Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Nonlinear Output Feedback Control for Linear Systems with Input Saturation

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Abstract: In this paper, nonlinear output feedback control design for linear systems subject to input saturation is addressed. Main issues of this paper are 1. stability analysis of the feedback system for both of state and output feedback and 2. design of stabilizing nonlinear feedback law. Sufficient global stability conditions for both of state and output feedback are derived. Furthermore, a design approach based on analytical solution of partial differential equations is proposed. It is shown that a class of feedback laws can be explicitly obtained, which is parameterized by nonlinear functions.

1. INTRODUCTION

Linear systems with input saturation are commonly encountered due to the inherent constraint on actuators. The study of such systems has received great attention because it is not only practically important, but also the input nonlinearity induces various interesting behaviours such as local stability, limit cycle, performance degradation. Linear controllers which are designed disregarding the nonlinearity cannot bring out expected performances. It is conceivable that nonlinear control may achieve better performance than linear control for systems subject to constraints. Recently, several nonlinear control methods have been developed. Major approaches are nonlinear state feedback control (Hu and Lin, 2003, Chen et al, 2003), gain-scheduling control, anti-windup control (Mulder et al, 2001) and model predictive control (Bemporad et al, 2002).

In this paper, we aim at designing nonlinear output feedback laws stabilizing linear systems with input saturation. First, we consider state feedback stabilization. A global stability condition for the closed loop system is derived, which is characterized by PDMI (partial differential matrix inequality) about the state feedback law. Then, we propose a design approach based on analytical solution of PDME (partial differential matrix equation). Next, observer based output feedback stabilization is discussed. A global stability condition for the closed loop system is derived and the design example of the proposed method is shown.

As the related research, Hu and Lin, 2003 proposed the composite quadratic Lyapuonv function. This approach gives a flexible method of constructing nonlinear feedback laws. Chen et al, 2003 proposed a composite nonlinear feedback control law which consists of a linear feedback law and a nonlinear feedback law without any switching element. The design approach in this paper is different from theirs in that the structure of the feedback law is explicitly obtained by solving a PDME analytically. The each parameters of the feedback law are determined by a matrix inequality including nonlinear design parameters.

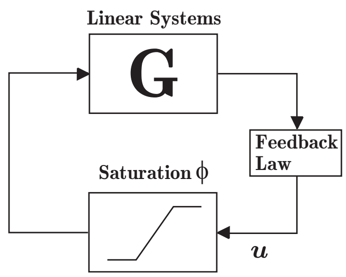

Fig. 1. Feedback control systems with saturation.

We consider n-th order single input linear systems with input saturation below:

$$G: \ \dot{x} = Ax + b\phi(u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^1 \tag{1}$$

where $\phi(\cdot)$ is a standard saturation function given by

$$\phi(u) = \mathrm{sgn}(u) \min\{|u|, u_m\} \tag{2}$$

where u_m denotes the maximum matnitude of the control input. Moreover, we make the following assumptions on the system (1).

Assumption 1. (A, b) is controllable.

Assumption 2. There exists a non-zero matrix $P = P^T \geq 0$ such that

$$A^T P + PA \leq 0. \tag{3}$$

Our purpose is to design globally stabilizing feedback laws. However, not all linear systems are globally stabilizable due to the input saturation. This assumption implies the poles of A are at least located on the closed left-half plane. Note that Assumption 2 does not necessarily require the boundness of open-loop impulse response since the double integrator is included in this class.

In this paper, the following notation will be used. Let A be an $n \times m$ matrix. A^+ denotes the pseudo-inverse matrix of A and $\ker A$ denotes the null space of A.

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10.3182/20080706-5-KR-1001.0495

2. STABILITY ANALYSIS - STATE FEEDBACK CASE

In this section, stability of the closed loop is analyzed for general state feedback case, i.e., $u = f(x)$. It is only assumed that the state feedback law $f(x)$ is at least C^0 for all x and piecewise C^1 with $f(0) = 0$. Then, the closed loop system is described as below:

$$\dot{x} = Ax + b\phi(f(x)) \tag{4}$$

2.1 Equilibria

The origin is not necessarily the unique equilibrium point under the nonlinear system (4). The following lemma clarifies the equilibria of the system (4).

Lemma 1. Consider the system (4). Then, the following statements hold:

1) If $\ker A = \{0\}$, the origin is the unique equilibrium point if and only if $\xi = 0$ is the unique solution of
$$f(-A^{-1}b\xi) = \xi, \quad \forall\xi \in [-u_m, u_m] \tag{5}$$
and $f(-A^{-1}bu_m) \le u_m$, $f(A^{-1}bu_m) \ge -u_m$.
2) If $\ker A \neq \{0\}$, the origin is the unique equilibrium point if and only if $\eta = 0$ is the unique solution of
$$f(\eta) = 0, \quad \forall\eta \in \ker A. \tag{6}$$

Proof The proof is based on the theory of linear matrix equations. First, we prove the case $\ker A = \{0\}$. The equilibrium point satisfies the equation $Ax + b\phi = 0$. Since A is nonsingular, $x = -A^{-1}b\phi$ holds. This means that the equilibrium point lies on a finite length line $\{-A^{-1}b\xi : \xi \in [-u_m, u_m]\}$. In the unsaturated region $|f(x)| \le u_m$, $\xi = f(x)$. Therefore, the equilibria exists iff there exists a $\xi \in [-u_m, u_m]$ such that $\xi = f(-A^{-1}b\xi)$. Obviously, the origin is the unique equilibrium point iff $\xi = 0$ is the unique solution of $\xi = f(-A^{-1}b\xi)$. In saturated region $|f(x)| > u_m$, $\xi = \pm u_m$. $f(-A^{-1}bu_m) \le u_m$ and $f(A^{-1}bu_m) \ge -u_m$ are necessary in order for the origin to be the unique equilibrium.

Next, we prove the case $\ker A \neq \{0\}$. $Ax + b\phi = 0$ has a solution x only if rank $[A\ b\phi]$ = rank A for $\phi \in [-u_m, u_m]$. Note that this rank condition is only a necessary condition because ϕ depends on $f(x)$. However, since (A, b) is controllable, rank $[A\ b\phi] = n$ when $\phi \neq 0$. Hence, rank $[A\ b\phi] = n >$ rank A. The solution x exists only if $\phi = 0$, i.e., $f(x) = 0$. As the result, the equilibrium point satisfies $Ax = 0$ and $f(x) = 0$. The origin is the unique equilibrium point iff $\eta = 0$ is the unique solution of $f(\eta) = 0$ for all $\eta \in \ker A$.

2.2 Stability Theorems

The stability analysis is based on a composite Lyapunov function, which is composed of a quadratic function and a potential energy function related to the actuator and the feedback law. Note that a simple quadratic form of Lyapunov function does not exist for the global stability if the plant is the double integrator. The following theorem guarantees the global stability of the system (4).

Theorem 1. Suppose that there exist a $P = P^T \ge 0$ and a $\lambda \in (0, 1)$ such that the following two conditions hold.

1) A PDMI (partial differential matrix inequality) below is satisfied.
$$\begin{bmatrix} A^TP + PA & Pb + A^T(\frac{\partial f}{\partial x})^T \\ b^TP + \frac{\partial f}{\partial x}A & \lambda\frac{\partial f}{\partial x}b + b^T(\frac{\partial f}{\partial x})^T\lambda \end{bmatrix} \le 0. \tag{7}$$
2) A Lyapunov function below is radially unbounded.
$$V(x) = x^TPx + 2\int_0^{f(x)} \phi(s)ds \tag{8}$$

Then, the state $x(t)$ of the system (4) converges globally to the largest invariant set $\mathcal{V}$ contained in

$$\Omega = \{x \in \mathbb{R}^n : Qx = 0,\ \frac{\partial f}{\partial x}b \cdot f(x) = 0\} \tag{9}$$

where Q satisfies

$$A^TP + PA = -Q^TQ. \tag{10}$$

Proof The proof is based on Lyapunov stability theory. By $P \ge 0$ and the condition 2), the Lyapunov candidate $V(x)$ given by (8) satisfies $V(x) \ge 0$ and is radially unbounded. Furthermore, $V(x)$ is continuously differentiable.

The time derivative of V is calculated as

$$\dot{V}(x) = \begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} A^TP+PA & Pb+A^T(\frac{\partial f}{\partial x})^T \\ b^TP+\frac{\partial f}{\partial x}A & \frac{\partial f}{\partial x}b+b^T(\frac{\partial f}{\partial x})^T \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix}. \tag{11}$$

Using $\lambda = \lambda_1\lambda_2$ with λ_1, $\lambda_2 \in (0, 1)$, we obtain

$$\dot{V}(x) = \begin{bmatrix} x \\ \phi \end{bmatrix}^T \begin{bmatrix} \lambda_1(A^TP+PA) & Pb+A^T(\frac{\partial f}{\partial x})^T \\ b^TP+\frac{\partial f}{\partial x}A & \lambda_2\frac{\partial f}{\partial x}b+b^T(\frac{\partial f}{\partial x})^T\lambda_2 \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \\ -(1-\lambda_1)\|Qx\|^2+(1-\lambda_2)\frac{\partial f}{\partial x}b\phi^2. \tag{12}$$

The negative semi-definiteness of the matrix inequality in the first term of (12) is equivalent to that there exists $\lambda \in (0, 1)$ such that the PDMI (7) holds. This equivalence can be easily proved by a congruence transformation $\mathrm{diag}\{\sqrt{\lambda_1}I, 1/\sqrt{\lambda_1}\}$. We can find λ_1, $\lambda_2 \in (0, 1)$ such that $\lambda = \lambda_1\lambda_2$ for any $\lambda \in (0, 1)$, and vice versa.

Hence, by the condition 1), $\dot{V}$ satisfies

$$\dot{V}(x) \le -(1 - \lambda_1)\|Qx\|^2 + (1 - \lambda_2)\frac{\partial f}{\partial x}b\phi^2 \le 0. \tag{13}$$

In addition, $\dot{V} = 0$ holds only in the region Ω.

By the above argument, we conclude $\dot{V}(x) \le 0$ for all $x \in \mathbb{R}^n$. According to LaSalle's invariant principle, x converges to the largest invariant set $\mathcal{V} \subset \Omega$.

Theorem 1 guarantees that the state converges to some invariant set. The following corollaries set forth that, if further conditions are added to Theorem 1, the convergence to the origin can be concluded.

Corollary 1. Assume that the conditions 1) and 2) of Theorem 1 are true. If $\frac{\partial f}{\partial x}b < 0$ for all $x \in \mathbb{R}^n/\{0\}$, then $\Omega = \{x \in \mathbb{R}^n : Qx = 0, f(x) = 0\}$. In particular, if a pair $(f(x), Ax)$ is zero-state observable or detectable, $\mathcal{V} = \{0\}$.

Proof By (9), if $\frac{\partial f}{\partial x}b < 0$ for all $x \in \mathbb{R}^n/\{0\}$, $\Omega = \{x \in \mathbb{R}^n : Qx = 0, f(x) = 0\}$. Moreover, Assume that the pair $(f(x), Ax)$ is zero-state observable or zero-state detectable. Then, $x(t) \in \mathcal{V} \subset \Omega$ converges to the origin. In conclusion, $\mathcal{V} = \{0\}$.

Corollary 2. Assume that the conditions 1) and 2) of Theorem 1 are true. If $PA + A^T P < 0$, then $\mathcal{V} = \{0\}$.

Proof By (9), $\Omega = \mathcal{V} = \{0\}$.

3. DESIGN OF NONLINEAR STATE FEEDBACK LAW

An important property of Theorem 1 is that the stability condition is characterized by a PDMI about the state feedback law $f(x)$. The PDMI (7) needs to be solved in order to design $f(x)$. In this section, a state feedback design method is proposed based on the PDMI condition in Theorem 1. The main idea is to focus on the kernel space of the matrix in (7) and (7) is reduced into a PDME (partial differential matrix equation) and a reduced-order PDMI. Then the PDME is solved analytically.

First, (7) is reduced into a PDME and a reduced-order PDMI. By Schur complement together with $PA + A^T P = -Q^T Q \le 0$, (7) is equivalent to

$$2\lambda\frac{\partial f}{\partial x}b + (b^T P + \frac{\partial f}{\partial x}A)(Q^T Q)^+(Pb + A^T(\frac{\partial f}{\partial x})^T) \le 0 \quad (14)$$

and

$$\ker Q \subset \ker (b^T P + \frac{\partial f}{\partial x}A). \quad (15)$$

Further, the kernel condition (15) is equilvalent to that there exists a $k(x) \in \mathbb{R}^n$ such that

$$b^T P + \frac{\partial f}{\partial x}A = -k(x)^T Q^T Q. \quad (16)$$

For fixed x, (16) is considered as a linear matrix equation about $k(x)$. (15) is a necessary and sufficient condition for the existence of the solution $k(x)$.

Substitution of (16) into (14) yields

$$\frac{\partial f}{\partial x}b + \frac{1}{2\lambda}k(x)^T Q^T Q(Q^T Q)^+ Q^T Q k(x) \le 0. \quad (17)$$

By the property $Q^T Q(Q^T Q)^+ Q^T Q = Q^T Q$ of pseudo-inverse, the above inequality equals to

$$\frac{\partial f}{\partial x}b + \frac{1}{2\lambda}k(x)^T Q^T Q k(x) \le 0. \quad (18)$$

Hence, the design problem boils down to finding $f(x)$ satisfying the PDME (16) subject to $\frac{\partial f}{\partial x}b \le -\frac{1}{2\lambda}\|Qk(x)\|^2$ and $A^T P + PA = -Q^T Q$.

The following theorem is one of the main results of this paper and gives the explicit state feedback law subject to some conditions, which is obtained via the PDME (16).

Theorem 2. Let $g(\cdot)$, $h(\cdot) \in \mathbb{R}$ be piecewise C^1 functions and k_0, k_1, l_0, $l_1 \in \mathbb{R}^n$ be constant vectors. The following state feedback laws $f(x)$ satisfy PDMI (7).

$$f(x) = -[(b^T P + k_0^T Q^T Q)A^+ + l_0^T(I - AA^+)]x - \int_0^{\alpha(x)} g(v)dv - \int_0^{\beta(x)} h(w)dw \quad (19)$$

where $\alpha(x)$, $\beta(x)$ are defined by

$$\alpha(x) = k_1^T Q^T Q A^+ x \quad (20)$$

$$\beta(x) = l_1^T(I - AA^+)x \quad (21)$$

if the integral kernel $g(\cdot)$, $h(\cdot)$ and the gain k_0, k_1, l_0, l_1 satisfy the following constraints:

(1) An inequality

$$-(b^T P + k_0^T Q^T Q)A^+ b - l_0^T(I - AA^+)b - k_1^T Q^T Q A^+ b \cdot g(\alpha(x)) - l_1^T(I - AA^+)b \cdot h(\beta(x)) + \frac{1}{2\lambda}\|Q\{k_0 + k_1 g(\alpha(x))\}\|^2 \le 0 \quad (22)$$

(2) Two algebraic equations

$$(b^T P + k_0^T Q^T Q)(I - A^+ A) = 0 \quad (23)$$

$$k_1^T Q^T Q(I - A^+ A) = 0 \quad (24)$$

Remark 1. The state feedback law (19) is parameterized by a constant gain vector and two integral kernels. Each parameters are freely chosen so that both of the constraints (22)∼(24) are satisfied.
It has been assumed that $g(x)$, $h(x) \in C^1$ in the derivation of (19). However, (19) is still the solution of (7) even if $g(\cdot)$, $h(\cdot)$ are only piecewise C^1 functions. Therefore, it is assumed that $g(\cdot)$, $h(\cdot)$ are at least piecewise C^1 functions in Theorem 2.

Proof of Theorem 2 In this proof, it is only confirmed that the state feedback law (19) satisfies the PDME (16). For the detailed derivation of solution, refer to Akasaka and Liu (2007).

We start with expanding the PDME (16) into coupled PDE's. Solving the linear matrix equation (16) with respect to $\partial S/\partial x$, the general solution is obtained as

$$\frac{\partial f}{\partial x} = -(b^T P + k(x)^T Q^T Q)A^+ + l(x)^T(I - AA^+) \quad (25)$$

where $l(x) \in \mathbb{R}^n$ is an arbitrary vector function and (25) is a solution of (16) iff

$$(b^T P + k(x)^T Q^T Q)(I - A^+ A) = 0. \quad (26)$$

Next, the compatibility of the coupled PDE's must be checked. The compatibility condition can be found in the textbooks of partial differential equations, for instance, Zwillinger (1992). It is verified that the compatibility condition is satisfied if $k(x)$, $l(x)$ is chosen to be

$$k(x) = k_0 + k_1 g(\alpha(x)), \quad l(x) = l_0 + l_1 h(\beta(x)). \quad (27)$$

We comfirm that (19) is a solution of (16) as follows. The partial derivative of (19) yields

$$\begin{aligned}\frac{\partial f}{\partial x} &= -[(b^T P + k_0^T Q^T Q)A^+ + l_0^T(I - AA^+)] \\ &\quad -g(\alpha(x))k_1^T Q^T Q A^+ - l_1^T(I - AA^+)h(\beta(x)) \\ &= -\{b^T P + (k_0 + k_1 g(\alpha(x)))^T Q^T Q\}A^+ \\ &\quad -(l_0 + l_1 g(\beta(x)))^T(I - AA^+)\end{aligned} \quad (28)$$

Hence, (19) is a solution of (16).

Finally, the inequality constraint (22) is derived from substituting (25) into (18). The algebraic equations are derived from (26).

4. STABILITY ANALYSIS - OUTPUT FEEDBACK CASE

In this section, observer based output feedback stabilization is studied based on the state feedback laws designed in the last section. Assume that $y = Cx \in \mathbb{R}^q$ and $\phi(u)$ can be measured as the feedback information and (C, A) is detectable.

An output feedback law is constructed as below:

$$\begin{aligned} \dot{\hat{x}} &= (A + LC)\hat{x} + B\phi - Ly \\ u &= f(\hat{x}) \end{aligned} \tag{29}$$

This system simply consists of a full-order Luenberger observer and the state feedback law $f(\cdot)$ obtained in the previous section. Let the estimated error be $e = \hat{x} - x$. Then, the closed loop system is described as

$$\begin{aligned} \dot{x} &= Ax + B\phi(f(x+e)) \\ \dot{e} &= (A + LC)e. \end{aligned} \tag{30}$$

In the sequel, the global stability of the system (30) will be discussed when the state feedback law is designed via the PDMI (7). In general, separation principle is not preserved for nonlinear systems. However, the system (30) is represented by a simple cascade system and $e(t)$ converges to zero exponentially provided that $A + LC$ is a Hurwitz matrix. This makes the stability analysis more easily. The next result gives a global stability condition for the system (30).

Theorem 3. Suppose that the conditions 1) and 2) of Theorem 1 are true and $A + LC$ is a Hurwitz matrix. If there exists positive constants ρ, M such that

$$\|\frac{\partial f}{\partial x}(x)\|\|x\| \leq \rho |f(x)| \tag{31}$$

for all $\|x\| \geq M$, then x, e converge globally to $\mathcal{V}$, 0, respectively.

Remark 2. The condition (31) in Theorem 3 requires the polynomial growth of $f(x)$ for $\|x\| \geq M$. If $f(x)$ is a polynomial function, (31) is automatically satisfied.

The following lemma is used in the proof of Theorem 3.

Lemma 2. If the condition (31) of Theorem 3 is true, then

$$\|\phi(f(\hat{x}))\frac{\partial f}{\partial \hat{x}}(\hat{x})\| \leq \frac{\rho}{2M} W(\zeta) \tag{32}$$

holds for $\|x\| \geq M$.

Proof See Appendix A.

Proof of Theorem 3 The idea of the proof is basically similar to Sepulchre et al. (1997). Let $\zeta = [x^T \ e^T]^T$ in the argument below.

First, it will be proved that $\zeta(t)$ is bounded for each $\zeta(0) \in \mathbb{R}^{2n}$ by using the function below.

$$W(\zeta) = x^T P x + 2\int_0^{f(x+e)} \phi(s) ds + e^T P_e e \tag{33}$$

where $P_e = P_e^T > 0$ is the solution of Lyapunov equaiton

$$P_e(A + LC) + (A + LC)^T P_e + Q_e = 0. \tag{34}$$

Since $A + LC$ is a Hurwitz matrix, there exists $P_e > 0$ for any $Q_e > 0$. By the condition 2) in Theorem 1 and $P_e > 0$, $W(\zeta)$ satisfies $W(\zeta) \geq 0$ and is radially unbounded.

The time derivative of $W(\zeta)$ satisfies

$$\dot{W}(\zeta) \leq 2\phi(f(\hat{x}))\frac{\partial f}{\partial \hat{x}}(\hat{x})(A + LC)e. \tag{35}$$

In above, the inequality follows from the condition 1) of Theorem 1 and $e^T Q_e e \geq 0$. Since $A + LC$ is Hurwitz, there exist $k_\epsilon(e(0)) > 0$ and $\epsilon > 0$ such that $\|e(t)\| \leq k_\epsilon(e(0)) \exp(-\epsilon t)$. Then,

$$\begin{aligned} \dot{W}(\zeta) &\leq 2\phi(f(\hat{x}))\frac{\partial f}{\partial \hat{x}}(\hat{x})(A + LC)e \\ &\leq 2\|\phi(f(\hat{x}))\frac{\partial f}{\partial \hat{x}}(\hat{x})\|\|A + LC\|\|e\| \\ &\leq 2\|\phi(f(\hat{x}))\frac{\partial f}{\partial \hat{x}}(\hat{x})\|\|A + LC\|k_\epsilon(e(0))\exp(-\epsilon t). \end{aligned} \tag{36}$$

Applying Lemma 2 to the above equation, for $\|x\| \geq M$,

$$\dot{W}(\zeta) \leq K_\epsilon(e(0))W(\zeta)\exp(-\epsilon t) \tag{37}$$

where $K_\epsilon(e(0)) = \rho\|A + LC\|k_\epsilon(e(0))/M > 0$. Then, the following bound of $W(\zeta(t))$ is obtained by Comparison Principle

$$\begin{aligned} W(\zeta(t)) &\leq W(\zeta(0))\exp\{K_\epsilon(e(0))\int_0^t e^{-\epsilon\tau} d\tau\} \\ &\leq W(\zeta(0))\exp\{K_\epsilon(e(0))/\epsilon\}. \end{aligned} \tag{38}$$

This implies that $\zeta(t)$ is bounded for each $\zeta(0) \in \mathbb{R}^{2n}$ because $W(\zeta)$ is continuous, bounded from below and radially unbounded.

Finally, we prove the convergence of the state as follows. The time derivative of $U(e) = e^T P_e e$ is $\dot{U}(e) = -e^T Q_e e < 0$ $(e \neq 0)$. The boundedness of the whole state has been already guaranteed by the above argument. Thus, $x(t)$, $e(t)$ globally converges to the largest invariant set contained in $\{(x, e) : e = 0\}$ according to LaSalle's invariant principle. In the largest invariant set, the dynamics of x is given by $\dot{x} = Ax + \phi(f(x))$. Since the condition 1) and 2) of Theorem 1 are true, then the largest invariant set must be $\{(x, e) : x \in \mathcal{V}, e = 0\}$. That concludes the proof.

5. DESIGN EXAMPLE

Global stabilizing nonlinear feedback law is concretely designed for the system (1) with a controllable canonical form.

5.1 Controllable Canonical Form

Consider the system (1) which is composed of integrators and stable poles as below:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ 0 & -\mu_2 & -\mu_3 & \cdots & -\mu_n \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{b} \phi \quad (39)$$

For the system (39), the matrices $I - A^+A$, $I - AA^+$ are calculated as

$$I - A^+A = \mathrm{diag}\{1, 0, \cdots, 0\} \quad (40)$$

$$I - AA^+ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \mu_2 & \mu_3 & \cdots & \mu_n & 1 \end{bmatrix}. \quad (41)$$

Using these matrices, we obtain the constraints (22)∼(24) in Theorem 2 as follows:

$$-l_1^T e_n \cdot h(\beta(x)) + \frac{1}{2\lambda}\|Q\{k_0 + k_1 g(\alpha(x))\}\|^2 \leq 0 \quad (42)$$

$$p_{n1} + k_0^T Q^T Q e_1 = 0 \quad (43)$$

$$k_1^T Q^T Q e_1 = 0 \quad (44)$$

where p_{ij} is the (i, j)-th element of P and $e_i \in \mathbb{R}^n$ is a unit vector whose i-th element is 1. In particular, if the constant gains are chosen to be $k_0 = k_1 = l_0 = 0$ and $l_1 = e_n$, the state feedback law (19) is

$$f(x) = -\sum_{j=1}^{n-1} p_{n(j+1)} x_j - \int_0^{\beta(x)} h(w)dw \quad (45)$$

where $\beta(x) = \mu_2 x_1 + \cdots + \mu_n x_{n-1} + x_n$, (42), (43) are reduced to $h(\cdot) \geq 0$, $p_{n1} = 0$ respectively and (44) is automatically satisfied.

Theorem 4. Assume that the system (39) consists of double integrators and stable poles. Let $\bar{A} \in \mathbb{R}^{(n-2)\times(n-2)}$, $q \in \mathbb{R}^{1\times n}$ be

$$\bar{A} = \begin{bmatrix} 0 & 1 & & & 0 \\ & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ -\mu_3 & -\mu_4 & & \cdots & -\mu_n \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ \mu_3 \\ \vdots \\ \mu_n \\ 1 \end{bmatrix}^T. \quad (46)$$

If P is given by

$$P = \kappa q^T q + \begin{bmatrix} 0_{2\times 2} & 0_{2\times(n-2)} \\ 0_{(n-2)\times 2} & \bar{P} \end{bmatrix} \quad (47)$$

where $\kappa > 0$ and $\bar{P} > 0$ is the solution of $\bar{P}\bar{A} + \bar{A}^T\bar{P} = -\bar{Q} < 0$, then the state feedback law (45) with $h(\cdot) > 0$ globally asymptotically stabilizes the equilibrium point $\mathcal{N} = \{0\}$ of the system (39).

Proof If $\bar{P} > 0$, P given by (47) is positive semi-definite. Furthermore, it is easy to see that $A^TP + PA \leq 0$ because

$$A^TP + PA = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}^T\bar{P} + \bar{P}\bar{A} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\bar{Q} \end{bmatrix} \leq 0. \quad (48)$$

Thus, P given by (47) is a solution of $A^TP + PA \leq 0$ and the constraint $p_{n1} = 0$ is satisfied.

To prove the globally asymptotic stability, we begin with the radial unboundedness of $V(x)$ given by (8). Let $x = \tau v$ in which $\tau > 0$ and $\|v\| = 1$. Then, $x^TPx = \tau^2 v^TPv = \tau^2(\kappa\|qv\|^2 + \bar{v}^T\bar{P}_2\bar{v})$ in which $\bar{v} = [v_3 \cdots v_n]^T$. Accordingly, $V(x)$ is not radially unbounded only if $\|qv\|^2 + \bar{v}^T\bar{P}_2\bar{v} = 0$. The unique solution v^* of this equation is obtained as $v^* = [1\ 0 \cdots 0]^T$ due to $\bar{P} > 0$. Therefore, the radial unboundedness of $V(x)$ is guaranteed iff $\lim_{\tau\to\infty} |f(\tau v^*)| = \infty$. By (45), $|f(\tau v^*)| = |\kappa\tau|$ and $\lim_{\tau\to\infty} |f(\tau v^*)| = \infty$ is satisfied. Hence, $V(x)$ is radially unbounded.

Finally, we prove that the maximum invariant set $\mathcal{V}$ in Theorem 1 equals to $\mathcal{N} = \{0\}$. Since $\frac{\partial f}{\partial x}b = -h(\cdot) < 0$,

$$\Omega = \{x \in \mathbb{R}^n : x_3 = \cdots = x_n = 0, f(x) = 0\}. \quad (49)$$

In the set Ω, $f(x)$, $\dot{f}(x)$ and $\ddot{f}(x)$ are respectively calculated as $f(x) = -p_{n2}x_1 - p_{n3}x_2$, $\dot{f}(x) = -p_{n2}x_2$ and $\ddot{f}(x(t)) = 0$. By (47), $p_{n2} = \kappa\mu_3 \neq 0$. Hence, $f(x) \equiv 0$ iff $x_1 = x_2 = 0$. This means that the maximum invariant set $\mathcal{V} = \{0\}$.

5.2 Numerical Example: Output Feedback Design

A numerical example is shown for the fourth order case of the system (39) consisting of double integrators and stable poles. Assume that all parameters of the plant (39) are known and the available feedback information is $y = x_1$ and $\phi(u)$. The control object is to track the output y to the constant reference signal r without windup phenomenon.

Concretely, we design the nonlinear output feedback law (29) as follows. $f(\hat{x})$ is designed based on Theorem 4 and the nonlinear kernel is chosen so that the condition of Theorem 3 is satisfied in order to ensure the global stability. Let us choose the nonlinear kernel as

$$h(w) = h_1 + h_2|w| \quad (50)$$

in which h_1, $h_2 > 0$. Then, the following state feedback law is obtained

$$f(\hat{x}) = -\sum_{i=1}^{3} p_{4(i+1)} x_i - h_1\beta(\hat{x}) - \frac{h_2}{2}|\beta(\hat{x})|\beta(\hat{x}) \quad (51)$$

where $\beta_2(x) = \mu_3 x_2 + \mu_4 x_3 + x_4$.

Fig. 2, 3 show the time responses of the system output $y = x_1$, the saturation output ϕ for the several values of the references $r = 1, 5, 10$, respectively. In the simulation, the plant parameters are set as $\mu_3 = 10$, $\mu_4 = 2$, i.e. the eigenvalues of A are 0, 0, $-1 \pm 3j$. The maximum value of the control input is $u_m = 5$. The observer gain is $L = [41\ 613\ 3639\ 1592]^T$. The initial state is $x(0) = [0\ 0.1\ 0.1\ 0.1]^T$ and $\hat{x}(0) = 0$. Each gain of (51) is set as

$$f(\hat{x}) = -15\Big\{10(\hat{x}_1 - r) - 3\hat{x}_2 - 1.6\hat{x}_3 - 0.2\beta(\hat{x}) - 0.1|\beta(\hat{x})|\beta(\hat{x})\Big\}. \quad (52)$$

In Fig 2, it is confirmed that the output y converges to each reference values without windup.

6. CONCLUSION

In this paper, a nonlinear output feedback control design method is proposed for linear systems subject to input

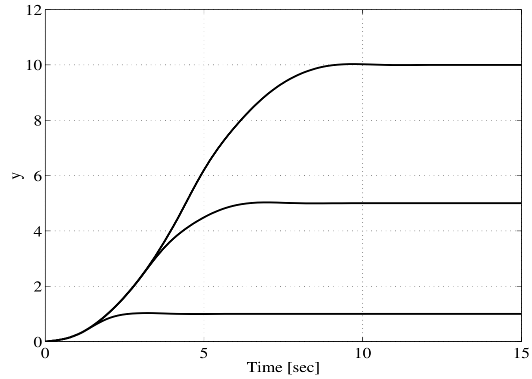

Fig. 2. The time responses of the plant output $y = x_1$ for the different values of the reference signal r (From above, the curve corresponds to the response for $r = 10, 5, 1$).

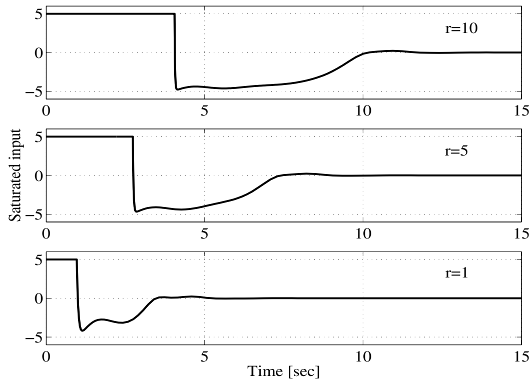

Fig. 3. The time responses of the saturation output ϕ for the different values of the reference signal r (From above, the curve corresponds to the response for $r = 10, 5, 1$).

saturation. The main results of this paper are as follows. First, global stability conditions are derived for both of state and output feedback cases. The sufficient condition is basically given by a partial differential matrix inequality and a growth order requirement for the feedback law. Secondly, a class of nonlinear feedback laws has been obtained by solving a partial differential matrix equation analytically. It is remarkable that the control laws are parameterized by nonlinear integral kernels.

The obtained state feedback law and the observer depend on the system parameters. As the next step, we need to develop robust design method for the parameter uncertainties. Further, achievable performances by the nonlinear control must be analyzed.

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Appendix A. PROOF OF LEMMA 2

First, we prove the following relationship:

$$|\phi(f(\hat{x}))||f(\hat{x})| \le W(\zeta) \tag{A.1}$$

The integral $\int_0^f \phi(s)ds$ satisfies

$$\int_0^f \phi(s)ds = \begin{cases} \frac{1}{2}|\phi(f)||f| & \text{if } |f| \le u_m \\ |\phi(f)||f| - \frac{1}{2}|\phi(f)|^2 & \text{if } |f| > u_m \end{cases}$$
$$\ge \frac{1}{2}|\phi(f)||f| \tag{A.2}$$

Hence,

$$|\phi(f(\hat{x}))||f(\hat{x})| \le 2\int_0^{f(\hat{x})} \phi(s)ds \le W(\zeta). \tag{A.3}$$

(32) is proved as follows.

$$\|\phi\frac{\partial f}{\partial \hat{x}}\| \le |\phi|\|\frac{\partial f}{\partial \hat{x}}\| \le \rho\frac{|\phi||f(\hat{x})|}{\|x\|} \le \frac{\rho W(\zeta)}{M} \tag{A.4}$$

for all $\|x\| \ge M$. The first inequality is obvious. The second inequality follows from (31) in Theorem 3. The third inequality holds due to (A.1) and $\|x\| \ge M$.