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Coordination of Multi-agent Systems with Communication Delays ⋆

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Abstract: In this paper, we investigate coordination of a network of second-order dynamic agents under communication delays. Based on the frequency-domain analysis and matrix theory, the necessary and sufficient conditions for the system converging to stationary consensus and dynamic consensus are obtained, respectively. The conditions depend on the communication delay, the eigenvalues of the Laplacian matrix, and the interconnection topology of the network. Moreover, we apply the consensus algorithm to the formation control of the multi-agent system with communication delays. The agents in the system can achieve arbitrary desired formation pattern, and the formation moves in a desired velocity. Simulation results illustrate the correctness of the results.

Keywords: Coordination; Consensus; Formation control; Multi-agent systems; Communication delays.

1. INTRODUCTION

Recently, coordination control of multi-agent systems has attracted a lot of interest in a variety of research communities including biology, robotics, communications and sensor networks, artificial intelligence, automatic control, etc (Pettersen et al., 2006). In the coordination control of multi-agent systems, each agent updates its state according to the target (if any) and the information of the states of its neighbors.

Consensus problem is one of the most important issues in the coordination control of multi-agent systems, which requires that the outputs of several spatially distributed agents or processors reach a common value without recurse to a central coordinator or global communication (Spanos et al., 2004). In 1995, Vicsek et al. (1995) proposed a simple discrete-time model of multi-autonomous agent systems, and provided various simulations which demonstrated the agents' heading consensus phenomena. Jadbabaie et al. (2003) studied the linearized Vicsek's model and proved that all the agents converge to a common steady state provided that the digraph formed by the agents is jointly connected. The consensus problem of the multi-agent system modeled by first-order integrators with fixed or switched communication topology has been extensively studied in Olfati-Saber and Murray (2004) and Moreau (2005). In reality, however, many vehicles can not be controlled directly by their speeds and their accelerations should be used as force-based control. So a double integrator is often needed to model the dynamics of the vehicles. Since both position consensus and speed consensus are involved, the consensus problem of the multi-agent system modeled by double integrators is more challenging. Ren (2006) proposed a consensus algorithm to solve the problem, and get some sufficient conditions for the system converging to a dynamical consensus, i.e., all the agents asymptotically converge to time-varying consensus value.

With non-negligible communication delays, the consensus analysis for multi-agent systems becomes much more difficult. For the system modeled by first-order integrators, some stability results for the consensus problem were obtained for the multi-agent system with identical communication delay (Olfati-Saber and Murray, 2004; Moreau, 2004). Base on the contraction theory and the wave variable method, Wang and Slotine (2006) studied the consensus problem of the multi-agent system with diverse communication delays. The topology graph in their analysis is connected and bidirectional or unidirectional formed in closed rings. Recently, Liu and Tian (2007) proposed a protocol to solve the consensus problem of the multi-agent system with diverse communication delays, and the results can be applied to networks with directed topology graphs with nonsymmetric weights. However, little attention has been payed to the consensus of the multi-agent system of double integrators with communication delays. Hu and Hong (2007) considered a leader-following consensus problem of multi-agent systems with identical communication delays. Under a certain bound of time delay, the connectedness condition was obtained for the convergence to the leader's state with fixed or switched topology. In their analysis, they didn't explicitly study the relationship between the communication delay and the consensus convergence.

Another important issue in the coordination control of multi-agent systems is the formation control, which requires each agent moves according to the prescribed trajectory, and all the agents keep a certain spatial formation pattern at the same time. To achieve this goal, many

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typical control strategies have been developed, such as leader-follower, virtual structure, potential functions, etc. Besides these methods, Lin et al. (2005) and Ren (2006) extended consensus algorithms to the formation control and get sufficient conditions for the multi-agent systems converging to arbitrary desired formation pattern.

The paper is organized as follows. Some definitions for directed graphs are given in Section 2. In Section 3, we present the formulation of consensus problem of multi-agent system modeled by double integrators with communication delays, and get the sufficient and necessary conditions for the system converging to stationary consensus and dynamical consensus respectively. In Section 4, consensus protocol is applied to the formation control of multi-agent systems with communication delays so that the system asymptotically achieves the desired geometric formation pattern in the plane, and the velocity of the formation asymptotically converges to the desired velocity. Simulations illustrate the correctness of the above results in Section 5. The conclusion is given in Section 6.

2. PRELIMINARIES

A weighted directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order n consists of a set of vertices $\mathcal{V} = \{v_1, ..., v_n\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in R^{n\times n}$ with nonnegative adjacency elements a_{ij}. The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, ..., n\}$. An edge of the weighted digraph $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j) \in \mathcal{E}$, i.e., e_{ij} is a directed edge from v_i to v_j. We assume that the adjacency elements associated with the edges of the digraph are positive, i.e., $a_{ij} > 0 \Leftrightarrow e_{ij} \in \mathcal{E}$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$.

In the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the out-degree of node i is defined as: $\deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}$. Let $\mathcal{D}$ be the diagonal matrix with the out-degree of each node along the diagonal, which is called as the degree matrix of $\mathcal{G}$. The Laplacian matrix of the weighted digraph is defined as $L = \mathcal{D} - \mathcal{A}$.

If there is a path in $\mathcal{G}$ from one node v_i to another node v_j, then v_j is said to be *reachable* from v_i, written $v_i \to v_j$. If not, then v_j is said to be not reachable from v_i, written $v_i \not\to v_j$. If a node is reachable from every other node in the digraph, then we say it *globally reachable*. A digraph is *strongly connected* if every two of its nodes, say v and u, are such that v is reachable from u and u is reachable from v. Thus, the connectedness of the digraph that has only one globally reachable node is much weaker than strong connectedness of the digraph.

In this paper, we just consider static topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, i.e., the connection of the nodes in the digraph $\mathcal{G}$ does not change with time.

3. CONSENSUS WITH COMMUNICATION DELAYS

In a multi-agent system with n agents, each agent can be considered as a node in a digraph, and information flow between two agents can be regarded as a directed path between the nodes in the digraph. Thus, the interconnection topology in a multi-agent system can be described as a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$.

Consider the multi-agent systems with each agent's dynamics described as

$$\begin{aligned} \dot{\xi}_i &= \zeta_i, \\ \dot{\zeta}_i &= a_i, \quad i \in \mathcal{I}, \end{aligned} \tag{1}$$

where $\xi_i \in R$, $\zeta_i \in R$, and $a_i \in R$ are the position, velocity and acceleration, respectively, of agent i.

3.1 Stationary consensus

Firstly, we investigate the stationary consensus of the system (1). We say that the system (1) asymptotically converges to a stationary consensus, if,

$$\lim_{t\to\infty} \xi_i(t) = c, \quad \lim_{t\to\infty} \zeta_i(t) = 0, \quad \forall i \in \mathcal{I},$$

where c is a constant.

Consider the following consensus protocol (Ren, 2006)

$$a_i = -\gamma\zeta_i - \sum_{v_j\in N_i} (a_{ij}(\zeta_i - \zeta_j) + \gamma a_{ij}(\xi_i - \xi_j)), \tag{2}$$

where $\gamma > 0$, N_i denotes the neighbors of agent i, and $a_{ij} > 0$ is the adjacency element of $\mathcal{A}$ in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. When there exist non-negligible communication delays for the information transmission between agents, the algorithm (2) turns to be

$$\begin{aligned} a_i = -\gamma\zeta_i &- \sum_{v_j\in N_i} a_{ij}(\zeta_i(t-\tau_{ij}) - \zeta_j(t-\tau_{ij})) \\ &-\gamma \sum_{v_j\in N_i} a_{ij}(\xi_i(t-\tau_{ij}) - \xi_j(t-\tau_{ij})), \end{aligned}$$

where communication delay $\tau_{ij} > 0$ corresponds to information flow from agent j to agent i, i.e., the edge $e_{ij} \in \mathcal{E}$ in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. In order to analyze the relationship between the communication delays and the Laplacian matrix L, we assume the delays are identical, i.e., $\tau_{ij} = \tau$, and get

$$\begin{aligned} a_i = -\gamma\zeta_i &- \sum_{v_j\in N_i} a_{ij}(\zeta_i(t-\tau) - \zeta_j(t-\tau)) \\ &-\gamma \sum_{v_j\in N_i} a_{ij}(\xi_i(t-\tau) - \xi_j(t-\tau)), \end{aligned} \tag{3}$$

With the consensus protocol (3), the closed-loop form of the system (1) becomes

$$\begin{aligned} \dot{\xi}_i &= \zeta_i, \\ \dot{\zeta}_i &= -\gamma\zeta_i - \sum_{v_j\in N_i} a_{ij}(\zeta_i(t-\tau) - \zeta_j(t-\tau)) \\ &\quad -\gamma \sum_{v_j\in N_i} a_{ij}(\xi_i(t-\tau) - \xi_j(t-\tau)). \end{aligned} \tag{4}$$

Theorem 1. Consider the network (4) of n agents with a static interconnection topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that has a globally reachable node. The system (4) asymptotically converges to a stationary consensus, if and only if

$$\tau < \min_{i\in\mathcal{I},\lambda_i\neq 0} \frac{1}{|\lambda_i|}(\frac{\pi}{2} - |\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})|), \tag{5}$$

where $\lambda_i, i \in \mathcal{I}$, is the eigenvalue of L. Moreover, when $\tau = \min_{i\in\mathcal{I},\lambda_i\neq 0} \frac{1}{|\lambda_i|}(\frac{\pi}{2} - |\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})|)$, the system (4) has periodic oscillations.

Proof. The system (4) can be rewritten as

$$\begin{aligned}\dot{\xi}(t) &= \zeta(t),\\ \dot{\zeta}(t) &= -\gamma\zeta(t) - L\zeta(t-\tau) - \gamma L\xi(t-\tau),\end{aligned} \tag{6}$$

where $\xi(t) = [\xi_1(t), \cdots, \xi_n(t)]^T$ and $\zeta(t) = [\zeta_1(t), \cdots, \zeta_n(t)]^T$. Taking the Laplace transform of the system (6), we get

$$\begin{aligned}s\xi(s) &= \zeta(s),\\ s\zeta(s) &= -\gamma\zeta(s) - \mathrm{e}^{-s\tau}L\zeta(s) - \gamma\mathrm{e}^{-s\tau}L\xi(s),\end{aligned}$$

where $\xi(s)$ and $\zeta(s)$ are the Laplace transforms of $\xi(t)$ and $\zeta(t)$ respectively. The above equation can also be written as

$$s^2\xi(s) = -\gamma s\xi(s) - \mathrm{e}^{-s\tau}Ls\xi(s) - \gamma\mathrm{e}^{-s\tau}L\xi(s).$$

Thus, we get the characteristic equation about $\xi(t)$ as follows:

$$\det(s^2 + (\gamma I + \mathrm{e}^{-s\tau}L)s + \gamma\mathrm{e}^{-s\tau}L) = 0, \tag{7}$$

i.e.,

$$\det[(sI + \gamma I)(sI + \mathrm{e}^{-s\tau}L)] = 0.$$

Therefore, (7) is equivalent to

$$\det(sI + \gamma I) = 0, \tag{8}$$

or

$$\det(sI + \mathrm{e}^{-s\tau}L) = 0. \tag{9}$$

Obviously, the equation (8) has n roots at $s = -\gamma$, and we will prove that all the roots of (9) are in the open left half plane or at $s = 0$ in the following.

Because $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a globally reachable node, 0 is the simple eigenvalue of L, i.e., $\mathrm{rank}(L) = n-1$, and all the other eigenvalues have positive real parts(Lin et al., 2005). Denote the eigenvalues of L as $\lambda_i, i = 1, \cdots, n$. We assume $\lambda_1 = 0$ and $\mathrm{Re}(\lambda_i) > 0, i = 2, \cdots, n$. Thus, (9) becomes

$$s\prod_{i=2}^{n}(s + \lambda_i\mathrm{e}^{-s\tau}) = 0. \tag{10}$$

It is obvious that (10) has a simple root at $s = 0$. When $s \neq 0$, we investigate the roots of the following equation for $\lambda_i, i = 2, \cdots, n$

$$s + \lambda_i\mathrm{e}^{-s\tau} = 0, \tag{11}$$

i.e.,

$$1 + \lambda_i\frac{\mathrm{e}^{-s\tau}}{s} = 0. \tag{12}$$

Based on the Nyquist stability criterion, the roots of (12) lie on the open left half complex plane, if and only if the Nyquist curve $\lambda_i\frac{\mathrm{e}^{-\mathrm{j}\omega\tau}}{\mathrm{j}\omega}$ does not enclose the point $(-1, \mathrm{j}0)$ for $\omega \in R$. Defining

$$\begin{aligned}g_i(\omega) &= \lambda_i\frac{\mathrm{e}^{-\mathrm{j}\omega\tau}}{\mathrm{j}\omega}\\ &= \frac{|\lambda_i|}{\omega}\mathrm{e}^{-\mathrm{j}(\frac{\pi}{2}+\omega\tau-\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}))},\end{aligned}$$

we get $|g_i(\omega)| = \frac{|\lambda_i|}{|\omega|}$.

When $\omega \in (0, +\infty)$, $|g_i(\omega)|$ and $\arg(g_i(\omega)) = -(\frac{\pi}{2} + \omega\tau - \arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}))$ are all monotonously decreasing, where $\arg(.)$ denotes the phase. $g_i(\omega)$ crosses the negative real axis for the first time at

$$\omega_{c1} = \frac{1}{\tau}(\frac{\pi}{2} + \arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}).$$

So we obtain

$$|g_i(\omega_{c1})| = \frac{\tau|\lambda_i|}{\frac{\pi}{2} + \arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})}.$$

When $\omega \in (-\infty, 0)$, $|g_i(\omega)|$ is monotonously increasing, and $\arg(g_i(\omega)) = \frac{\pi}{2} - \omega\tau + \arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})$ is monotonously decreasing. $g_i(\omega)$ crosses the negative real axis for the last time at

$$\omega_{c2} = \frac{1}{\tau}(\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}) - \frac{\pi}{2}).$$

So we get

$$|g_i(\omega_{c2})| = \frac{\tau|\lambda_i|}{\frac{\pi}{2} - \arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})}.$$

From Nyquist stability criterion, the roots of (12) all have negative real part, if and only if

$$|g_i(\omega_{c1})| < 1 \text{ and } |g_i(\omega_{c2})| < 1,$$

holds. Thus, (9) has its roots in the open left half complex plane except for a root at $s = 0$, if and only if

$$|g_i(\omega_{c1})| < 1 \text{ and } |g_i(\omega_{c2})| < 1, \quad \forall i = 2, \cdots, n, \tag{13}$$

holds. By simple computation, (13) is equivalent to (5).

As proved above, the roots of (7) all have negative real parts except for a root at $s = 0$. Therefore, the state $\xi_i(t)$ of the system (4) converges to a steady state, i.e., $\lim_{t\to\infty}\xi_i(t) = \xi_i^\star, i \in \mathcal{I}$, and $\lim_{t\to\infty}\zeta_i(t) = 0, \forall i \in \mathcal{I}$ holds for (4). Thus, it is obtained from (4) that $L[\xi_1^\star, \cdots, \xi_n^\star]^T = 0$. Since $\mathrm{rank}(L) = n-1$ and $L[1, \cdots, 1]^T = 0$ based on the definition of the Laplacian matrix L, the roots of $L\xi^\star = 0$ can be expressed as $\xi^\star = c[1, \cdots, 1]^T$, where c is a constant. Therefore, the system (4) converges to a stationary consensus.

Similar to the proof of Theorem 10 in Olfati-Saber and Murray (2004), we can prove (7), i.e., (9), has roots on the imaginary axis except for $s = 0$ when $\tau = \min_{i\in\mathcal{I},\lambda_i\neq 0}\frac{1}{|\lambda_i|}(\frac{\pi}{2} - |\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})|)$, but we omitted the proof here for the length of the paper. Thus, the system (4) has periodic oscillations. Theorem 1 is proved.

3.2 Dynamical consensus

In the following, we investigate the dynamical consensus of system (1). We say that system (1) asymptotically converges to a dynamical consensus, if

$$\lim_{t\to\infty} \xi_i(t) = c + \int_0^t v(t)dt, \quad \lim_{t\to\infty} \zeta_i(t) = v(t), \quad \forall i \in \mathcal{I},$$

where c is a constant, and $v(t) \in R$ is the desired velocity of all the agents in the system (1).

The consensus algorithm for the dynamic consensus is taken as (Ren, 2006)

$$a_i = \dot{v} - \gamma(\zeta_i - v) - \sum_{v_j \in N_i} (a_{ij}(\zeta_i - \zeta_j) + \gamma a_{ij}(\xi_i - \xi_j)). \quad (14)$$

With communication delay τ between agents, the consensus algorithm (14) turns to be

$$a_i = \dot{v} - \gamma(\zeta_i - v) - \sum_{v_j \in N_i} a_{ij}(\zeta_i(t-\tau) - \zeta_j(t-\tau)) \\ -\gamma \sum_{v_j \in N_i} a_{ij}(\xi_i(t-\tau) - \xi_j(t-\tau)). \quad (15)$$

With (15), the closed-loop form of (1) is

$$\dot{\xi}_i = \zeta_i, \\ \dot{\zeta}_i = \dot{v} - \gamma(\zeta_i - v) - \sum_{v_j \in N_i} a_{ij}(\zeta_i(t-\tau) - \zeta_j(t-\tau)) \\ -\gamma \sum_{v_j \in N_i} a_{ij}(\xi_i(t-\tau) - \xi_j(t-\tau)). \quad (16)$$

Theorem 2. Consider the network (16) of n agents with a static interconnection topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that has a globally reachable node. If and only if

$$\tau < \min_{i\in\mathcal{I},\lambda_i\neq 0} \frac{1}{|\lambda_i|}\left(\frac{\pi}{2} - \left|\arctan\left(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)}\right)\right|\right), \quad (17)$$

where $\lambda_i, i \in \mathcal{I}$ is the eigenvalue of L, the system (16) asymptotically converges to a dynamical consensus.

Proof. Take the variable transformation as follows

$$\bar{\zeta}_i(t) = \zeta_i(t) - v(t), \quad \bar{\xi}_i(t) = \xi_i(t) - \int_0^t v(t)dt. \quad (18)$$

Hence, the dynamics of $\bar{\xi}_i(t)$ and $\bar{\zeta}_i(t)$ are given by

$$\dot{\bar{\xi}}_i = \bar{\zeta}_i, \\ \dot{\bar{\zeta}}_i = -\gamma\bar{\zeta}_i - \sum_{v_j \in N_i} a_{ij}(\bar{\zeta}_i(t-\tau) - \bar{\zeta}_j(t-\tau)) \\ -\gamma \sum_{v_j \in N_i} a_{ij}(\bar{\xi}_i(t-\tau) - \bar{\xi}_j(t-\tau)). \quad (19)$$

Form the proof of Theorem 1, if and only if (17) holds, the system (19) converges to a stationary consensus, i.e.,

$$\lim_{t\to\infty} \bar{\xi}_i(t) = c, \quad \lim_{t\to\infty} \bar{\zeta}_i(t) = 0, \quad \forall i \in \mathcal{I},$$

where c is a constant. Therefore, we get from (18)

$$\lim_{t\to\infty} \xi_i(t) = c + \int_0^t v(t)dt, \quad \lim_{t\to\infty} \zeta_i(t) = v(t), \quad \forall i \in \mathcal{I}.$$

Theorem 2 is proved.

4. FORMATION CONTROL WITH COMMUNICATION DELAYS

In this section, we apply the consensus analysis in the previous section to the formation control of multi-agent systems with communication delays. Consider the following multi-agent system modeled by double integrators

$$\dot{q}_i = p_i, \\ \dot{p}_i = u_i, \quad i \in \mathcal{I}, \quad (20)$$

where $q_i \in R^2$ denotes the position of agent i, $p_i \in R^2$ denotes its velocity, and $u_i \in R^2$ denotes its control input.

For multi-agent systems, there exist two ways to describe the geometric pattern in the plane (Lin et al., 2005). The first one is using the inter-agent distance $d_{ij} = \| q_i - q_j \|, v_j \in N_i$. In this way, the formation geometric pattern is determined by the constraints of the desired distance between linked agents, as in the rigid formation framework of Olfati-Saber and Murray (2002). The other way is specifying the position vector, $c_i^\star \in R^2$, of each agent with respect to a common coordinate frame (Ren, 2006; Lin et al., 2005). We adopt the latter way to describe the formation pattern in this paper. For simplicity of statement we just consider the time-invariant position vector $c_i^\star$. However, our discussions can be extended to the time-varying case directly (Ren, 2006). The control objectives are as follows.

a. All the agents asymptotically converge to a prescribed geometric pattern in the plane, which is characterized by $\{c_i^\star \in R^2, i \in \mathcal{I}\}$.

b. Each agent's velocity asymptotically approaches to a desired function $s(t)$.

For the system (20), the formation control algorithm based on the consensus protocol (14) is

$$u_i = \dot{s} - \gamma(p_i - s) - \sum_{v_j \in N_i} a_{ij}(p_i - p_j) \\ -\gamma \sum_{v_j \in N_i} a_{ij}((q_i - c_i^\star) - (q_j - c_j^\star)), \quad (21)$$

where $\gamma > 0$, N_i denotes the neighbors of agent i, and $a_{ij} > 0$ is the adjacency element of $\mathcal{A}$ in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. With communication delays between agents, the above algorithm turns to be

$$u_i = \dot{s} - \gamma(p_i - s) - \sum_{v_j \in N_i} a_{ij}(p_i(t-\tau) - p_j(t-\tau)) \quad (22) \\ -\gamma \sum_{v_j \in N_i} a_{ij}((q_i(t-\tau) - c_i^\star) - (q_j(t-\tau) - c_j^\star)),$$

where τ is the uniform communication delay.

With (22), the closed-loop form of the system (20) is

$$\dot{q}_i = p_i, \tag{23}$$
$$\dot{p}_i = \dot{s} - \gamma(p_i - s) - \sum_{v_j \in N_i} a_{ij}(p_i(t-\tau) - p_j(t-\tau))$$
$$-\gamma \sum_{v_j \in N_i} a_{ij}((q_i(t-\tau) - c_i^\star) - (q_j(t-\tau) - c_j^\star)).$$

Theorem 3. Consider the network (23) of n agents with a static interconnection topology $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that has a globally reachable node. If and only if

$$\tau < \min_{i \in \mathcal{I}, \lambda_i \neq 0} \frac{1}{|\lambda_i|}\left(\frac{\pi}{2} - |\arctan(\frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)})|\right), \tag{24}$$

where $\lambda_i, i \in \mathcal{I}$ is the eigenvalue of L, the system (23) asymptotically achieves the desired geometric formation pattern in the plane, and the velocity of the formation asymptotically approaches to $s(t)$.

Proof. Take the variable transformation as follows

$$\overline{p}_i(t) = p_i(t) - s(t), \quad \overline{q}_i(t) = q_i(t) - \int_0^t s(t)dt - c_i^\star.$$

Thus, the dynamics of $\overline{p}_i(t)$ and $\overline{q}_i(t)$ are given by

$$\dot{\overline{q}}_i = \overline{p}_i,$$
$$\dot{\overline{p}}_i = -\gamma\overline{p}_i - \sum_{v_j \in N_i} a_{ij}(\overline{p}_i(t-\tau) - \overline{p}_j(t-\tau))$$
$$-\gamma \sum_{v_j \in N_i} a_{ij}(\overline{q}_i(t-\tau) - \overline{q}_j(t-\tau)).$$

The above equation can be rewritten as

$$\dot{\overline{q}} = \overline{p}, \tag{25}$$
$$\dot{\overline{p}} = -\gamma\overline{p} - (L \otimes I_{2\times 2})\overline{p}(t-\tau) - \gamma(L \otimes I_{2\times 2})\overline{q}(t-\tau),$$

where $\otimes$ denotes the Kronecker product, $\overline{q} = [\overline{q}_1^T, \cdots, \overline{q}_n^T]^T$ and $\overline{p} = [\overline{p}_1^T, \cdots, \overline{p}_n^T]^T$. From the proof of Theorem 1 and the properties of Kronecker product, if and only if (24) holds, the system (25) asymptotically converges to a stationary consensus. Therefore, the system (23) asymptotically achieves the desired formation pattern, and the velocity of the formation asymptotically converges to $s(t)$. Theorem 3 is proved.

5. SIMULATION

In order to illustrate the correctness of the above discussions, we consider the consensus problem and formation control of the multi-agent systems respectively in the following simulations.

Example 1. Consensus problem. Consider a network of five agents described by (4). The interconnection topology is described in Figure 1. The globally reachable node set of the digraph is $\{1, 2, 5\}$. The weights of the directed edges are: $a_{15} = 0.10$, $a_{21} = 0.15$, $a_{32} = 0.20$, $a_{43} = 0.25$, $a_{52} = 0.30$. The eigenvalues of Laplacian matrix L are: $\lambda_1 = 0$, $\lambda_2 = 0.25$, $\lambda_3 = 0.2$, $\lambda_{4,5} = 0.275 \pm \mathrm{j}0.1199$. Based on Theorem 1, the system (4) asymptotically converges to a stationary consensus if and only if $\tau < 3.865$(s). Choosing $\tau = 1.5$(s), $\gamma = 0.5$, and the initial states generated randomly, we obtain that the system converges to a

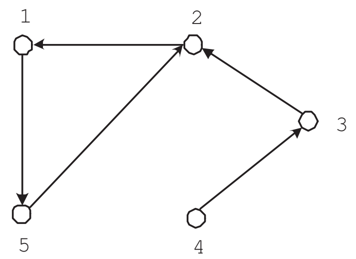

Fig. 1. The digraph composed of 5 agents.

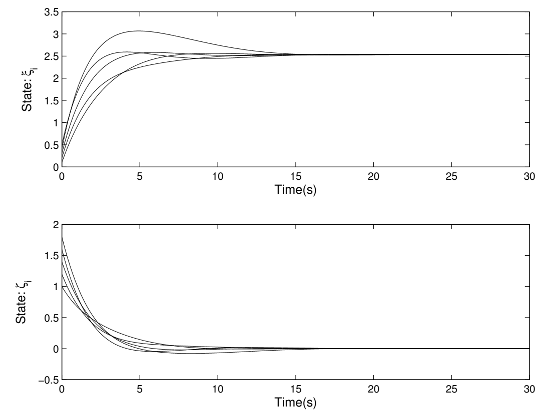

Fig. 2. Stationary consensus of the multi-agent systems.

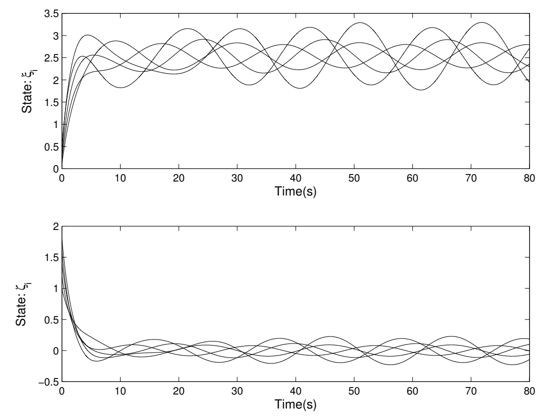

Fig. 3. Periodic oscillations in the multi-agent systems.

stationary consensus (see, Figure 2). When $\tau = 3.865$(s), the system has periodic oscillations (see, Figure 3). When $\tau > 3.865$(s), the system diverges, but we don't demonstrate the simulation for short.

With the same interconnection topology and link weights as above, we obtain from Theorem 2 that the system (16) asymptotically converges to a dynamical consensus if and only if $\tau < 3.865$(s). However, we don't demonstrate the simulation here for the length of the paper.

Example 2. Formation control. Consider a network of four agents described by (23). The interconnection topology is described in Figure 4. The globally reachable node set of the digraph is $\{1, 2, 4\}$. The weights of the directed edges are: $a_{14} = 0.20$, $a_{21} = 0.50$, $a_{32} = 0.60$, $a_{42} = 1.00$. The eigenvalues of the Laplacian matrix L are: $\lambda_1 = 0$, $\lambda_2 = 0.60$, $\lambda_{3,4} = 0.85 \pm \mathrm{j}0.2784$. From Theorem 3, the system (23) asymptotically achieves the desired geometric

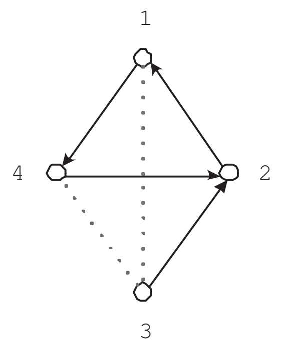

Fig. 4. The desired formation pattern: diamond.

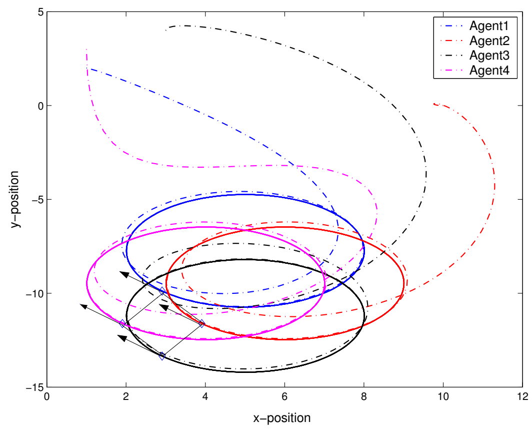

Fig. 5. Trajectories of the agents.

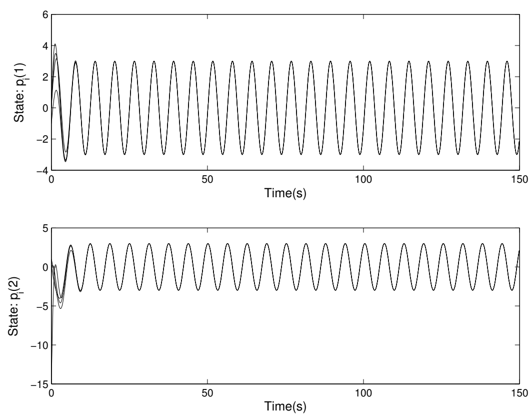

Fig. 6. Velocities of the agents.

formation pattern in the plane, and the velocity of the formation asymptotically converges to $s(t)$, if and only if $\tau < 1.402$(s). The desired formation pattern is a diamond (see, Figure 4), and $c_1^\star = [1, \sqrt{3}]^T, c_2^\star = [2, 0]^T, c_3^\star = [1, -\sqrt{3}]^T, c_4^\star = [0, 0]^T$. Choose $\tau = 0.5$(s), $\gamma = 0.5$, and the initial states generated randomly, and we assume the desired velocity $s(t) = [3 \sin t, 3 \cos t]^T$. Then, each agent's dynamic trajectory is illustrated in Figure 5, and the agents,velocities asymptotically converge to the desired velocity (see, Figure 6).

6. CONCLUSION

In this paper, the coordination control is investigated for the multi-agent system modeled by double integrators with directed interconnection topology. We study a simple consensus protocol for the consensus problem of the multi-agent system with communication delays. Using the frequency-domain analysis and matrix theory, we obtain the necessary and sufficient condition for the system asymptotically converging to stationary consensus and dynamical consensus, respectively. It is shown that under a fixed directed topology that has a globally reachable node, the system exhibits periodic oscillations when the delay equals the critical value, which is a function of the eigenvalues of the Laplacian matrix. Furthermore, we apply the consensus protocol to the formation control of the multi-agent system with communication delays. The agents can asymptotically achieve arbitrary prescribed geometric formation pattern in the plane, and the agent asymptotically converges to the desired velocity.

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