leq \pi, 0 \leq [\omega_{l0}, \omega_{r0}] \leq \omega_{max}$.

5. SIMULATION RESULTS

Simulation results of the closed loop neural network controller with different initial positions are shown hereafter. As already mentioned, the employment of techniques of mirroring and rotating states during the on-line control allows to extend the control to any initial state. For all the considered experiments, the model parameters and the employed constraints are those shown in Tables 1 and 2. Further, the target position is always $x = 0$ and $y = 0$ with null velocities, while the yaw angle is free.

In Figure 4 and 5 simulation results for the case where the initial position is $x_0 = 2.2$, $y_0 = 2$, the initial yaw angle is $\Phi_0 = \pi/2$ and the initial angular velocities are $\omega_{l0} = 5$ and $\omega_{r0} = 10$ are given. The robot attains its final target state at time $t = 5.9$ s. In order to confirm the effectiveness of the new control strategy another example is described in Figures 6 and 7. In this case the initial state is $x_0 = -1.8$, $y_0 = -0.2$, the initial yaw angle is $\Phi_0 = \pi$ and the initial angular velocities are $\omega_{l0} = 15$ and $\omega_{r0} = 15$. The robot attains its target final state at time $t = 5.8$ s. It has to be noted that, obviously, the neural network feedback controller approximates the minimum-time control law, as it can be deduced from the presented results, where the torque signals slightly exceed the posed constraints for a small time interval (see Figures 5 and 7). However, the approximations are negligible from a practical point of view and it can be concluded that the neural network feedback controller is capable to implement the minimum-time control law with a good accuracy.

6. CONCLUSIONS

In this paper we have proposed a solution for the minimum-time control of a two-wheeled differentially driven mobile robot in the presence of slip between the wheels and the ground. The solution is based on determining the minimum-time trajectories by means of a Chebyshev approach and by subsequently implementing on line a neural network feedback controller that has been trained with the data resulting from the optimisation

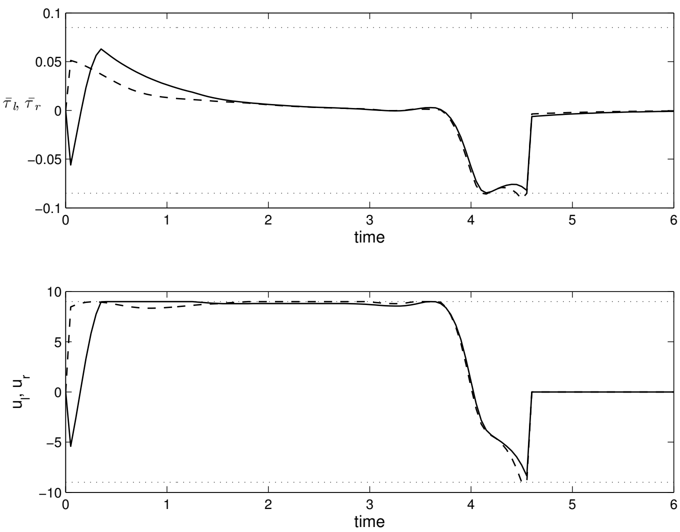

Fig. 5. Torques and voltages applied to the motor by the NN controller (first example). Solid line: left wheel; dashed line: right wheel.

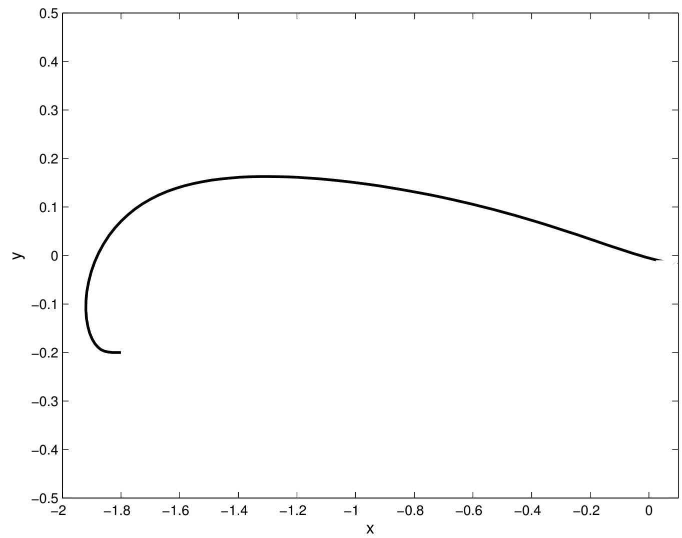

Fig. 6. Trajectory of the robot with the NN controller (second example).

procedure. Simulation results have shown the effectiveness of the methodology.

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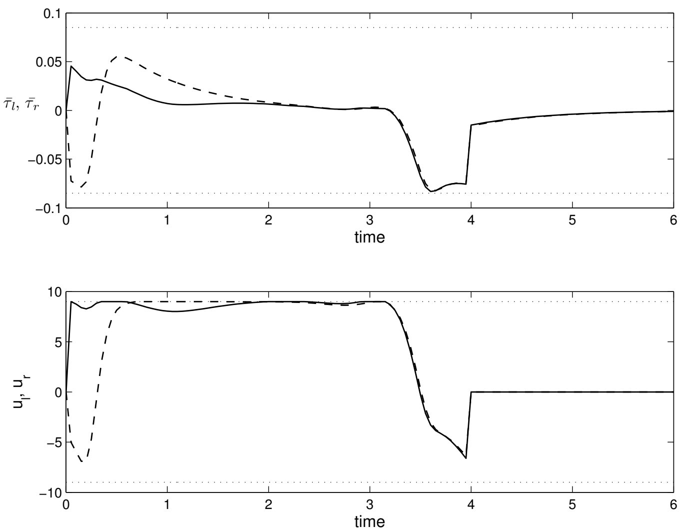

Fig. 7. Torques and voltages applied to the motor by the NN controller (second example). Solid line: left wheel; dashed line: right wheel.

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