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Stability of Discrete Impulsive Hybrid Systems via Comparison Principle ⋆

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Abstract: This paper studies discrete impulsive hybrid systems. The comparison principle and uniform stability are established for such hybrid systems. Moreover, the attraction region is estimated. As applications, the comparison principle is used to study the robust stability problem for linear interval discrete impulsive hybrid systems and a class of nonlinear uncertain discrete impulsive hybrid systems.

1. INTRODUCTION

It is now recognized that the theory of impulsive hybrid systems provides a natural framework for mathematical modelling of many real world phenomena. Impulses can not only lead to the failure of stability for a stable continuous system, but also be used to stabilize an unstable system. It is, therefore, very important to investigate the stability problem for impulsive hybrid systems.

In recent years, significant progress has been made in the stability and robust stability theory of impulsive hybrid systems, in which the impulses occur in a continuous systems at some instances, see Lakshmikantham et al. (1989), Michel (1999), Michel et al. (1995), Ye et al. (1998), Li et al. (2000), Li et al. (2001a) and (2001b), Li et al. (2002), Liu et al. (1994), Liu et al. (2001), Liu et al. (2006), Li et al. (2003), Guan et al. (2005), Zhang et al. (2005), and Liu et al. (2003)-(2006). However, the corresponding theory for discrete impulsive hybrid systems, in which the impulses occur in a discrete system at some instances, has not been fully developed. More recently, in Liu et al. (2007a)-(2007b), the robust stability and ISS (input-to-state stability) property for discrete impulsive hybrid systems has been investigated. In this paper, we will analyze the stability property for this kind of systems via comparison approach.

Among the methods contributed to the study of the stability problem for dynamical systems, the comparison principle is an interesting and efficient method. The stability of the original system can be derived by comparing to a simpler system, with known stability properties. The comparison principle method has been applied successfully to study of stability for continuous systems, impulsive systems and switched systems, see Lakshmikantham et al. (1989), Isidori (1999), Phat (2005), Zhang et al. (2001), Liao (2001), Yang et al. (1997), and Chatterjee et al. (2006).

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In this paper, we shall establish the comparison principle for discrete impulsive hybrid systems. Then, the comparison principle is used to investigate the uniform stability properties of discrete impulsive systems. As applications, the comparison principle is used to study the robust stability problem for linear interval discrete impulsive hybrid systems and a class of nonlinear uncertain discrete impulsive hybrid systems.

2. PRELIMINARIES

Let R^n denote the n-dimensional real vector space and $||A||$ the norm of a matrix A induced by the Euclidean norm, i.e., $||A|| = [\lambda_{\max}(A^T A)]^{\frac{1}{2}}$. Let N denote the set of nonnegative integers, i.e., $N = \{0, 1, 2, \cdots\}$, and $R_+ = [0, +\infty)$. Let $\lambda_{\max}(X)$ (respectively, $\lambda_{\min}(X)$) the maximum (respectively, minimum) eigenvalue of the matrix X.

For $A = (a_{ij})_{n\times m}$, denote: $|A| = (|a_{ij}|)_{n\times m}$, and $A \geq 0$ if and only if $a_{ij} \geq 0$ for all $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. Let $x = (x_1, x_2, \cdots, x_n)^T, y = (y_1, y_2, \cdots, y_n)^T \in R^n$, $x \leq y$ if and only if $x_i \leq y_i, i = 1, 2, \cdots, n$.

A function $\gamma : R_+ \to R_+$ is of class-$\mathcal{K}$ ($\gamma \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. A vector function $l(r) = (l_1(r), \cdots, l_m(r))^T : R_+ \to R_+^m$ is of class-$\mathcal{K}_m\mathcal{R}$ ($l \in \mathcal{K}_m\mathcal{R}$) if $l(r) \in C[R_+, R_+^m]$, $l_i(0) = 0$, $l_i(r) > 0$, $r > 0$, and $l_i(r) \to \infty$, when $r \to +\infty$, $i = 1, 2, \cdots, m$.

Consider the following discrete impulsive hybrid systems:

$$S_1 : \quad \begin{cases} x(k+1) = f_c(k, x(k)),\ k \neq N_i, \\ \Delta x(k+1) = f_d(k, x(k)),\ k = N_i, \\ x(k_0) = x_0,\ k \in N, k \geq k_0, \end{cases} \tag{1}$$

$$S_2 : \quad \begin{cases} w(k+1) = g_c(k, w(k)),\ k \neq N_i, \\ \Delta w(k+1) = g_d(k, w(k)),\ k = N_i, \\ w(k_0) = w_0,\ k \in N, k \geq k_0, \end{cases} \tag{2}$$

$$S_3 : \quad \begin{cases} r(k+1) = h_c(k, r(k)),\ k \neq N_i, \\ \Delta r(k+1) = h_d(k, r(k)),\ k = N_i, \\ r(k_0) = r_0,\ k \in N, k \geq k_0, \end{cases} \tag{3}$$

where $x \in R^n$, $w \in R_+^m$, $r \in R_+$; $\Delta x(k+1) = x(k+1) - x(k)$, $\Delta w(k+1) = w(k+1) - w(k)$, $\Delta r(k+1) = r(k+1) - r(k)$; and the following assumptions are satisfied:

(A_1): The sequence $\{N_i\}$ satisfies: $N_i \in N$ and $k_0 \leq N_0 < N_1 < \cdots < N_i < \cdots$, with $N_{i+1} - N_i > 1, i \in N$.

(A_2): $f_c, f_d \in C[N_+ \times R^n, R^n]$, $g_c, g_d \in C[N_+ \times R_+^m, R_+^m]$, $h_c, h_d \in C[N_+ \times R_+, R_+]$;

(A_3): Every solution of systems $S_1 - S_3$ exists globally and uniquely on N, respectively.

Let $x(k) \triangleq x(k, k_0, x_0)$, $w(k) \triangleq w(k, k_0, w_0)$, $r(k) \triangleq r(k, k_0, r_0)$ be the solution of systems $S_1 - S_3$ with initial condition $x(k_0) = x_0$. We give the following standard definitions.

Definition 2.1. System S_1 is said to be uniformly stable (US) if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, such that when $\|x_0\| \leq \delta$, the following inequality holds:

$$\|x(k, k_0, x_0)\| < \epsilon, \quad k \geq k_0, k \in N. \tag{4}$$

Definition 2.2. System S_1 is said to be uniformly asymptotically stable (UAS) if it is US, and moreover the following equality holds:

$$\lim_{k\to\infty} \|x(k, k_0, x_0)\| = 0. \tag{5}$$

Definition 2.3. System S_1 is said to be uniformly exponentially stable (UES) if there exist positive constants $\alpha > 0, K \geq 1$ such that

$$\|x(k)\| \leq K\|x_0\|e^{-\alpha(k-k_0)}, \quad k \geq k_0, k \in N. \tag{6}$$

Definition 2.4. A set $D(x) \subseteq R^n$ is called an attractive region of system S_1 if, for any $x_0 \in D(x)$, the solution $x(k, k_0, x_0)$ of system S_1 satisfies (5).

Lemma 2.1. (Liu et al. (2004)) Let $X \in R^{n\times n}$ be a positive definite matrix and $Q \in R^{n\times n}$ a symmetric matrix. Then for any $x \in R^n$, the following inequality holds

$$\lambda_{\min}(X^{-1}Q) \cdot x^T Xx \leq x^T Qx \leq \lambda_{\max}(X^{-1}Q) \cdot x^T Xx \tag{7}$$

Lemma 2.2. (Liu et al. (2004)) For any $A \in N[P, Q]$, where $N[P, Q] = \{A = (a_{ij}) \in R^{n\times n} : p_{ij} \leq a_{ij} \leq q_{ij},\ i, j = 1, 2, \cdots, n.\}$, then A can be formulated as follows:

$$A = A_0 + E\Sigma F, \tag{8}$$

where $A_0 = \frac{1}{2}(P + Q), H = (h_{ij}) = \frac{1}{2}(Q - P)$, $\Sigma \in \Sigma^* = \{\Sigma \in R^{n^2\times n^2} : \Sigma = \mathrm{diag}(\epsilon_{11}, \cdots, \epsilon_{nn}), |\epsilon_{ij}| \leq 1; i, j = 1, 2, \cdots, n.\}$, $EE^T = \mathrm{diag}\{\sum_{j=1}^n h_{1j}, \cdots, \sum_{j=1}^n h_{nj}\}$, and $F^T F = \mathrm{diag}\{\sum_{j=1}^n h_{j1}, \cdots, \sum_{j=1}^n h_{jn}\}$.

3. COMPARISON PRINCIPLE AND STABILITY

In this section, we shall establish the comparison principle and stability criteria for discrete impulsive hybrid systems.

Theorem 3.1. Assume that functions $g_c(k, \upsilon), g_d(k, \upsilon)$ are nondecreasing with respect to υ for any $k \in N$, and furthermore suppose that there are functions $V(k, x) \in C[N \times R^n, R_+^m]$ and $l(r) \in \mathcal{K}_m\mathcal{R}$, with $l(a+b) \geq l(a)+l(b)$ for any $a, b \in R_+$, such that the following conditions hold:

(i) for any $k \neq N_i$, then

$$V(k + 1, x(k + 1)) \leq g_c(k, V(k, x(k))); \tag{9}$$

(ii) for $k = N_i$, then

$$\Delta V(k + 1, x(k + 1)) \leq g_d(k, V(k, x(k))), \tag{10}$$

where $\Delta V(k+1, x(k+1)) = V(k+1, x(k+1))-V(k, x(k))$;

(iii) for any $k \neq N_i$, then

$$g_c(k, l(r)) \leq l(h_c(k, r)), \quad r \in [0, r^*], \tag{11}$$

where r^* is some positive constant or $r^* = +\infty$;

(iv) for any $k = N_i$, then

$$g_d(k, l(r)) \leq l(h_d(k, r)), \quad r \in [0, r^*], \tag{12}$$

(v) for any $(k_0, r_0) \in N \times [0, r^*]$, then

$$r(k, k_0, r_0) \in [0, r^*], \quad k \in N. \tag{13}$$

Then, $V(k_0, x_0) \leq w_0 \leq l(r_0)$ implies that

$$V(k, x(k)) \leq w(k) \leq l(r(k)), \quad k \geq k_0, k \in N. \tag{14}$$

Proof. We prove (14) by using induction on k:

When $k = 0$, then (14) holds obviously. Now we assume that (14) holds for the case of k. We show (14) also holds for the case of $k + 1$.

For $k \neq N_i$, by conditions (i), (iii), (v), and the induction assumption, we get that

$$\begin{aligned} V(k + 1, x(k + 1)) &\leq g_c(k, V(k, x(k))) \\ &\leq g_c(k, w(k)) = w(k + 1) \\ &\leq g_c(k, l(r(k))) \\ &\leq l(h_c(k, r(k))) = l(r(k + 1)),\ k \neq N_i, \end{aligned} \tag{15}$$

which means that (14) holds for $k + 1$ and $k \neq N_i$.

For $k = N_i$, by conditions (ii) and (iv)-(v) and induction assumption, we have

$$\begin{aligned} V(k + 1, x(k + 1)) &\leq V(k, x(k)) + g_d(k, V(k, x(k))) \\ &\leq w(k) + g_d(k, w(k)) = w(k + 1) \\ &\leq l(r(k)) + g_d(k, l(r(k))) \\ &\leq l(r(k)) + l(h_d(k, r(k))). \end{aligned} \tag{16}$$

It follows from (16) and the fact that $l(a) + l(b) \leq l(a + b)$ for any $a, b \in R_+$, that

$$V(k+1, x(k+1)) \leq w(k+1) \leq l(r(k+1)),\ k \neq N_i, \tag{17}$$

which means that (14) holds for $k + 1$ and $k = N_i$.

Thus, by (15) and (17), we obtain that (14) holds for the case $k + 1$ and by the induction principle (14) holds for all $k \in N$. □

Corollary 3.1. Assume that functions $g_c(k, \upsilon), g_d(k, \upsilon)$ are nondecreasing with respect to υ for any $k \in N$, and furthermore suppose that there exists a function $V(k, x) \in C[N\times R^n, R_+^m]$ such that the conditions (i)-(ii) of Theorem 3.1 hold, then $V(k_0, x_0) \leq w_0$ implies that

$$V(k, x(k)) \leq w(k), \quad k \geq k_0, k \in N. \tag{18}$$

Proof. It is a direct consequence of Theorem 3.1. □

Corollary 3.2. Assume that all assumptions except that for function $l(r)$ in Theorem 3.1 are satisfied. If $l(r)$ is a smooth vector function satisfying $l''(r) \geq 0$ for all $r \in R_+$, then the result of Theorem 3.1 still holds.

Proof. By Theorem 3.1, we only need to prove that

$$l(a) + l(b) \leq l(a+b),\ a, b \in R_+. \tag{19}$$

For any fixed $a \in R_+$, let $F(b) = l(a+b) - l(a) - l(b)$, $b \in R_+$, then $F(b)$ is a smooth vector function. Moreover, $F(0) = 0$, and $F'(b) = l'(a+b) - l'(b)$. By $l''(r) \geq 0$, we get that function $l'(r)$ is nondecreasing and hence $l'(a+b) \geq l'(b)$. Thus, we get that $F'(b) \geq 0$ which means that function $F(b)$ is nondecreasing. Hence, $F(b) \geq F(0) = 0$, for any $b \in R_+$. Thus, (19) holds and hence the proof is complete. □

In the following, in order to investigate the stability of systems $S_1 - S_3$, we assume that $f_c(k,0) \equiv 0$, $f_d(k,0) \equiv 0$, $g_c(k,0) \equiv 0$, $g_d(k,0) \equiv 0$, $h_c(k,0) \equiv 0$, and $h_d(k,0) \equiv 0$. Hence, systems $S_1 - S_3$ all admit the trivial solution.

Theorem 3.2. Assume that systems $S_1 - S_3$ satisfy all conditions of Theorem 3.1 and also assume that there exist functions $\varphi_1, \varphi_2 \in \mathcal{K}$ such that

$$\varphi_1(\|x\|) \leq \|V(k,x)\| \leq \varphi_2(\|x\|), \tag{20}$$

then, the US (UAS) properties of system S_3 imply that the same US (UAS) properties hold for system S_1. Moreover, $D(x)$ is an attractive region of system S_1, where

$$D(x) = \big\{x \in R^n : V(k,x) < \sup_{0 \leq r < r^*} \{l(r)\},\ k \in N\big\}.$$

Proof. Firstly, if system S_3 is US, we show that system S_1 is also US.

Since vector function $l(\cdot)$ is continuous and $l_i(0) = 0$, $l_i(r) > 0\,(r > 0)$, $i = 1, 2, \cdots, m$, for any positive number $\epsilon > 0$, there exists a $\epsilon_1(\epsilon)$ with $\epsilon_1(\epsilon) < r^*$ such that when $0 \leq r < \epsilon_1(\epsilon)$, we have

$$\|l(r)\| \leq \varphi_1(\epsilon). \tag{21}$$

From the uniform stability of system S_3, for $\epsilon_1(\epsilon) > 0$, there exists a $\delta_1(\epsilon) > 0$ such that when $0 \leq r_0 < \delta_1(\epsilon)$, we get that

$$r(k, k_0, r_0) < \epsilon_1(\epsilon), \quad k \in N. \tag{22}$$

Let $\bar{\delta}_i(\epsilon) \triangleq \sup_{0 \leq r < \epsilon_1(\epsilon)} \{l_i(r)\}$, $\delta_2(\epsilon) \triangleq \frac{1}{2} \min_{1 \leq i \leq m} \{\bar{\delta}_i(\epsilon)\}$, $\delta(\epsilon) \triangleq \varphi_2^{-1}(\delta_2(\epsilon))$.

By the continuity of function $l(r)$, we obtain that there exists a r_0 with $0 < r_0 < \delta_1(\epsilon)$ such that

$$\delta_2(\epsilon) \leq l_i(r_0) \leq \bar{\delta}_i(\epsilon), \quad i = 1, 2, \cdots, m. \tag{23}$$

Let $V(k_0, x_0) = w_0$, then when $\|x_0\| \leq \delta(\epsilon)$, we get that

$$\|w_0\| \leq \varphi_2(\|x_0\|) \leq \delta_2(\epsilon), \tag{24}$$

which implies that

$$V(k_0, x_0) = w_0 < l(r_0). \tag{25}$$

Thus, by Theorem 3.1, we get

$$\begin{aligned} \|x(k, k_0, x_0)\| &\leq \varphi_1^{-1}(\|V(k, x(k))\|) \\ &\leq \varphi_1^{-1}(\|w(k, k_0, w_0)\|) \\ &\leq \varphi_1^{-1}(\|l(r(k, k_0, r_0))\|) \\ &< \epsilon, \quad k \geq k_0, k \in N. \end{aligned} \tag{26}$$

Hence, systems S_1 is US.

In the following, we show that system S_1 is UAS if system S_3 is UAS. From the above proof, we only need to prove that $\lim_{k\to\infty} \|x(k, k_0, x_0)\| = 0$ holds uniformly for $k_0 \in N$.

By the UAS of system S_3, we get that $\lim_{k\to\infty} r(k, k_0, x_0) = 0$ holds uniformly for $k_0 \in N$. From the continuity of function $l(r)$, it leads to $\lim_{k\to\infty} \|l(r(k, k_0, x_0))\| = 0$ holds uniformly for $k_0 \in N$. Thus, by Theorem 3.1, we obtain that

$$\begin{aligned} \lim_{k\to\infty} \|x(k, k_0, x_0)\| &\leq \lim_{k\to\infty} \varphi_1^{-1}(\|V(k, x(k, x(k)))\|) \\ &\leq \lim_{k\to\infty} \varphi_1^{-1}(\|w(k, k_0, w_0)\|) \\ &\leq \lim_{k\to\infty} \varphi_1^{-1}(\|l(r(k, k_0, r_0))\|) = 0, \end{aligned} \tag{27}$$

holds uniformly for $k_0 \in N$.

Hence, system S_1 is UAS.

Moreover, for any $x_0 \in D(x)$, there exists a r_0 with $0 \leq r_0 < r^*$ such that $V(k_0, x_0) \leq l(r_0)$. Hence, we can choose that $w_0 = V(k_0, x_0)$ and hence by Theorem 3.1, we get (27) holds. Therefore, $D(x)$ is an attractive region of system S_1. The proof is complete. □

Theorem 3.3. Assume that systems $S_1 - S_3$ satisfy all conditions of Theorem 3.1 and also assume that the following conditions hold:

(i) there exist constants $\lambda_1 > 0, \lambda_2 > 0, p > 0$ such that

$$\lambda_1 \|x\|^p \leq \|V(k,x)\| \leq \lambda_2 \|x\|^p, \quad k \in N; \tag{28}$$

(ii) there exist a constant $q \geq 1$ and $\Theta_1, \Theta_2 \in R_+^m$, where $\Theta_1 = (c_1\ c_2\ \cdots\ c_m)^T$, $\Theta_2 = (d_1\ d_2\ \cdots\ d_m)^T$, with $c_i > 0, d_i > 0, i = 1, 2, \cdots, m$, such that

$$r^q \Theta_1 \leq l(r) \leq r^q \Theta_2, \quad r \in [0, r^*]. \tag{29}$$

Then, the UES of system S_3 implies that the UES of system S_1. Moreover, $E(x)$ is an attractive region of system S_1, where

$$E(x) \triangleq \big\{x \in R^n : V_i(k,x) < c_i r^*, k \in N, i = 1, 2, \cdots, m\big\},$$

where $V = (V_1, V_2, \cdots, V_m)^T$.

Proof. Suppose that system S_3 is UES, then there exist positive constants $K \geq 1, \alpha > 0$ such that

$$r(k, k_0, r_0) \leq K r_0 e^{-\alpha(k-k_0)}, \quad k \geq k_0, k \in N. \tag{30}$$

For any $x_0 \in E(x)$, let $w_0 = V(k_0, x_0)$, and $r_0 = \max_{1 \leq i \leq m} \left\{\frac{V_i(k_0, x_0)}{c_i}\right\}^{\frac{1}{q}}$, then, by condition (ii), we get that $0 \leq r_0 < r^*$ and $0 \leq V(k_0, x_0) = w_0 \leq l(r_0)$. Thus, by Theorem 3.1, we get, for any $k \geq k_0, k \in N$,

$$\|V(k, x(k))\| \leq \|l(r(k))\| \leq \Big(\sum_{j=1}^{m} d_j^2\Big)^{\frac{1}{2}} K^q r_0^q e^{-\alpha q(k-k_0)}. \tag{31}$$

It follows from (31) and condition (i), we have

$$\|x(k)\| \leq \left(\frac{\left(\sum_{j=1}^{m} d_j^2\right)^{\frac{1}{2}} K^q r_0^q}{\lambda_1}\right)^{\frac{1}{p}} e^{-\frac{\alpha q}{p}(k-k_0)}$$
$$\leq \left(\frac{K^q \lambda_2 \left(\sum_{j=1}^{m} d_j^2\right)^{\frac{1}{2}}}{\lambda_1 \min_{1\leq i\leq m}\{c_i\}}\right)^{\frac{1}{p}} \|x_0\| e^{-\frac{\alpha q}{p}(k-k_0)}. \quad (32)$$

Hence, system S_1 is UES and $E(x)$ is an attractive region of system S_1. □

In the following, we investigate the stability properties of system S_1 by using the stability properties of system S_2. Let $\Omega(w)$ be an attractive region of system S_2.

Theorem 3.4. Assume that systems $S_1 - S_2$ satisfy all conditions of Corollary 3.1 and (20) holds for some functions $\varphi_1, \varphi_2 \in \mathcal{K}$, then, the US (UAS) properties of system S_2 implies that the same US (UAS) properties of system S_1. Moreover, $D(x)$ is an attractive region of system S_1, where $D(x) = \big\{x \in R^n : V(k,x) \in \Omega(w),\ k \in N\big\}$.

Proof. If system S_2 is US, then for any $\epsilon > 0$, there exists $\delta_1(\epsilon) > 0$ such that for any w_0 satisfying $\|w_0\| \leq \delta_1(\epsilon)$, we have

$$\|w(k,k_0,w_0)\| \leq \varphi_1(\epsilon), \quad k \geq k_0, k \in N. \quad (33)$$

Let $\delta(\epsilon) \triangleq \varphi_2^{-1}(\delta_1(\epsilon))$. For any x_0 satisfying $\|x_0\| \leq \delta(\epsilon)$, we let $w_0 = V(k_0, x_0)$, then, by (20), we have

$$\|w_0\| = \|V(k_0,x_0)\| \leq \varphi_2(\|x_0\|) \leq \delta_1(\epsilon), \quad (34)$$

which implies (33) holds for all $k \in N$. Thus, by Corollary 3.1, we get that

$$\|x(k,k_0,x_0)\| \leq \varphi_1^{-1}(\|V(k,x(k))\|)$$
$$\leq \varphi_1^{-1}(\|w(k,k_0,w_0)\|) < \epsilon, \quad k \geq k_0, k \in N. \quad (35)$$

Hence, system S_1 is US. Moreover, if system S_2 is UAS with attractive region $\Omega(w)$, then, by similar proof of Theorem 3.2, we obtain that system S_1 is UAS with attractive region $D(x)$. □

Theorem 3.5. Assume that systems $S_1 - S_2$ satisfy all conditions of Corollary 3.1 and also assume that the condition (i) in Theorem 3.3 holds, then, the UES of system S_2 implies that the UES of system S_1. Moreover, $D(x)$ is an attractive region of system S_1, where $D(x) = \big\{x \in R^n : V(k,x) \in \Omega(w),\ k \in N\big\}$.

Proof. Suppose that system S_2 is UES, then there exist positive constants $K \geq 1, \alpha > 0$ such that

$$\|w(k,k_0,r_0)\| \leq K\|w_0\|e^{-\alpha(k-k_0)}, \quad k \geq k_0, k \in N. \quad (36)$$

For any $x_0 \in D(x)$, let $w_0 = V(k_0,x_0)$, then by Corollary 3.1, we get $V(k,x(k)) \leq w(k)$ for all $k \geq k_0, k \in N$. Using condition (i) in Theorem 3.3, we get

$$\|x(k)\| \leq \left(\frac{K\lambda_2}{\lambda_1}\right)^{\frac{1}{p}} \|x_0\| e^{-\frac{\alpha}{p}(k-k_0)}, \quad k \geq k_0, k \in N. \quad (37)$$

Hence, system S_1 is UES and $D(x)$ is an attractive region of system S_1. □

4. APPLICATIONS TO ROBUST STABILITY.

In this section, we apply the comparison principle Theorems 3.1-3.5 established in Section 3 to robust stability analysis of linear and nonlinear uncertain discrete impulsive hybrid systems.

Case 1. Consider the linear interval discrete impulsive hybrid system:

$$\begin{cases} x(k+1) = Ax(k), & k \neq N_i, \\ \Delta x(k+1) = B_k x(k), & k = N_i, i \in N, \\ x(k_0) = x_0, \end{cases} \quad (38)$$

where $A \in N[P,Q]$ and $B_k \in N[P_k,Q_k]$, $k \in N$, with $B_0 = 0$, and $P = (p_{ij}), Q = (q_{ij}), P_k = (p_{ij_k}), Q_k = (q_{ij_k})$ are $n \times n$ known matrices.

By Lemma 2.2, (38) can be rewritten as

$$\begin{cases} x(k+1) = A_0 x(k) + E\Sigma F x(k),\ k \neq N_i, \\ \Delta x(k+1) = B_{k_0} x(k) + E_k \Sigma_k F_k x(k),\ k = N_i, i \in N, \\ x(k_0) = x_0, \end{cases} \quad (39)$$

where $A = A_0 + E\Sigma F, B_k = B_{k_0} + E_k\Sigma_k F_k, k \in N$.

Theorem 4.1. Suppose Assumption (A_1) holds. Then, the system (38) is robust UAS if there exists a constant $\alpha > 0$ such that for any $k \in (N_i, N_{i+1}]$, $k, i \in N$,

$$\ln\big(\|A_0\| + \|E\|\|F\|\big) + \frac{\sum_{j=1}^{i} \ln\big(\|I + B_{N_{j0}}\| + \|E_{N_j}\|\|F_{N_j}\|\big)}{k-i} \leq -\alpha. \quad (40)$$

Moreover, if there exists a positive constant $0 < \beta < \frac{1}{2}$ such that

$$\inf_{k\in(N_i,N_{i+1}]} \Big\{\frac{i}{k-k_0}\Big\} \geq \beta > 0, \quad i \in N, \quad (41)$$

then, the system (38) is robust UES.

Proof. Let $V(k,x) = \|x\|$, then $V(k,x) \in C[N \times R^n, R_+]$ and $V(k,x) \geq 0$. For any $w, r \in R_+$, let: $g_c(k,w) = \|A\|w$, $g_d(k,w) = \big(\|I+B_k\|-1\big)w$, $h_c(k,r) = (\|A_0\|+\|E\|\|F\|)r$, $h_d(k,r) = (\|I + B_{k_0}\| + \|E_k\|\|F_k\| - 1)r$, and $l(r) = r$, then, we get

$$V(k+1,x(k+1)) \leq g_c(k,V(k,x(k))),\ k \neq N_i,$$
$$\Delta V(k+1,x(k+1)) \leq g_d(k,V(k,x(k))),\ k = N_i.$$
$$g_c(k,l(r)) \leq l(h_c(k,r)), \quad k \neq N_i, r \in R_+,$$
$$g_d(k,l(r)) \leq l(h_d(k,r)), \quad k = N_i, r \in R_+,$$

which implies that all conditions of Theorem 3.1 hold.

For any $x_0 \in R^n$, let $r_0 = w_0 = V(k_0,x_0) = \|x_0\|$, then by Theorem 3.1, we have that

$$\|x(k)\| = V(k,x(k)) \leq w(k) \leq r(k), \quad k \in N, \quad (42)$$

where $w(k)$ and $r(k)$ are the solutions to following systems, respectively:

$$\begin{cases} w(k+1) = \|A\|w(k),\ k \neq N_i, \\ \Delta w(k+1) = (\|I + B_k\| - 1)w(k),\ k = N_i, \\ w(k_0) = w_0 = \|x_0\|,\ k \in N, , k \geq k_0, \end{cases} \quad (43)$$

and

$$\begin{cases} r(k+1) = (\|A_0\| + \|E\|\|F\|)r(k),\ k \neq N_i, \\ \Delta r(k+1) = (\|I + B_{k_0}\| + \|E_k\|\|F_k\| - 1)r(k),\ k = N_i, \\ r(k_0) = r_0 = \|x_0\|,\ k \in N, k \geq k_0. \end{cases} \quad (44)$$

Thus, by Theorem 3.2 that the US (UAS) properties of system (44) implies that the same US (UAS) properties of system (38). Hence, in the following, we only need to prove that the system (44) is UAS under the condition (40).

Denote $a \triangleq \|A_0\| + \|E\|\|F\|, b_k \triangleq \|I + B_{k_0}\| + \|E_k\|\|F_k\|$. For any $k \in (N_i, N_{i+1}], k, i \in N$, by (44), we get

$$r(k) = ar(k-1) = a^{k-(N_i+1)}r(N_i+1) = a^{k-N_i-1}b_{N_i}r(N_i), \tag{45}$$

which implies

$$r(N_{i+1}) = a^{N_{i+1}-N_i-1}b_{N_i}r(N_i),\ i \in N. \tag{46}$$

It follows from (45)-(46) that

$$\begin{aligned} r(k) &= a^{k-N_i-1}b_{N_i}r(N_i), \\ &= a^{k-N_i-1}b_{N_i}a^{N_i-N_{i-1}-1}b_{N_{i-1}}r(N_{i-1}) \\ &= a^{k-N_i-1}b_{N_i}\prod_{j=i}^{1} a^{N_j-N_{j-1}-1}b_{N_{j-1}}r_0 \\ &= a^{k-i}\prod_{j=1}^{i} b_{N_j}r_0 = e^{(k-i)\ln a + \sum_{j=1}^{i}\ln b_{N_j}}r_0. \end{aligned} \tag{47}$$

On the other hand, it follows from Assumption (A1) that

$$N_i \geq N_{i-1} + 2 \geq N_{i-2} + 4 \geq \cdots \geq N_0 + 2i \geq 2i. \tag{48}$$

Hence, for any $k \in (N_i, N_{i+1}]$, we get

$$k > N_i \geq 2i. \tag{49}$$

Therefore, by (40), (47) and (49), for any $k \in (N_i, N_{i+1}]$, we obtain that

$$0 \leq r(k) \leq e^{-\alpha(k-i)}r_0 < e^{-\alpha i}r_0. \tag{50}$$

Obviously, $0 \leq r(k) < r_0$, which implies that the system (44) is US. Moreover, from (50) and the fact that $k \to \infty$ if and only if $i \to \infty$, we obtain the system (44) is UAS. Hence, by Theorem 3.2, system (38) is robust UAS.

Moreover, if (41) holds, then, it follows from (50) that

$$0 \leq r(k) < e^{-\alpha i}r_0 \leq e^{-\alpha\beta(k-k_0)}r_0,\ k \geq k_0, \tag{51}$$

which means that system (44) is UES. Thus, by Theorem 3.3, system (38) is robust UES. The proof is complete. □

Case 2. Consider a class of nonlinear uncertain discrete impulsive hybrid system in form of (1):

$$\begin{cases} x(k+1) = f(k, x(k)) + \varphi(k, x(k)),\ k \neq N_i, \\ \Delta x(k+1) = B_k x(k),\ k = N_i, \\ x(k_0) = x_0,\ k \geq k_0, k \in N, \end{cases} \tag{52}$$

under the following assumptions:

(B_1): for any $k \in N$ and $x, y \in R^n$, there exist matrices $A_k \in R^{n\times n}$ such that

$$\big|f(k,x) - f(k,y)\big| \leq A_k|x-y| \tag{53}$$

(B_2): The functions φ represent structural uncertainty or uncertain perturbation characterized by: there exist some matrix $C_k \in R^{n\times n}$, such that

$$|\varphi(k,x)| \leq C_k|x|, k \in N. \tag{54}$$

Theorem 4.2. Suppose that Assumptions (A_1)-(A_3) and (B_1)-(B_2) hold. Furthermore, assume that there exists a positive definite matrix $P \in R^{n\times n}$ such that

$$\sum_{k=0}^{\infty} \ln \gamma_k = -\infty, \tag{55}$$

where $\gamma_k = \begin{cases} \alpha_k, & \text{if } \quad k \neq N_i, \\ \beta_k, & \text{if } \quad k = N_i, k, i \in N, \end{cases}$ where $\alpha_k = \lambda_{\max}\big(P^{-1}(|A_k + C_k|)^T P(|A_k + C_k|)\big)$, and $\beta_k = \lambda_{\max}\big(P^{-1}(|I + B_k|)^T P(|I + B_k|)\big)$.
Then, system (52) is robust UAS.

Moreover, if there exist $k_1 \in N$ with $k_1 \geq k_0$ and a positive constant $\alpha > 0$ such that

$$\sup_{k \geq k_1, k \in N}\left\{\frac{\sum_{j=0}^{k-1}\ln\gamma_j}{k-k_0}\right\} \leq -\alpha, \tag{56}$$

then, the system (52) is robust UES.

Proof. Let $V(k,x) = |x|$, then $V(k,x) \in C[N \times R^n, R_+^n]$ and $V(k,x) \geq 0$. For any $w \in R^n$ with $w \geq 0$, denote: $g_c(k,w) \triangleq (A_k + C_k)w$, and $g_d(k,w) \triangleq \big(|I + B_k| - I\big)w$, then, by Assumptions (B_1)-(B_2), we get

$$\begin{aligned} &V(k+1, x(k+1)) \leq g_c(k, V(k, x(k))),\ k \neq N_i, \\ &\Delta V(k+1, x(k+1)) \leq g_d(k, V(k, x(k))),\ k = N_i. \end{aligned}$$

Thus, under Assumptions (B_1)-(B_2) and by using $|.|$, we can linearize the system (52) into the following linear comparing system:

$$\begin{cases} w(k+1) = (A_k + C_k)w(k),\ k \neq N_i, \\ \Delta w(k+1) = (|I + B_k| - I)w(k),\ k = N_i, \\ w(k_0) = w_0 = |x_0|,\ k \in N. \end{cases} \tag{57}$$

It follows from $V(k_0, x_0) = |x_0| \leq w_0$ and Corollary 3.1 that

$$|x(k)| = V(k, x(k)) \leq w(k), \quad k \in N. \tag{58}$$

Thus, by Theorem 3.4 that the US (UAS) properties of system (57) implies that the same US (UAS) properties of system (52). Hence, in the following, we only need to prove that the system (57) is UAS under the condition (55).

Let Lyapunov function be $W(x) = w^T P w$, then by Lemma 2.1 and Assumptions (B_1)-(B_2), for any $k \neq N_i$, we get

$$\begin{aligned} W(w(k+1)) &= w(k)^T\big[A_k + C_k\big]^T P\big[A_k + C_k\big]w(k) \\ &\leq \lambda_{\max}\big(P^{-1}(|A_k + C_k|)^T P(|A_k + C_k|)\big)W(w(k)) \\ &= \alpha_k W(w(k)); \end{aligned} \tag{59}$$

and for $k = N_i$, by Lemma 2.1, we get

$$\begin{aligned} W(w(k+1)) &= \big[|I + B_k|w(k)\big]^T P\big[|I + B_k|w(k)\big] \\ &\leq \lambda_{\max}\big(P^{-1}(|I + B_k|)^T P(|I + B_k|)\big)W(w(k)) \\ &\leq \beta_k W(w(k)), \end{aligned} \tag{60}$$

that is,

$$W(w(N_k + 1)) \leq \beta_{N_k} W(w(N_k)). \tag{61}$$

From (59)-(61), for any $k \in (N_i, N_{i+1}]$, we obtain that

$$W(w(k)) \leq \Big(\prod_{j=0}^{k-1}\gamma_j\Big)W(w_0) = e^{\sum_{j=0}^{k-1}\ln\gamma_j}W(w_0), \tag{62}$$

which implies that

$$\|w(k)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\, e^{\frac{1}{2}\sum_{j=0}^{k-1} \ln \gamma_j} \|w_0\|,\ k \geq k_0, k \in N. \tag{63}$$

Hence, if $\sum_{j=0}^{\infty} \ln \gamma_j = -\infty$, then by (63), there exists a positive constant $K > 0$ such that $\|w(k)\| \leq K\|w_0\|$, which leads to the US of system (57). Moreover, $\lim_{k\to\infty} \|w(k)\| = 0$. Thus, system (57) is UAS. Hence, by Theorem 3.4, system (52) is robust UAS.

Moreover, if (56) holds, then, by (63), there exists a positive constant $K_1 > 0$ such that

$$\|w(k)\| < K_1 \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\, e^{-\frac{\alpha}{2}(k-k_0)} \|w_0\|,\ k \geq k_0, \tag{64}$$

which means that system (57) is UES. Thus, by Theorem 3.5, system (52) is robust UES. The proof is complete. □

Remark 4.1. Obviously, if the conditions of Theorems 4.1-4.2 hold, then, the attractive region of systems (38) and (52) is R^n.

5. CONCLUSIONS

In this paper, we have established the comparison principle for discrete impulsive hybrid systems. Based on the comparison principle, we derived uniform stability (US, UAS and UES) criteria and the region of attraction for this kind of systems. As applications, the comparison principle has been used to investigate robust stability for linear interval discrete impulsive hybrid systems and a class of nonlinear uncertain discrete impulsive hybrid systems. The robust stability criteria obtained are verifiable via solving algebraic inequalities.

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