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Frequency Domain Identification of Linear, Deterministically Time-Varying Systems

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Abstract: This paper proposes an extension of the framework of linear time invariant system identification to linear slowly time-varying system identification. The class of systems considered is described by ordinary differential equations with coefficients varying piecewise polynomially with time. The estimation is performed in the frequency domain using multisine excitation signals within an errors-in-variables framework. It is also discussed how multisine excitations provide estimates of, on the one hand, the speed of variation of the system and, on the other hand, an estimation of a non-parametric noise model.

Keywords: Time-varying systems, frequency domain, multisine excitations

1. INTRODUCTION

The framework of LTI (Linear Time Invariant) systems has been shown to provide good approximating models for a lot of real-life applications. For some classes of problems, however, the assumption of time-invariance may not always be satisfied. Consider for instance nonlinear systems with a time-varying setpoint. These systems can locally be approximated by LTI models, which are a function of one or more so-called scheduling parameters (Bamieh B., Giarré L., 2002). Examples are: a moving robot arm, flight flutter analyses with the flight velocity and altitude as scheduling parameters, structures with varying loads, etc. Other systems that have a time-varying behavior are systems with parameters that actually vary with time, e.g. the impedance of a metal subjected to pitting corrosion. However, from a black box identification point of view, both setpoint varying systems and systems with time-varying parameters can be seen as belonging to the same class of systems.

A very common method for identifying time-varying systems where the time-variations are identified in a non-parametric fashion is recursive identification, which is thoroughly studied in (Niedzwiecki M., 2000) and in (Ljung and Söderström, 1983). This method is, of course, mostly applicable when the time variations are random and relatively slow w.r.t. the dynamics of the system. This paper will only consider the parametric identification of time-variations.

Classically, setpoint varying systems could be identified using the LTI framework, in which case multiple experiments would be carried out on a well-defined grid of the scheduling parameters. An interpolation algorithm could then describe the parameters as functions of the setpoint, resulting in so-called polytopic models. This method has the following important disadvantages: i) the scheduling parameters must have the ability to be "frozen" (which is not always possible, e.g. in chemical processes), ii) multiple experiments may last a long time, iii) the quality of the model depends on the interpolation method. Making use of linear time-varying models (LTV) provides an important benefit in this case. The whole range of the setpoints of interest is identified in only one well-designed experiment.

In (Fujimori and Ljung, 2006) polytopic models of linear time-varying systems are identified. In this case, measurements of time-varying setpoint systems are matched with interpolated LTI models. An interpolation method is chosen and the parameters of the LTI models on a well-defined grid of the scheduling parameters are identified.

This paper handles the identification of linear, SISO (Single-Input, Single-Output), lumped, continuous time systems which can be described by ordinary differential equations with time dependent parameters. These parameters are approximated by piecewise low-order polynomials in time, as illustrated on Fig. 1. Extra constraints are applied to the polynomials to ensure that the parameters are continuous and have continuous derivative(s). Note that any time-variation can be approximated by such piecewise polynomials. Fast variations will, however, require the time-pieces to be short, resulting in a low resolution and eventually prohibiting a good identification.

As mentioned in (Bamieh and Giarré, 2002), in many processes it is a realistic assumption that, for systems with a varying setpoint, the scheduling parameters are measurable as functions of time (the position of the robot arm or the speed of the airplane in the case of flight flutter analysis, etc.). As a consequence, when the parameters are identified as functions of time (as will be done in this paper), they are also known as functions of the scheduling parameters.

Contrary to most previous work (Bamieh and Giarré, 2002), (Niedzwiecki, 2000), (Poulimenos and Fassois, 2006), the identification will be performed in the frequency domain. This has two main advantages. The first is that a non-parametric model of the colored input/output noise can easily be

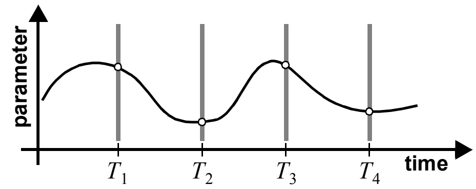

Fig. 1 The parameters are approximated by low-order piecewise polynomials. Between two grey vertical lines the parameter is described by one polynomial. In the small black circles '**O**' the function and its derivative(s) are constrained to be continuous.

used. The second is that the identification is performed in a well-defined frequency band of interest. This band of interest is chosen such that it contains the dynamics of the system which are important for the intended application.

Time-domain methods on the other hand have the advantage that modelling time-varying parameters as a linear combination of arbitrary basis functions is more easily implemented (chapter 6 in Niedzwiecki, 2000) and (Poulimenos and Fassois, 2006). For their frequency domain counterparts the Laplace transforms of the system equations are needed. These will contain the convolutions of the Laplace transforms of the basis functions used and the derivatives of the input and output signals. These are usually not easily implemented. An exception to this is the use of polynomials as basis functions, as will be illustrated in this paper.

Multisines will be used as excitation signals. These signals have been shown to be very useful in identification. They are designed to cover a well-defined chosen frequency band and, for time invariant systems, they allow for a clear distinction between system noise and nonlinear distortions (De Locht et al., 2004), (Schoukens et al., 2005). For linear time-varying systems they will be shown to provide an idea of the speed of the time-variations and an approximate non-parametric model of colored input/output noise.

The remainder of this paper is organized as follows. Section 2 gives a formal description of the class of systems considered. Section 3 considers setting up a cost function and its minimization for the identification. The use of multisines as an excitation signal for time-varying systems to estimate the speed of the time-variations and the power spectrum of the noise is discussed in Section 4. Section 5 shows simulation results and conclusions are drawn in Section 6.

1.1. Notational conventions

Continuous time-domain signals are designated by lower case letters (e.g. $u(t)$) and their Laplace transforms by the respective uppercase letters ($U(s)$). $L\{\bullet\}$ denotes the Laplace transform and s the Laplace variable. $U(s_k)$ denotes the k-th bin of the sampled Laplace transform on the $j\omega$-axis, i.e. $s_k = j2\pi k f_s/N$ where $f_s = 1/T_s$ is the sampling frequency and N is the number of equidistant time-samples inside one time-record.

2. PROBLEM FORMULATION

The time-varying systems in the considered class are described by ordinary differential equations in the time domain with parameters varying polynomially with time, as given by:

$$\sum_{n=0}^{N_a}\left(\sum_{p=0}^{N_p} a_{n,p}t^p\right)\frac{d^n}{dt^n}y_0(t) = \sum_{n=0}^{N_b}\left(\sum_{p=0}^{N_p} b_{n,p}t^p\right)\frac{d^n}{dt^n}u_0(t) \qquad (1)$$

where $u_0(t)$ and $y_0(t)$ are the *noiseless* input and output signals, respectively. By using the property

$$L\{t^r x(t)\} = (-1)^r\frac{d^r}{ds^r}L\{x(t)\} \qquad (2)$$

equation (1) can easily be converted to the Laplace domain (see Appendix A):

$$\sum_{r=0}^{N_p} A_r(s_k)\frac{d^r}{ds^r}Y_T(s_k) = \sum_{r=0}^{N_p} B_r(s_k)\frac{d^r}{ds^r}U_T(s_k) + \sum_{n=0}^{\max(N_a,N_b)-1} s_k^n tr_n \qquad (3)$$

with tr_n the transient terms, $U_T(s)$ and $Y_T(s)$ the Laplace transforms of the *windowed* (uniform) input and output signals, respectively:

$$X_T(s) = L\{x(t)w(t)\}$$

with x replaced by u and y and:

$$w(t) = \begin{cases} 1 \text{ for } t \in [0, T] \\ 0 \text{ elsewhere} \end{cases}, \quad T = NT_s$$

$A_r(s)$ and $B_r(s)$ in (3) are defined as:

$$A_r(s) = \sum_{p=r}^{N_p}\sum_{n=0}^{N_a} a_{n,p}(-1)^p\binom{p}{r}n(n-1)\ldots(n-p+r+1)s^{n-p+r}$$

$$B_r(s) = \sum_{p=r}^{N_p}\sum_{n=0}^{N_b} b_{n,p}(-1)^p\binom{p}{r}n(n-1)\ldots(n-p+r+1)s^{n-p+r}$$

where $\binom{p}{r} = p!/(p-r)!r!$. Clearly, $A_r(s)$ and $B_r(s)$ are polynomials in s with coefficients which are linear in $a_{n,p}$ and $b_{n,p}$. Notice that setting $N_p = 0$ in (3) gives the classical input/output equation for an LTI system. The transient terms tr_n are functions of the differences between the initial and end conditions. For LTI systems excited by periodic signals and measured in steady-state, this difference would be zero since the output signal would also be periodic. For time-varying systems on the other hand, the response to a periodic signal will, in general, not be periodic, making the transient terms in (3) mandatory. Note that the expression for the transient terms in (3) is only valid on the sampled frequency lines s_k (as follows from the appendix).

To be of practical use when working with sampled band-limited signals, the Laplace transform should be approximated by the Discrete Fourier Transform (DFT). The k-th bin of the DFT of the time-domain signal $x_0(t)$ will be written as $X(k)$. The errors due to aliasing made during the identification can be reduced by allowing a higher order of the transient polynomial in (3) (Pintelon and Schoukens, 1997 and 2001). Note that, due to the non periodicity of the output signal, these aliasing errors will always be present. When working with the DFT's, an approximation of the derivatives of the Laplace transforms needed in (3) are readily calculated as

$$\begin{aligned} \frac{d^m}{ds_k^m}X_T(s_k) &= (-1)^m\int_0^{NT_s} t^m x(t)e^{-s_k t}dt \\ &\cong \frac{-1^m}{f_s^m}\sum_{t_d=0}^{N-1} t_d^m x(t_d T_s)e^{-\frac{j2\pi k t_d}{N}} = \frac{d^m}{ds_k^m}X(k) \end{aligned} \qquad (4)$$

with x to be replaced by u and y, and where t_d denotes the normalized discrete time. Equation (4) is computed very efficiently by using the Fast Fourier Transform (FFT) algorithm.

For the parameters to be piecewise polynomially varying, the system equation (3) must be satisfied for each time-piece. In addition, some continuity constraints on the parameters and their derivatives w.r.t. t may be imposed at the boundaries of the time-pieces, as explained in Fig. 1. For instance, if the parameters are approximated by third order polynomials, one may impose the derivatives to be continuous up to the second order. This is equivalent to a cubic spline approximation. The practical implementation of these constraints is discussed in section 3.4.

3. TRANSLATION TO AN IDENTIFICATION PROBLEM

3.1. Measurement and noise assumptions

The identification problem consists of estimating the parameters $a_{n,p}$, $b_{n,p}$ and tr_n from available measurements of the discrete Fourier transforms of the input and the output signals. The latter are assumed to satisfy the following:

$$\begin{aligned} U(k) &= U_0(k) + N_u(k) \\ Y(k) &= Y_0(k) + N_y(k) \end{aligned} \tag{5}$$

where $N_u(k)$ and $N_y(k)$ are the noise on the k-th frequency bin of, respectively, input and output signals. For this paper, the noise is assumed to be circular complex normally distributed and uncorrelated over the frequency:

$$\begin{aligned} E\{N_u(s_k)\overline{N_u(s_v)}\} &= \delta_{kv}\sigma_u^2(s_k) \\ E\{N_y(s_k)\overline{N_y(s_v)}\} &= \delta_{kv}\sigma_y^2(s_k) \\ E\{N_u(s_k)\overline{N_y(s_v)}\} &= \delta_{kv}\sigma_{uy}(s_k) \end{aligned} \tag{6}$$

with $\delta_{kv} = 1$ for $k = v$ and 0 otherwise, and $E\{\bullet\}$ denotes an expected value. Note that coloring of the noise is allowed (i.e. the variances and co-variances may be frequency-dependent). The noise assumptions are summarized in Fig. 2.

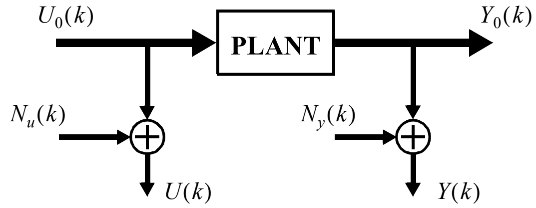

Fig. 2 Schematic summarizing the noise assumptions, as described by (5) and (6). N_u and N_y are noise sources which may be correlated.

3.2. Weighted nonlinear least squares cost function

A cost function must be set up. A good candidate is the weighted nonlinear least squares cost function, defined as:

$$V(\theta) = \sum_k \frac{|e(k,\theta)|^2}{\sigma_e^2(k,\theta)} \equiv \sum_k |\varepsilon(k,\theta)|^2 \tag{7}$$

where the set of summation indices k is chosen such as to cover the whole frequency band of interest. In this equation, e is the so-called *equation error* calculated as the difference between the left and the right hand side of (3) where $U_T(s_k)$ and $Y_T(s_k)$ are replaced by the input/output DFT spectra $U(k)$ and $Y(k)$. θ is a vector containing all the parameters to be estimated ($a_{n,p}$, $b_{n,p}$ and $tr_{n,p}$). σ_e^2 is the variance of e and will be discussed later on.

Cost functions formed like (7) as the sum of squared errors weighted by their uncertainty have been shown to provide "healthy" estimators. This cost function is inspired on the Maximum Likelihood cost function for LTI systems (see Chapter 7.11 of Pintelon R., Schoukens J., 2001) and can easily be shown to be correct. Consistency is not proven in a straightforward fashion because longer measurements imply more time-variations to be taken into account. This implies more time-pieces and thus a growing number of parameters. Intuitively, the uncertainty on the estimated parameters can be decreased when experiments are repeatable. Note that accurate reproduction of a measurement with time-varying systems is not an easy job, especially when considering the synchronization of the scheduling parameters with the excitation signal.

3.3. Variance of the equation error

As mentioned earlier, the cost function (7) needs the variance of the equation error. It can be calculated as:

$$\begin{aligned} \sigma_e^2 = {} & \sum_{r=0}^{N_p} |A_r|^2 \mathrm{var}\left\{\frac{d^r Y}{ds^r}\right\} + 2\mathrm{Re}\left\{\sum_{r=0}^{N_p}\sum_{l=r+1}^{N_p} A_r\bar{A}_l \mathrm{cov}\left\{\frac{d^r Y}{ds^r}, \frac{d^l Y}{ds^l}\right\}\right\} \\ & + \sum_{r=0}^{N_p} |B_r|^2 \mathrm{var}\left\{\frac{d^r U}{ds^r}\right\} + 2\mathrm{Re}\left\{\sum_{r=0}^{N_p}\sum_{l=r+1}^{N_p} B_r\bar{B}_l \mathrm{cov}\left\{\frac{d^r U}{ds^r}, \frac{d^l U}{ds^l}\right\}\right\} \\ & + 2\mathrm{Re}\left\{\sum_{r=0}^{N_p}\sum_{l=r+1}^{N_p} A_r\bar{B}_l \mathrm{cov}\left\{\frac{d^r Y}{ds^r}, \frac{d^l U}{ds^l}\right\}\right\} \end{aligned} \tag{8}$$

where $\bar{x}$ denotes the complex conjugate of x. The variances of the derivatives and covariances between derivatives of the signals needed in this equation are easily calculated from the variances and covariances of the signals:

$$\begin{aligned} & \mathrm{cov}\left\{\frac{d^m}{ds_v^m}X(v), \frac{d^l}{ds_v^l}X(v)\right\} \\ & = \frac{(-1)^{m+l}}{N^2 f_s^{m+l}} \sum_{k=0}^{N-1} \sigma_x^2(k) K_m(k-v)\overline{K_l(k-v)} \end{aligned} \tag{9}$$

with $K_m(\bullet) = \sum_{t=0}^{N-1} t^m e^{\frac{j2\pi(\bullet)t}{N}}$, X replaced by U and Y, σ_x by σ_u and σ_y (proof in Appendix B). The $\mathrm{cov}\left\{\frac{d^m}{ds_v^m}U(v), \frac{d^l}{ds_v^l}Y(v)\right\}$ is obtained by replacing σ_x^2 by the covariance σ_{uy} in (9).

In this section, $\sigma_x^2(k)$, $\sigma_y^2(k)$ and $\sigma_{uy}(k)$ have been assumed to be known. If this isn't the case, section 4.2 discusses how one can obtain estimates from the measured signals.

3.4. Minimizing the cost function with the constraints

Since the cost function $V(\theta)$, given by (7) is non-quadratic in θ, minimizing it will require starting values. A linear least squares or total least squares algorithm (TLS, bootstrapped TLS or weighted generalized TLS) can provide those values (Pintelon R. et al., 1998).

$V(\theta)$, being a sum of squares, is well-suited to be minimized by the Levenberg-Marquardt algorithm (Fletcher R., 1987, chapter 6). This algorithm requires the so-called *Jacobian* of the cost function to be calculated:

$$J_\theta(k) = \frac{d}{d\theta}\varepsilon(k,\theta) \tag{10}$$

Note that in ε, defined in (7), only the polynomials A_r and B_r are functions of θ. Additionally, since they are linear in θ, the derivatives of A_r and B_r have to be evaluated only once, at the start of the iteration.

For the parameters to be identified as piecewise polynomials, the cost function (7) must be minimized for each time-piece. In addition, the constraints that ensure the continuity of these polynomials must be integrated in this minimization. Assume that some parameter $a(t)$ can be written as

$$\begin{aligned} a(t) &= \sum_{p=0}^{N_p} a_{1,p} t^p \text{ for } t < t_b \\ a(t) &= \sum_{p=0}^{N_p} a_{2,p} t^p \text{ for } t \geq t_b \end{aligned}.$$

Expressing that its r-th derivative in $t = t_b$ is continuous implies:

$$\sum_{p=r}^{N_p} a_{1,p} \frac{p!}{(p-r)!} t_b^{p-r} = \sum_{p=r}^{N_p} a_{2,p} \frac{p!}{(p-r)!} t_b^{p-r}.$$

This expression is linear and homogeneous in the coefficients of the polynomials. As a consequence, the constraints can be written in matrix notation:

$$C\theta_{ext} = 0 \tag{11}$$

where C is a matrix with the number of rows equal to the number of constraints n_c. θ_{ext} is the extended vector of parameters, obtained by vertical concatenation of the parameters of all the time-pieces. Let n_θ be the length of θ_{ext}, the total number of parameters. Any solution to (11) can be written as:

$$\theta_{ext} = U_{null}\psi \tag{12}$$

where the columns of U_{null} span the null-space of the matrix C. U_{null} is now calculated using a singular value decomposition (SVD) of C^T, which is expressed as

$$C^T = U\Sigma V^T \tag{13}$$

where U and V are unitary matrices with dimensions $n_\theta \times n_\theta$ and $n_c \times n_c$ respectively and Σ is a diagonal matrix containing the singular values, padded with zeros to fulfill the dimensions. U_{null} is easily shown to be formed by the last $n_\theta - n_c$ columns of U (Golub and Van Loan, 1989).

Expressing θ_{ext} as in (12) ensures the parameters to be piecewise polynomials with continuous derivatives up to a given order. Plugging (12) into the set of cost functions (expressing (7) for all the time-pieces) and minimizing this w.r.t. ψ provides estimates of the parameters, which fulfill the constraints. Note that once the Jacobian w.r.t. θ_{ext} ($J_{\theta_{ext}}$) is known, the Jacobian w.r.t. ψ is readily calculated as (arguments k and θ omitted for convenience):

$$J_\psi = \frac{d\varepsilon}{d\psi} = \frac{d\varepsilon}{d\theta}\frac{d\theta}{d\psi} = J_{\theta_{ext}} U_{null}$$

4. USE OF MULTISINE EXCITATIONS

Although (3) is satisfied for arbitrary excitations, random phase multisines will be used in this paper. Random phase multisines have the following form:

$$u_0(t) = (1/\sqrt{K}) \sum_{\kappa=0}^{N/2-1} A(\kappa) \sin\left(\frac{2\pi f_s \kappa}{N} t + \varphi_\kappa\right) \tag{14}$$

where the φ_κ's are stochastic variables w.r.t. κ, uniformly distributed in the interval $[0, 2\pi[$, such that $E\{e^{j\varphi_\kappa}\} = 0$. The constant K is equal to the number of excited frequency lines

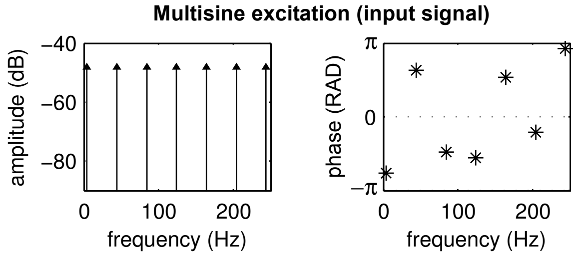

Fig. 3 Frequency domain representation of the input signal: a random phase multisine with uniform spectrum.

(as explained further), so that the root mean square (RMS) value of the signal would be independent of the number of excited lines. $u_0(t)$ consists of a sum of sines with frequencies all multiples of the same so-called *fundamental frequency* f_s/N. Due to its periodicity, its Fourier spectrum is discrete (as shown in Fig. 3). The excited frequency band can be chosen arbitrarily by determining the fundamental frequency and the amplitudes $A(\kappa)$ of the individual lines. Choosing $A(\kappa) = 1$ for frequency lines inside the band of interest (so-called *excited* frequency lines) and 0 outside (the *non-excited* frequency lines) provides a perfectly band-limited *white* excitation inside the chosen band as shown in Fig. 3. An arbitrary coloring can be obtained by choosing $A(\kappa)$ accordingly. When one period of this signal is sampled by N equidistant time samples at a sampling rate of f_s, neither aliasing nor leakage occurs.

4.1. Estimating the speed of variations

Exciting LTI systems with multisines gives also a multisine as output signal in steady-state with the same excited frequencies as the input signal, but shaped by the frequency response of the system (as represented by the black arrows on Fig. 4, left). When exciting a (non-cyclically) time-varying system with a multisine, the output signal will not be periodic anymore and one can no longer talk about a steady-state. When the time-variations are sufficiently slow, the most important contributions will also be found at the excited frequency lines of the output spectrum. But contrary to time invariant systems, the non-excited lines will also contain contributions (the grey dots on Fig. 4, right). The spectrum is built up of peaks and valleys, which give a rough non-parametric idea of the speed of the time-variations. For slower variations, the valleys are deeper and the peaks are sharper. This is as expected since for time invariant systems, the

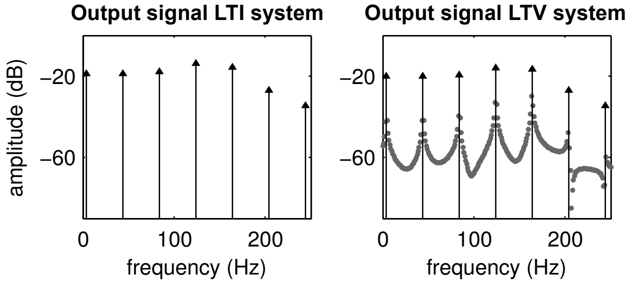

Fig. 4 Frequency domain of steady-state output signal (absolute value) of a linear time-invariant (left) and a linear slowly time-varying system (right) to the multisine excitation. The tops of the black arrows give the signal at the excited frequency lines. The grey dots represent the non-excited lines. The latter don't appear in the time-invariant case. The output signal of the time-varying system consists of valleys and sharp peaks.

peaks are actually infinitely sharp. One could say that, as for LTI systems, the energy is concentrated on the excited frequency lines; for slowly time-varying systems, the energy from the excited lines was slightly spread out over the non-excited lines. As a consequence, when identifying time-varying systems using multisines, one should not only consider the excited lines but also use the non-excited ones. For slow variations, the excited lines will mostly contain information of a "mean" system over time, while the non-excited lines tell a lot about the time-variations.

4.2. Noise estimation

In section 3 an estimator was presented which needed the frequency-dependent variances and covariance of the input and output signals. If those are a priori unknown (as is usually the case), one could try to extract them from the signal. The use of multisines will prove to be helpful for this task.

As was mentioned in Section 4.1, when exciting a time-invariant system with a multisine, the signal at the output is supposed to be only present at the excited frequency lines. Therefore, for a purely linear system, the energy on the non-excited lines is entirely due to noise. As shown in Fig. 5 left, a non-parametric frequency-dependent noise model is thus easily extracted from these non-excited lines. This technique is, however, not directly applicable to time-varying systems because the non-excited lines are a mix of the signal and the noise, as illustrated in Fig. 5 on the right. A similar technique is, however, possible, since for sufficiently slow variations, peaks and valleys are present (as discussed in Section 4). The peaks correspond to regions with high Signal-to-Noise Ratio (high SNR), whereas the signal in the valleys has a low SNR. In fact, it is perfectly possible that the signal in the valleys gets close to the noise floor (as the second valley in Fig. 5, right, for instance) or even is submerged by the noise. If one can assume that the power spectrum of the noise is varying slowly w.r.t. the frequency resolution used - meaning that the power spectrum is nearly white inside each valley - one can estimate its variance from neighbouring points that have a low SNR.

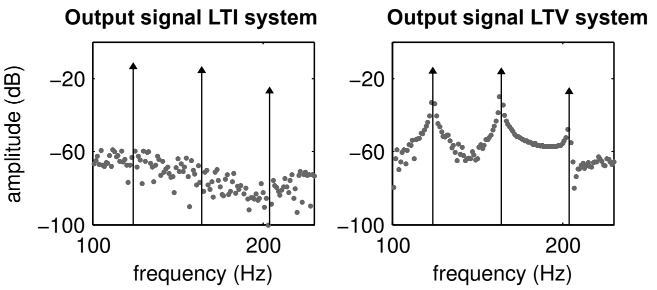

Fig. 5 Frequency domain of noisy output signal: left for an LTI system, right for an LTV system (zoomed in). In the LTI case, the non-excited lines (grey dots) can immediately serve as a non-parametric model of the variance of the excited frequency lines (black arrows). This is less straightforward for the LTV case. Remark: one and the same noise realization was used in both the LTI and the LTV cases.

5. SIMULATION RESULTS

The estimator described in the previous sections has been verified on a simulated continuous-time, linear time-varying system. This system was of order 2 in the numerator and 4 in the denominator and with parameters varying piecewise polynomially with time. The polynomials were of 3rd degree and the derivatives of the parameters w.r.t time were constrained to be continuous up to the second order. The system was excited with a multisine containing 41 frequencies. The input and output signals were distorted by colored noise. The mean SNR was 54dB.

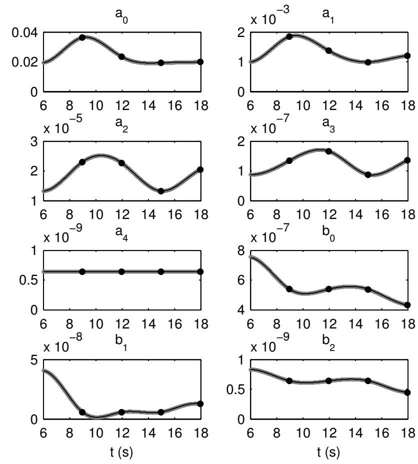

Fig. 6 Comparison of the denominator (a_i) and numerator (b_i) coefficients. The real values are given by the thin black curves while the identified parameters are given by the thick grey curves. In between two black dots •, each parameter is represented by one polynomial of third order.

In Fig. 6 the resulting identified parameters (grey thick line) are compared with the real parameters (thin black line) as functions of time. As is clear, they coincide very well. As shown, the time-record was split into 4 time-pieces (the boundaries of each piece being given by the black circles). Note that extrapolating these parameter values outside the considered time-record is very dangerous since polynomials diverge very quickly. When comparing the real noiseless output signal with the output signal simulated using the identified parameters, the mean squared error lies 66dB below the RMS value of the output signal, which is an improvement of 12dB w.r.t. the SNR of the noisy signals. This is in accordance with the rule of thumb (Ljung, 1999) that the improvement should be about

$$\sqrt{\frac{\text{number of independent parameters}}{\text{number of data points}}}.$$

A total number of 56 independent parameters were identified (taking into consideration the constraints) using 1376 frequency domain data points (excited and non-excited lines).

Setting up Fig. 6 for real-life applications is of course impossible because neither the real parameter values nor the noiseless output signal are known. Another method for determining the quality of the identification is illustrated in Fig. 7. After minimization of the cost function (7), the left hand side (black dots '•') and the right hand side (black circles 'o') of the system equation (3) should coincide at each frequency line. The difference between both (given by the broken grey line) should lie around the standard deviation of the equation error (thick black line), calculated using (8). As a consequence, the residues ε in (7) (given by the grey crosses 'x'), which are precisely the ratio of the equation error and its standard deviation, should lie around 1 (or 0 dB). This is the case in Fig. 7.

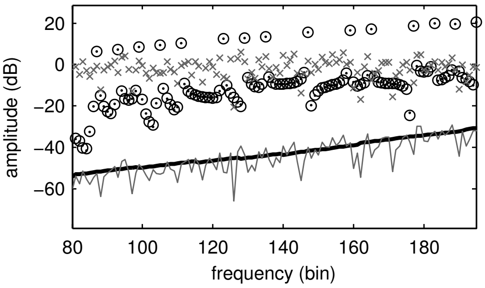

Fig. 7 Left hand side and right hand side (given by the black dots '•' and circles 'o', respectively) of the system equation (3) after minimization of the cost function (7). The grey line is the difference between the left and the right hand side while the black thick line is an estimation of the standard deviation of the equation error. The grey crosses 'x' are the residues ε in (7).

6. Conclusion

This paper presents an identification method for linear, continuous time, time-varying dynamic systems. The identification is performed in the frequency domain, revealing the important benefits of using multisines as excitation signals. A non-parametric model of the colored disturbing noise was extracted from only one measurement and a rough idea of the speed of variation is given.

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Appendix A. Laplace transform of the system equation

For windowed signals (rectangular window), one can show (see Appendix 5.B of Pintelon and Schoukens, 2001):

$$\begin{aligned} L\left\{\frac{d^n}{dt^n}x(t)\right\} &= \int_0^{NT_s}\frac{d^n}{dt^n}x(t)dt \\ &= s^nX(s)+\sum_{r=0}^{n-1}s^r(e^{-sNT_s}x^{(n-1-r)}(NT_s\text{-})-x^{(n-1-r)}(0+)) \end{aligned} \quad (15)$$

where $x^{(\bullet)}$ denotes the derivative of order •. When evaluated along the $j\omega$-axis at the DFT frequencies $s_k = j2\pi f_s k/N$, the sum in (15) is a polynomial of order $n-1$ in s_k. Combining property (2) and eq. (15) gives:

$$\begin{aligned} L\left\{t^p\frac{d^n}{dt^n}x(t)\right\} &= (-1)^p\frac{d^p}{ds^p}L\left\{\frac{d^n}{dt^n}x(t)\right\} \\ &= (-1)^p\frac{d^p}{ds^p}(s^nX(s))+I(s) \end{aligned} \quad (16)$$

where $I(s)$ is still a polynomial of order $n-1$ when evaluated at the DFT frequencies. Applying (16) to the right hand side of (1) gives:

$$\sum_{n=0}^{N_b}\sum_{p=0}^{N_p}b_{n,p}(-1)^p\frac{d^p}{ds^p}(s^nU_T(s))+I_b(s)$$

which can be rewritten as (by working out the derivatives and isolating the factors in which the signal is present):

$$\begin{aligned} &\sum_{n=0}^{N_b}\sum_{p=0}^{N_p}\sum_{r=0}^{p}b_{n,p}(-1)^p\binom{p}{r}\frac{d^{p-r}}{ds^{p-r}}s^n\frac{d^r}{ds^r}U_T(s)+I_b(s) \\ &= \sum_{r=0}^{N_p}\frac{d^r}{ds^r}U_T(s)\sum_{p=r}^{N_p}\sum_{n=0}^{N_b}b_{n,p}(-1)^p\binom{p}{r}\frac{d^{p-r}}{ds^{p-r}}s^n+I_b(s) \end{aligned}$$

Applying the same derivation to the left hand side of (1), combining the transient terms and evaluating the whole expression in the DFT frequencies gives (3).

Appendix B. Covariances of derivatives of signals

The inverse discrete fourier transform of $X(k)$ is:

$$x(t_dT_s) = \frac{1}{N}\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi k}{N}t_d}$$

with $t_d = 0, \ldots, N-1$. Substituting this equation into (4) gives:

$$\begin{aligned} \frac{d^m}{ds_\nu^m}X(\nu) &= \frac{(-1)^m}{Nf_s^m}\sum_{t_d=0}^{N-1}t_d^m\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi k}{N}t_d}e^{-j\frac{(2\pi\nu)}{N}t_d} \\ &= \frac{(-1)^m}{Nf_s^m}\sum_{k=0}^{N-1}X(k)\sum_{t_d=0}^{N-1}t_d^me^{j\frac{2\pi(k-\nu)}{N}t_d} \end{aligned} \quad (17)$$

Expression (9) is then obtained by evaluating

$$\varepsilon\left\{\left(\frac{d^m}{ds_\nu^m}X(\nu)-\varepsilon\left\{\frac{d^m}{ds^m}X(\nu)\right\}\right)\overline{\left(\frac{d^l}{ds_\nu^l}X(\nu)-\varepsilon\left\{\frac{d^l}{ds_\nu^l}X(\nu)\right\}\right)}\right\}$$

using (17) and by making the assumption that the noise is uncorrelated over the frequency, as expressed by (6).