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Controller Implementation for a Class of Spatially-Varying Distributed Parameter Systems ⋆

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Abstract: In this paper we discuss fast implementation of the model based centralized controllers using fractional Fourier transform for large scale plant models coming from spatial discretization of a certain type of linear spatially-varying distributed parameter systems. This fast implementation reduces the computational time delay significantly when the dimension of the system is higher than $512 = 2^9$. Compared to direct implementation, the proposed method allows faster sampling. If the control design objectives are demanding fast closed loop modes, then slower sampling required by direct implementation leads to instability. The results are illustrated by an example.

Keywords: Distributed Parameter Systems; Fractional Fourier Transform; Spatially-Varying Systems; Control Implementation; Time Delay.

1. INTRODUCTION

Spatially invariant distributed systems have been studied extensively in the literature, see e.g. Bamieh et al. (2002), D'Andrea and Dullerud (2003). Adaptive control of spatially varying distributed parameter systems is studied in Bentsman and Orlov (2001). Distributed control of systems on lattices is discussed in Jovanovic and Bamieh (2005). By using the Fourier transform with respect to the spatial variable Bamieh et al. (2002) converted an infinite dimensional control problem into a parameterized finite dimensional problem. From the computational point of view, this approach may be more attractive than the more classical ways of attacking these types of systems. There exists a large body of literature on designing controllers for distributed parameter systems using approximations of the plant, or the controller, see for example Atwell et al. (2001), Banks et al. (2000), Efe and Ozbay (2004), Ito and Morris (1998), Morris (2001), Xiao and Basar (1999), and their references.

Many interesting modeling and approximation techniques for spatially distributed systems lead to large-scale finite dimensional dynamical systems. There are further approximation techniques for such large-scale systems, Antoulas (2005), Gugercin and Antoulas (2004).

In this paper we consider a class of linear spatially-varying systems for which a modal approximation in the spatial coordinates lead to a large scale dynamical model for the temporal coefficients. The A-matrix of this model is assumed to have a special structure: fractional Fourier transform (FrFT) of order $a \in (0, 1)$ is diagonalizing this matrix. In the special case $a = 1$ this corresponds to the usual Fourier transform in the spatial coordinates, and linear spatially-invariant systems are covered in this manner.

We use a fast computation of the FrFT and its inverse in the implementation of the model based controllers to reduce the computational time delays. There can be significant savings in the computation time this way when the dimension of the system is larger than $512 = 2^9$. It is well known that such unmodeled time delays may have a destabilizing effect if they are large or the controller design is too aggressive, see Section 4 for further discussion and the results/references of Logemann (1998) and Rebarber and Townley (1998).

In Section 2 we define the class of linear spatially-varying systems to be considered. Section 3 contains a brief overview of the fractional Fourier transform and filtering using this transform. Main results appear in Section 4 with an example. Concluding remarks are made in Section 5, where some extensions and open problems are also mentioned.

2. SPATIALLY-VARYING SYSTEMS

In this paper we deal with the control of distributed parameter systems described by the dynamical equation (1), with input u and output y:

$$\frac{\partial y(x,t)}{\partial t} = \mathcal{A}(y(x,t)) + u(x,t) \tag{1}$$

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where $t \in \mathbb{R}_+$ represents the time and $x \in \mathbb{R}$ is the spatial variable. We assume that $y(x, 0) = 0\ \forall x \in \mathbb{R}$. Depending on the definition of the operator $\mathcal{A}$ we may be considering different classes of plants. For example, if

$$\mathcal{A}(y(x,t)) = \kappa \frac{\partial^2 y(x,t)}{\partial x^2} \tag{2}$$

then we have the heat equation, which is a linear spatially-invariant (LSpI) system. There is a rich literature on LSpI systems, in particular on those described by the PDEs. Here, we will study a subclass of linear spatially-varying (LSpV) systems defined by the integral operator

$$\mathcal{A}(y(x,t)) = \int_{-\infty}^{\infty} h(x,x')y(x',t)dx'. \tag{3}$$

If the kernel h satisfies $h(x,x') = h(x-x')$ then the plant is LSpI, and these types of systems have been considered earlier, see e.g. Bamieh et al. (2002).

As far as the feedback control is concerned we will consider a centralized structure. More precisely, we assume that at each time instant $t_1 > 0$ the output $\{y(x,t),\ x \in \mathbb{R}, t \in [0, t_1]\ \}$ is available by an infinite dimensional sensor, and the distributed control signal $u(x, t_1)$ is generated by an infinite dimensional actuator. Such infinite dimensional actuators and sensors may be realized approximately by using arrays of closely spaced (mini-micro-nano etc.) sensors and actuators as shown in Figure 1.

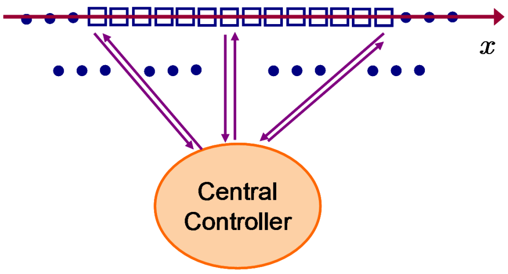

Fig. 1. A Distributed Parameter System with Arrays of Actuators and Sensors and Centralized Controller

Let a basis for $\mathcal{D}(\mathcal{A})$ be $\{\phi_k(x)\ ,\ k = 1, 2, \ldots\}$ and assume that the input and the corresponding output can be expressed in the form

$$u(x,t) = \sum_{k=1}^{\infty} \beta_k(t)\phi_k(x)\ , \quad y(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\phi_k(x)$$

where

$$\tilde{\alpha}(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \end{bmatrix} \quad \text{and} \quad \tilde{\beta}(t) = \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \\ \vdots \end{bmatrix}$$

are time coefficients and they satisfy

$$\dot{\tilde{\alpha}}(t) = \mathbf{A}\tilde{\alpha}(t) + \tilde{\beta}(t)$$

with the operator $[\mathbf{A}]_{\ell k} = a_{\ell k}$ defined by

$$\mathcal{A}(\phi_k(x)) = \sum_{\ell=1}^{\infty} a_{\ell k}\phi_\ell(x).$$

We now consider a spatial discretization and finite dimensional approximation in the form

$$u(x,t) \approx \sum_{k=1}^{n} \beta_k(t)\phi_k(x)\ , \quad y(x,t) \approx \sum_{k=1}^{n} \alpha_k(t)\phi_k(x)$$

where $\alpha(t) = [\alpha_1(t), \ldots, \alpha_n(t)]^{\mathrm{T}}$, $\beta(t) = [\beta_1(t), \ldots, \beta_n(t)]^{\mathrm{T}}$, and they satisfy

$$\dot{\alpha}(t) = A\alpha(t) + \beta(t) \tag{4}$$

where A is an $n \times n$ matrix whose entries are $a_{\ell k}$, $1 \leq \ell, k \leq n$.

The structure of A depends on the choices of ϕ_k's and the nature of the operator $\mathcal{A}$. For example, standard finite element methods on the heat equation, (2), leads to an A which is tri-diagonal, see e.g. Gockenbach (2002). For LSpI systems, (3) with $h(x,x') = h(x-x')$, the matrix A can be put in the form of a Toeplitz matrix. On the other hand, for general LSpV systems the matrix A may have arbitrary structure.

In this paper we will be considering LSpV plants for which the matrix A is represented, or approximated, by the form

$$A = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a}) \tag{5}$$

where $\mathcal{F}^a$ is the ath power fractional Fourier transform (FrFT) operator, with $0 \leq a \leq 1$, Λ is a diagonal matrix and $\Re$ takes the real part. If A contains an imaginary part, then a better representation (or approximation) can be found in the form $\mathcal{F}^{-a}\Lambda\mathcal{F}^{a}$. But here we consider real systems only where the operator A is real, so the use of $\Re$ is necessary. Also, note that in (5), if $a = 0$, then A is diagonal; and $a = 1$ means the system is LSpI and can be diagonalized by using the usual Fourier transform as a similarity transform. In this sense, the representation (5) captures these two extreme cases. We should also point out that for the LSpI systems in the form

$$\frac{\partial y(x,t)}{\partial t} = \int_{-\infty}^{\infty} h(x-x')y(x',t)dx' + u(x,t)$$

by applying the Fourier transform

$$\mathcal{F}(\mathcal{A}(y(x,t)) = H(\omega_x)Y(\omega_x,t)$$

we get

$$\frac{\partial Y(\omega_x,t)}{\partial t} = H(\omega_x)Y(\omega_x,t) + U(\omega_x,t)$$

which is a classical first order plant parameterized by ω_x. Controller design for these types of plants has been studied by Bamieh et al. (2002). In their approach, rather than dealing with a large size Toeplitz matrix, a scalar parameter ω_x is introduced. When we deal with an LSpV system we cannot use this approach.

Q1. What type of systems are in the form (5) for some diagonal Λ and an $a \in (0\ , 1)$?

Q2. What is the advantage of putting A in the form (5)?

The above questions will be discussed in the next section. Then in Section 4 we use this particular implementation of A in realtime implementation of state feedback controllers.

3. FRACTIONAL FOURIER TRANSFORM AND FILTERING

We can see the operator A as a filter in the spatial coordinates; its realization in the form (5) is a filtering in the fractional Fourier domain. Let ν be an $n \times 1$ vector and define $\mu = A\nu$. For a general $n \times n$ matrix A, once ν is given, we need to complete n^2 multiplication operations to compute μ. On the other hand, when A is in the form (5), by using the fast computation algorithms of the fractional

Fourier transform the number of computations can be reduced from n^2 to the order of $n \log_2(n)$. This gives a partial answer to Q2. Later, in the next section, we will see how this can be important in realtime implementation of the state feedback controllers for the class of LSpV systems considered.

3.1 The Fractional Fourier Transform (FrFT)

The ath order FrFT of a function $f(x)$ is given by

$$f_a(x') = \mathcal{F}^a(f(x)) = \int_{-\infty}^{\infty} K_a(x,x') f(x) dx \qquad (6)$$

where

$$K_a(x,x') = M_a \; exp[i\pi(x^2 cot(\bar{a}) - 2xx' csc(\bar{a}) + x'^2 cot(\bar{a}))]$$

$M_a = \sqrt{1 - icot(\bar{a})}$ and $\bar{a} = a\;\pi/2$.

The samples of the transform of a signal can be computed fast if most of the signal energy is confined in an ellipse of the Wigner plane (space-frequency plane) as in Figure 2, see e.g. Ozaktas et al. (1996), Ozaktas et al. (2001).

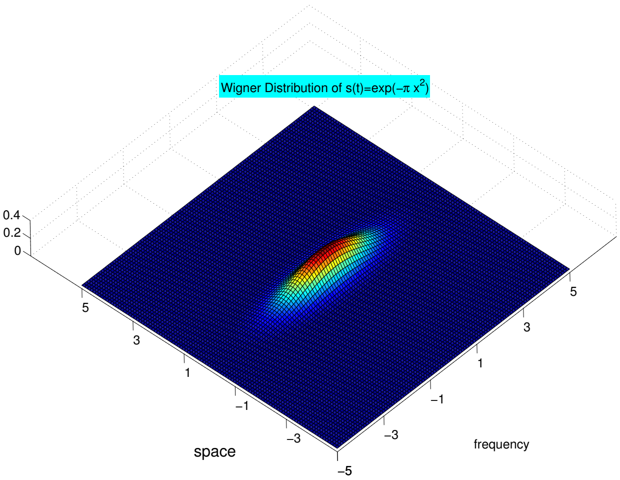

Fig. 2. Example of a signal whose energy is confined to a circle in the Wigner plane

There are different methods for fast calculation of the FrFT, see e.g. Ozaktas et al. (1996). Samples of the transform of a signal can be written as

$$f_a(\frac{m}{2R}) \approx \frac{M_a}{2R} exp[i\pi(cot(\bar{a}) - csc(\bar{a}))(m/2R)^2]$$
$$.\sum_{n=-N}^{N} exp[i\pi csc(\bar{a})((m-n)/2R)^2]$$
$$.\; exp[i\pi(cot(\bar{a}) - csc(\bar{a}))(n/2R)^2] f(\frac{n}{2R})$$

Here R is the diameter of the circle in the Wigner plane containing the signal, $2N+1$ is the number of samples and $N = R^2$. Last equation is a vector multiplication followed by a convolution, followed by a multiplication: all these can be computed fast. Since there are finitely many samples in the above, some error is introduced. The error can be reduced by increasing the number of samples and therefore decreasing the energy falling outside of the circle/ellipse in Wigner plane. We refer the reader to Ozaktas et al. (1996, 2001) for further details.

3.2 Filtering in the Fractional Fourier Domains

The basic filtering configuration in the ath Fourier domain is given by

$$\widehat{f}(x) = \mathcal{F}^{-a} \Lambda \mathcal{F}^a(f(x))$$

The above operations are done in three steps: (i) transform the input signal to ath fractional Fourier domain, (ii) element-wise multiply it with the diagonal of Λ, and (iii) transform it back to the original domain. Since FrFT can be computed fast, filter can be implemented fast. For a given real f the filtered output $\widehat{f}$ is not necessarily real. We can see the diagonal matrix Λ as the filter in the fractional Fourier domain. Furthermore, it can be shown that

$$\widehat{f}(x) = \mathcal{F}^{-a} \Lambda \mathcal{F}^a(f(x)) = \int_{-\infty}^{\infty} L(x,x'') f(x'') dx''$$

where

$$L(x,x'') = \int_{-\infty}^{\infty} K_a(x',x) g(x') K_{-a}(x'',x') dx'$$
$$= \frac{1}{|sin(\bar{a})|} \; e^{-i\pi cot(\bar{a})(x^2 - x''^2)} \; G(-csc(\bar{a})(x - x''))$$

and $\mathcal{F}(g) = G$. Here we can see the multiplication with g as filtering in the ath Fourier domain (Λ operator). Note that $L(x,x'')$ is not a function of $(x - x'')$.

Real part of the filter kernel can be given as
$L_r(x,x'') = H_c\; G_r(csc(\bar{a})(x-x'')) - H_s\; G_i(csc(\bar{a})(x-x''))$
Similarly, the imaginary part is
$L_i(x,x'') = H_s\; G_r(csc(\bar{a})(x-x'')) + H_c\; G_i(csc(\bar{a})(x-x''))$
where G_r and G_i are real and imaginary parts of G, respectively, and

$$H_s(x,x'') = \frac{1}{|sin(\bar{a})|} sin(\pi cot(\bar{a})(x^2 - x''^2))$$
$$H_c(x,x'') = \frac{1}{|sin(\bar{a})|} cos(\pi cot(\bar{a})(x^2 - x''^2))$$

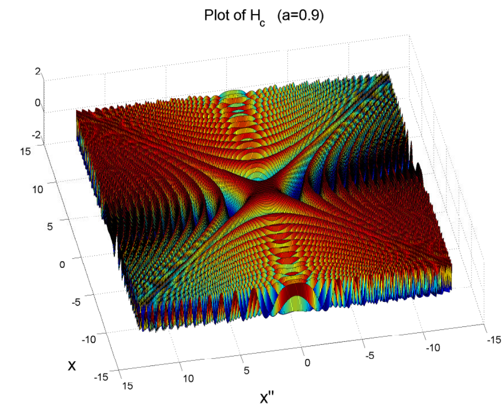

Fig. 3. The function $H_c(x,x'')$ for $a = 0.9$.

The above discussion answers the first question, Q1, posed in the previous section. That is, LSpV systems for which the matrix A is in the form (5) are characterized by H_c and H_s. The real part of the filter kernel L_r is determined by these two functions and G, which is the free parameter in this characterization.

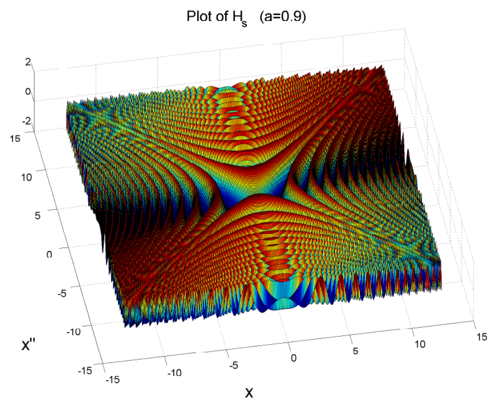

Fig. 4. The function $H_s(x, x'')$ for $a = 0.9$.

4. CONTROLLER IMPLEMENTATION

Now consider the feedback control of the system represented by (4) and (5). We restrict ourselves to state feedback control in the form

$$\beta(t) = -K\alpha(t) + \vartheta(t)$$

where K is the $n \times n$ matrix to be determined from control objectives, and ϑ represents the disturbance. Let us design K for pole placement and decoupling, i.e. choose

$$K = A - \Lambda_D \quad (7)$$

where Λ_D is a diagonal matrix and its entries are the desired closed loop poles in $\mathbb{R}_-$. In this case the feedback system satisfies

$$\dot{\alpha}(t) = \Lambda_D \alpha(t) + \vartheta(t).$$

4.1 Computational Issues

Implementation of this controller means that at each time instant t we need to compute

$$-K\alpha(t) = \Lambda_D \alpha(t) - A\alpha(t)$$

But instantaneous implementation is not possible. So, consider sampling time instants t_k, $k = 0, 1, 2, \ldots$, and note that at best we can generate control signal at time t_{k+1} from measurements up to t_k, i.e.

$$\beta(t_{k+1}) = -K\alpha(t_k) = \Lambda_D \alpha(t_k) - A\alpha(t_k)$$

Can we perform the multiplication $-K\alpha(t_k)$ in one time step? Recall that for an arbitrary K, in direct implementation, we need n^2 multiplication operations. Let us assume that each multiplication can be done in τ_m amount of time (typically today's DSP chips perform 2×10^8 multiply and accumulate operations in one second; for such an example we can think $\tau_m = 5 \times 10^{-9}$sec). Therefore, to complete this direct controller implementation within one sampling time, τ_s, we must satisfy

$$\tau_{\text{direct}} := n^2 \ \tau_m \leq \tau_s.$$

In other words, there is a limit on how fast sampled-data implementation can be. On the other hand, when K is in the form (7) with A given by (5), fast implementation of A can be done using FrFT as outlined in Section 3. If we follow this fast implementation approach the number of multiplication operators necessary to perform is $8n + 4n\log_2(n)$. Thus, in this case the lower bound for τ_s can be reduced to $\tau_{\text{fast}} := (4n\log_2(n) + 8n)\ \tau_m$. In general, if $n = 2^m$ then we have

$$\rho := \frac{\tau_{\text{direct}}}{\tau_{\text{fast}}} = \frac{2^{m-2}}{m+2}.$$

m	9	10	11	15	20
ρ	11.6	21.3	46.5	482	11915

The main benefit of fast implementation is captured by ρ. Compared to direct implementation, fast implementation of (7) allows ρ times faster sampling time.

Let us now leave the sampling time issue aside and discuss the effect of time delay due to number of multiplications involved in direct and fast implementations. For simplicity, consider the continuous time case and assume that computational delay is τ, as shown in Figure 5. We can think that in direct implementation τ is ρ times higher than that of the fast implementation. Assume $\Lambda_D = -r\ I$, $r > 0$.

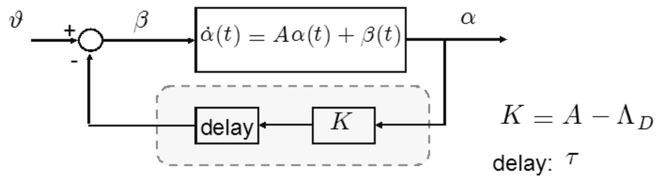

Fig. 5. State Feedback Controller with Computational Delay

The small gain theorem says that the feedback system is stable if $\|K\| < 1/\tau$, i.e. if there exists $r > 0$ such that $\|A + rI\| < \tau^{-1}$ then this system can be made stable. It is clear that r should not be too large compared to τ^{-1}.

On the other hand, if r is large, there is a danger for instability as shown below. To see this, let us examine the characteristic equation

$$\det(\ I - (I + r^{-1}A)\ \frac{r}{s+r}\ (1 - e^{-\tau s})\ ) = 0.$$

If r is large enough that $I + r^{-1}A$ has an eigenvalue $\lambda \in (0.5\ ,\ 1)$, with the corresponding eigenvector v, then we have

$$\left(s + r - \lambda r + \lambda r e^{-\tau s}\right) v = 0$$

Now check the stability of the feedback system formed by

$$G(s) = \frac{\lambda r}{s + r(1 - \lambda)}$$

and the delay element $e^{-\tau s}$. A simple delay margin analysis (see e.g. Ozbay (2000)) shows that the feedback system is stable if and only if

$$\frac{\pi - \tan^{-1}\left(\frac{\sqrt{2\lambda - 1}}{1 - \lambda}\right)}{r\sqrt{2\lambda - 1}} > \tau$$

When $r \gg \|A\|$ then $0 < (1 - \lambda) \ll 1$ and the above stability condition simplifies to $\frac{\pi}{2} > r\ \tau$.

Clearly, if the design is too aggressive (i.e. r is too large) or delay, τ, is too large, then the feedback system becomes unstable. Therefore, in addition to the limit on the sampling time, there is a limit on how aggressive the control design can be. Thus we can say that the fast implementation using FrFT allows the controller to be ρ times more aggressive (i.e. r can be ρ times larger) compared to the controller which can be used in the direct implementation.

4.2 Example

Let us consider a system for which the A matrix has the structure shown in Figure 6. It corresponds to a diagonal operator Λ, whose (k,k)th entry is in the form $e^{-c(k-512)^2}$, where c is a positive constant, and the fractional Fourier power is chosen as $a = 0.7$.

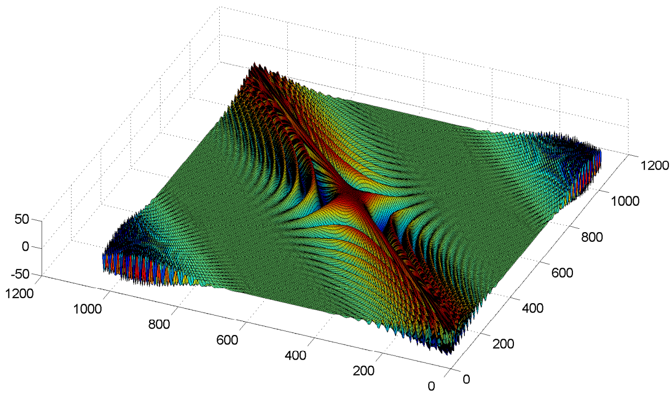

Fig. 6. Structure of the matrix A

In this case $n = 1024 = 2^{10}$, which means that the fast implementation allows more than $\rho \approx 21$ times faster sampling. Accordingly, we choose sampling time in such a way that the computation of the control signal is done within one sampling period, but in direct implementation control signal can be generated in about 20 sampling periods. The desired closed loop poles are chosen as -200, i.e. $\Lambda_D = -200\, I$.

The closed loop simulations are done for the disturbance $\vartheta(t) = [\vartheta_1(t), \ldots, \vartheta_{1024}(t)]^{\mathrm{T}}$, where $\vartheta_k(t) = \sin(\frac{2k\pi}{1024})\mathbb{U}(t)$, where $\mathbb{U}(t)$ represents the unit step function. The resulting output norm is shown as solid line in Figure 7. We see that the feedback system is stable. While the direct implementation is 20 times slower, it leads to an unstable feedback system.

The same simulation is done for the case where fast implementation of K is taken, ($K = A - \Lambda_D$ with $A = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$), but the A matrix of the original plant is uncertain (entries of this matrix are perturbed by uniformly distributed random numbers with values in $\pm 50\%$ of the nominal values), see dashed lines in Figure 7. From this figure we see that the closed loop system is robustly stable under this type of uncertainty.

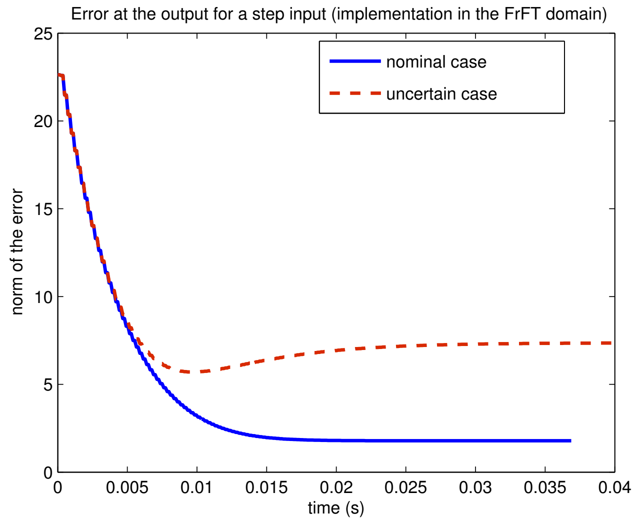

Fig. 7. Norm of the output under fast implementation: nominal and uncertain cases.

Other examples of A matrices that can be represented in the form $A_p = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$ are shown in Figures 8, 9. Similar time domain results can be obtained for these and other plants in this class.

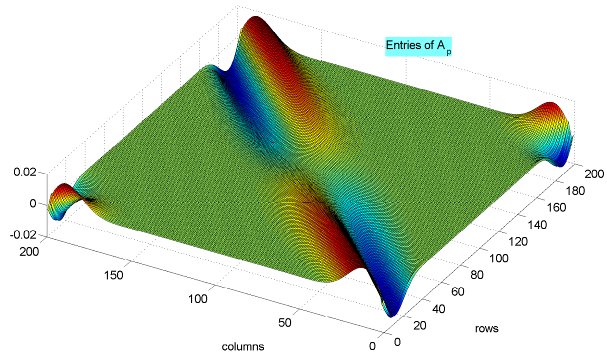

Fig. 8. Example of $A_p = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$

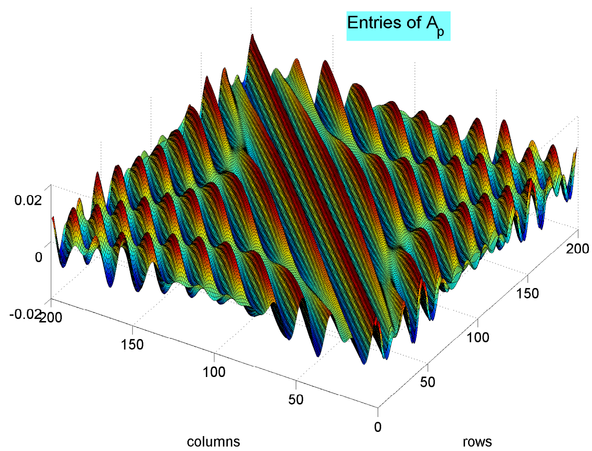

Fig. 9. Example of $A_p = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$

5. CONCLUSIONS AND FUTURE DIRECTIONS

We considered large scale systems (the number of states larger than $n = 2^9$) which come from spatial discretization of a class of linear spatially-varying distributed parameter systems. Model based controller require n^2 multiplication during one sampling period. Clearly, this puts a restriction on how small the sampling time can be. On the other hand, if the desired closed loop eigenvalues have large magnitude, then the sampling time have to be small in order to avoid instability. To solve this problem we proposed a fast implementation which is based on the fractional Fourier transform. The computational gain in this approach ranges from a factor of $\rho > 11$ for $n = 512$, to $\rho = 482$ for $n = 32768$. Therefore, the alternative solution of using ρ parallel processors can be costly or infeasible for larger values of n.

The class of plants considered here are assumed to have an A-matrix which is in the form $A = \Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$. What type of physical systems can be written in this form is a subject of future study. In fact, one may perhaps approximate a large class of A by $\Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$. Of course when such an approximation is used in the controller implementation, the approximation error must be explicitly computed and taken into account at the design stage.

In this paper we did not go into a detailed discussion of what type of distributed parameter systems admit a large scale system structure considered in the paper (the spatial variable x was assumed to be in $\mathbb{R}$). We believe that different geometries and periodic structures in the spatial variable are also possible.

The time dynamics of the plant considered here was first order. But with standard state-space methods, higher order time dynamics can also be handled in this setting. The control objective here was simple pole-placement. A large class of general model-based controllers (observer + state feedback) can also be considered in this framework. The key is again implementation of the matrix multiplication when the A-matrix can be written in the form $\Re(\mathcal{F}^{-a}\Lambda\mathcal{F}^{a})$.

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