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Single-Parameter Tuning of PI Controllers: From Theory to Practice

Radek Matušů, Roman Prokop

Department of Automation and Control Engineering
Faculty of Applied Informatics, Tomas Bata University in Zlín
Nad Stráněmi 4511, 760 05 Zlín, Czech Republic
Tel: +420 57 603 3015, fax: +420 57 603 5279
e-mail: {rmatusu; prokop}@fai.utb.cz

Abstract: The paper deals with influence of a single scalar positive tuning parameter on performance properties of the closed control loop which contains algebraically designed PI controller while the response quality is evaluated by the size of first under- or overshoots. The controller coefficients are calculated from general solutions of diophantine equations in the ring of proper and Hurwitz stable rational functions. Subsequently, these controllers can be tuned by the only parameter. The contribution brings simple tuning rules and, moreover, it presents their possible practical application during control of real laboratory model assumed as system with parametric uncertainty.

1. INTRODUCTION

The issue of appropriate controller parameters adjustment has been very momentous and attractive topic for an array of decades. It is obvious that a large amount of tuning methods with their own specifications has been developed during such a long time of interest. However, simple setting rules are still desired and valuable. From the practical applications point of view, still the most important role play the controllers of PID and PI type, which are well known, widespread and easily utilizable.

An elegant and effective tool for control design is adopted from algebraic approach (Vidyasagar, 1985), (Kučera, 1993), (Prokop and Corriou, 1997). This technique is based on general solutions of diophantine equations in the ring of proper and Hurwitz stable rational functions (R_{PS}). It supposes the utilization of the known Youla-Kučera parameterization, which allows generating infinite amount of possible stabilizing controllers, while the choice of final one depends on the desired properties mathematically represented by conditions of divisibility in the specific ring. Anyway, the selected controller can be further tuned. And one of advantages of this approach is that behavior of regulators can be influenced by the only scalar tuning parameter $m > 0$. There are some methods how to choose the appropriate m for obtaining a controller which fulfil requirements of robustness, but there is a conspicuous lack of rules for nominal systems.

The contribution is focused on proper choice of single tuning parameter for simple PI controllers. The synthesis technique itself is adopted from above mentioned algebraic approach and the main aim of the paper is to study and investigate relations between a scalar tuning parameter $m > 0$ and the desired behavior of the control loop. The basic considerations and analyses are based on the first order stable SISO systems and its response in a closed loop. However, the practical part shows the utilization of the proposed method on the higher order systems, even with uncertain parameters.

2. AN ALGEBRAIC APPROACH TO CONTROL DESIGN

There are many traditional as well as modern methods for design of PI or PID-like controllers (Åström and Hägglund, 1995). However, the fractional approach developed in (Kučera, 1993) and (Vidyasagar, 1985) enables relatively deep insight into control tuning and a more elegant expression of all suitable controllers. This synthesis supposes the description of linear systems in R_{PS} as a ratio of two rational fractions:

$$G(s) = \frac{b(s)}{a(s)} = \frac{\dfrac{b(s)}{(s+m)^n}}{\dfrac{a(s)}{(s+m)^n}} = \frac{B(s)}{A(s)} \qquad (1)$$

$$n = \max\{\deg(a), \deg(b)\}, \; m > 0$$

The scalar positive parameter $m > 0$ can be later conveniently used as a "tuning knob" for control behavior.

A general feedback system is shown in Fig. 1. It represents for $C(s) = \frac{Q(s)}{P(s)}$ a classical feedback one-degree-of-freedom (1DOF) control loop. Ergo, the signal u is generated according to the control law:

$$P(s)u = Q(s)\left[w - y\right] + P(s)n \qquad (2)$$

In a two-degree-of-freedom (2DOF) control system, the controller $C(s)$ consists of two transfer functions $\frac{Q(s)}{P(s)}$ and $\frac{R(s)}{P(s)}$. The control law is governed by:

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$$P(s)u = R(s)w - Q(s)y + P(s)n \tag{3}$$

The objective is to design controller transfer functions such that the feedback system is internally BIBO stable, the reference error tends asymptotically to zero and the disturbances v and n are asymptotically eliminated from the plant output. All transfer functions of the closed control system (Fig. 1) have common denominator $AP + BQ$. One of the nice and convenient results of the algebraic philosophy is that this denominator should be a unit in the ring R_{PS}. In other words, the term $(AP + BQ)^{-1}$ resides in R_{PS} and the feedback system is BIBO stable. If the elements A and B are coprime in R_{PS} then all stabilizing controllers are given through an arbitrary solution P_0, Q_0 of diophantine (Bézout) equation:

$$AP + BQ = 1 \tag{4}$$

in a parametric form:

$$\frac{Q}{P} = \frac{Q_0 - AT}{P_0 + BT} \tag{5}$$

where T varies over R_{PS} while satisfying $P_0 + BT \neq 0$.

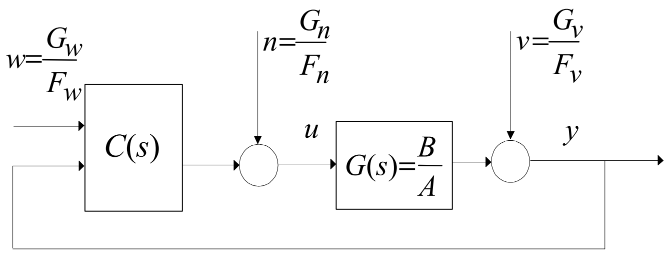

Fig. 1. General feedback system

From the practical point of view, it is often desirable to ensure more than stability. Probably the most frequent problem is that of reference tracking. Under assumption of zero disturbances $(n = v = 0)$, algebraic analysis results in the fact that the tracking error e tends to zero if

a) F_w divides AP for 1DOF
b) F_w divides $(1 - BR)$ for 2DOF

The last condition gives the second diophantine equation in the form:

$$F_w Z + BR = 1 \tag{6}$$

This approach is described in detail for example in (Prokop and Corriou, 1997), (Prokop *et al.*, 2002).

The design process is demonstrated for first order system. A nominal transfer function is supposed as:

$$G(s) = \frac{b_0}{s + a_0} \tag{7}$$

Further, step-wise reference with $F_w = \frac{s}{s+m}$ and no disturbances are assumed. The diophantine equation (4) takes the form:

$$\frac{s + a_0}{s + m} p_0 + \frac{b_0}{s + m} q_0 = 1 \tag{8}$$

Multiplying by $(s + m)$ and comparing coefficients give the general stabilizing solution in the form:

$$P(s) = p_0 + \frac{b_0}{s + m} T; \quad Q(s) = q_0 - \frac{s + a_0}{s + m} T \tag{9}$$

where $q_0 = \frac{m - a_0}{b_0}$; $p_0 = 1$ and T is free in R_{PS}. The asymptotic tracking for a stepwise reference w will be given by divisibility of $F_w = \frac{s}{s+m}$ and AP. It is achieved for $T = t_0 = -\frac{m}{b_0}$ so that $P(s)$ has zero absolute coefficient in the numerator. Then inserting t_0 into (9) gives

$$P(s) = \frac{s}{s + m}; \quad Q(s) = \frac{\tilde{q}_1 s + \tilde{q}_0}{s + m} \tag{10}$$

where

$$\tilde{q}_1 = \frac{2m - a_0}{b_0}; \quad \tilde{q}_0 = \frac{m^2}{b_0} \tag{11}$$

Finally, the 1DOF controller has the transfer function:

$$\frac{Q}{P} = \frac{\tilde{q}_1 s + \tilde{q}_0}{s} \tag{12}$$

which is a traditional PI control law governed by:

$$u(t) = \tilde{q}_1 \left[w(t) - y(t)\right] + \tilde{q}_0 \int \left[w(t) - y(t)\right] dt \tag{13}$$

The 2DOF structure brings the second equation (6) in the form:

$$\frac{s}{s + m} z_0 + \frac{b_0}{s + m} r_0 = 1 \tag{14}$$

with the general solution:

$$R(s) = r_0 + \frac{s}{s + m} \tilde{T}, \quad r_0 = \frac{m}{b_0} \tag{15}$$

where $\tilde{T}$ is again free in R_{PS}. The choice $\tilde{T} = \tilde{t}_0 = 0$ gives the feedforward part as $\frac{R}{P} = \frac{r_0(s + m)}{s}$ and yields a general

2DOF controller while the choice $\tilde{t}_0 = -r_0$ represents the PI controller operating with control error in the integrating part (feedforward part is then described by $\frac{R}{P} = \frac{r_0 m}{s}$).

3. TUNING OF PI CONTROLLERS

From the methodology based on R_{PS} representation follows the fact that controller parameters and control response can be simply tuned by a scalar parameter $m > 0$. The question how to select or reject this parameter from the available set is very topical. Moreover, the task what criterion should be chosen is also important. The analysis of both questions is proposed in this part.

For the first outline and from the point of view of control engineers a reasonable criterion can be seen in the overshooting and undershooting of control responses. This analysis is visualized for three couples of $\{b_0, a_0\}$ in transfer function (7), while $a_0 > 0$, i.e. stable system is assumed:

$$b_0 = 1; \quad a_0 = 0.5 \tag{16}$$

$$b_0 = 1; \quad a_0 = 1 \tag{17}$$

$$b_0 = 1; \quad a_0 = 2 \tag{18}$$

Supposing the 1DOF configuration, PI controllers (12) with parameters (11) were designed and tuned by $m \in \langle 0.05; 15 \rangle$ for these three systems. Fig. 2 shows relations between the parameter *m* and the percentage of the first undershoot. Fig. 3 represents a similar dependence for the overshoots. Graph in Fig. 2 is zoomed for better view. Typical shapes of the control responses with first undershoot or overshoot can be seen in Fig. 4.

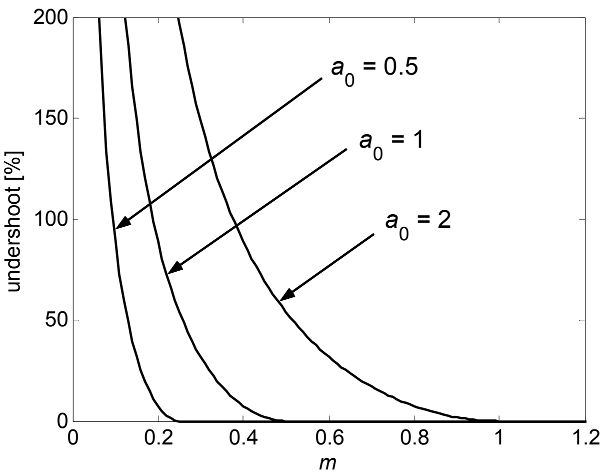

Fig. 2. Relations between *m* and undershoot for (16) – (18)

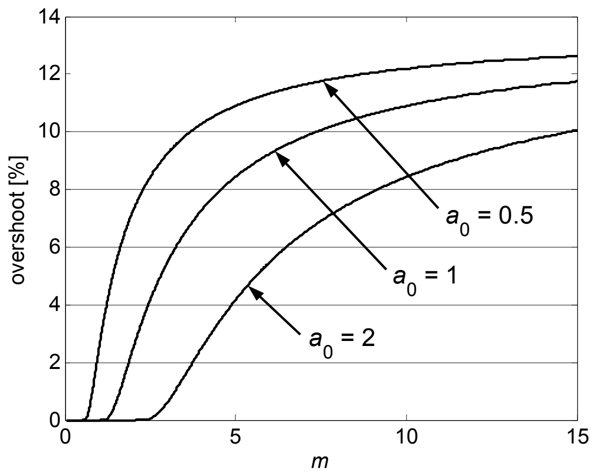

Fig. 3. Relations between *m* and overshoot for (16) – (18)

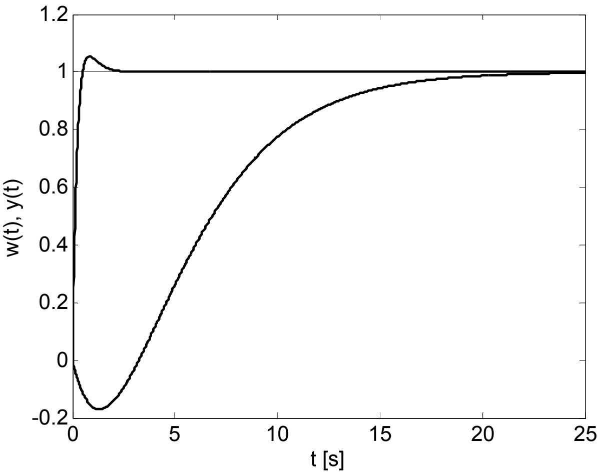

Fig. 4. Typical control responses with first undershoot or overshoot

The deeper insight into this analysis gives an important result. If the ratio of *m* and a_0 is a constant then the control loop produces response with the same size of undershoot or overshoot. In other words, the control results of "same quality", from the point of view of selected criterion, can be obtained if:

$$\frac{m}{a_0} = k \tag{19}$$

where *k* is a constant.

The closed-loop system produces the control responses with first undershoot if:

$$k < 0.5 \tag{20}$$

However, the bulk of industrial processes requires the regulation with shorter settling time (and without undershoots). This requirement can fulfil a higher value of *k*. The corresponding constant *k* for several values of first overshoot in percentage can be found in Tab. 1.

Tab. 1. Relation between *k* and overshoot

Overshoot [%]	*k*
0	1.00
1	1.62
2	1.87
3	2.14
4	2.44
5	2.80
6	3.25
7	3.81
8	4.58
9	5.67
10	7.38

Probably the most significant consequence from this analysis is that it exists some interval of m which does not produce any overshoot or undershoot. The "optimal" choice of m seems to be if the response is as fast as possible but still without overshoot. For this case, constant k from (19) equals to 1 (see also Tab. 1), thus:

$$m = a_0 \tag{21}$$

By further simulations, it was found out that the value of parameter b_0 does not influence the choice of $m > 0$. The curves in Fig. 2 and Fig. 3 would be the same for every b_0.

Putting (21) into (11) gives the "optimal" parameters of PI controller:

$$\tilde{q}_1 = \frac{a_0}{b_0}; \quad \tilde{q}_0 = \frac{a_0^{\,2}}{b_0} \tag{22}$$

If the controlled system is assumed in the form:

$$G(s) = \frac{K}{Ts+1} \tag{23}$$

where $K = \frac{b_0}{a_0}$ and $T = \frac{1}{a_0}$, then equations (19), (21) and (22), respectively, change into:

$$Tm = k \tag{24}$$

$$m = \frac{1}{T} \tag{25}$$

$$\tilde{q}_1 = \frac{1}{K}; \quad \tilde{q}_0 = \frac{1}{KT} \tag{26}$$

These ideas and simulation results can be confirmed also by analysis of the closed loop transfer function (see Fig. 1 and suppose 1DOF configuration):

$$K_{w,y} = \frac{GC}{1+GC} = \frac{\frac{BQ}{AP}}{1+\frac{BQ}{AP}} = \frac{BQ}{AP+BQ} = \\ = \frac{\frac{b_0\left(\tilde{q}_1 s + \tilde{q}_0\right)}{\left(s+m\right)^2}}{\frac{\left(s+a_0\right)s + b_0\left(\tilde{q}_1 s + \tilde{q}_0\right)}{\left(s+m\right)^2}} = \frac{b_0\left(\tilde{q}_1 s + \tilde{q}_0\right)}{\left(s+a_0\right)s + b_0\left(\tilde{q}_1 s + \tilde{q}_0\right)} \tag{27}$$

Assuming controller parameters (11), it holds for the numerator of (27):

$$b_0\left(\tilde{q}_1 s + \tilde{q}_0\right) = b_0\left(\frac{2m - a_0}{b_0} s + \frac{m^2}{b_0}\right) \tag{28}$$

For example, it can be seen, that the closed-loop system has non-minimum phase behaviour (first undershoot for input signal positive step change) if:

$$m < 0.5 a_0 \tag{29}$$

which concurs with equations (19) and (20).

The simulation examples, which indicates applicability of the derived rules for first as well as higher order systems can be found in (Matušů *et al.*, 2006).

4. REAL LABORATORY EXPERIMENTS

4.1 Model description

The controlled plant has been represented by laboratory model of hot-air tunnel constructed in VŠB – Technical University of Ostrava (Smutný *et al.*, 2002). Generally, this object can be seen as multi-input multi-output (MIMO) system, however, the experiments have been done on two selected SISO loops. The model is composed of the bulb, primary and secondary ventilator and an array of sensors covered by tunnel. The bulb is powered by controllable source of voltage and serves as the source of light and heat energy while the purpose of ventilators is to ensure the flow of air inside the tunnel. All components are connected to the electronic circuits which adjust signals into the voltage levels suitable for CTRL 51 unit. Finally, this control unit is connected with the personal computer (PC) via serial link RS232. The diagram of the plant and whole control system is shown in Fig. 5.

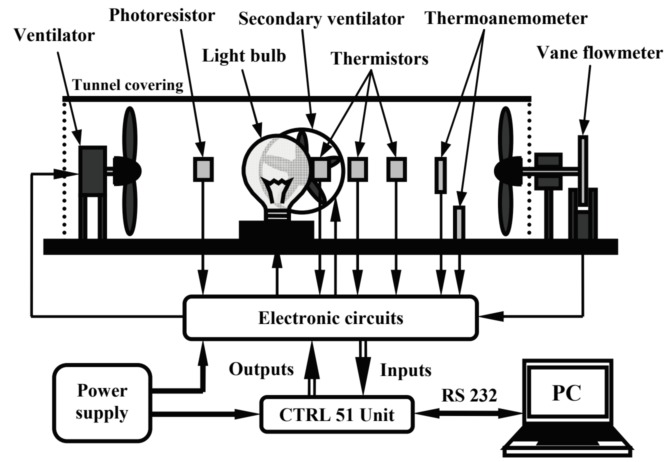

Fig. 5. Scheme of hot-air tunnel and whole control system

The CTRL 51 unit has been produced by Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic (Klán *et al.*, 2003). All presented control experiments were performed using the notebook HP Compaq nc6120, Windows XP and MATLAB 6.5.1. The communication between MATLAB and CTRL 51 unit was arranged through four user function (for initialization, reading and writing of data and for closing) and the synchronization of the program with real time was done via „semaphore" principle. To ensure the sufficient emulation of the continuous-time control algorithms, the sampling time 0.1 s was set. The detailed information about utilization of serial link under MATLAB including mentioned user routines, program synchronization mechanism and several tests can be found in (Dušek and Honc, 2002). The discretization of integrative parts of control laws was carried out by left rectangle approximation method.

4.2 Control experiments

The first considered loop covers bulb voltage $u1$ (control signal), which influences temperature of the bulb $y3$ (controlled variable). The other, at given moment unutilized, actuating signals were preset to constant values – primary ventilator voltage $u2$ to 2 V and secondary ventilator voltage $u3$ to 0 V. The designation of variables corresponds to real connection of input and output signals of CTRL 51 unit (Smutný *et al.*, 2002).

This controlled system can be described by mathematical model with parametric uncertainty, i.e. its parameters lie within given intervals:

$$G(s,K,\tau,T_1,T_2)=\frac{K(\tau s+1)}{(T_1 s+1)(T_2 s+1)}= \\ =\frac{[0.2;\,0.7]([25;\,130]s+1)}{([3;\,14]s+1)([70;\,210]s+1)} \tag{30}$$

The nominal system is obtained with selection of fixed parameters and subsequent simple approximation:

$$\frac{0.5(100s+1)}{(9s+1)(150s+1)}\approx\frac{0.5}{159s+1}=\frac{0.003145}{s+0.006289}=G_N(s) \tag{31}$$

The tuning parameter $m=0.01761$, which correspond to 5% of first overshoot for the nominal case, has been selected (see Tab. 1). The 1DOF PI controller computed according to (11) is:

$$\frac{Q}{P}=\frac{\tilde{q}_1 s+\tilde{q}_0}{s}=\frac{9.199s+0.0986}{s} \tag{32}$$

Assuming the family of plants (30) and the controller (32), the family of closed-loop characteristic polynomials can be easily formulated as:

$$\begin{aligned} p(s,K,\tau,T_1,T_2) &= (T_1 s+1)(T_2 s+1)s+ \\ &+K(\tau s+1)(\tilde{q}_1 s+\tilde{q}_0)= \\ &=T_1T_2s^3+(T_1+T_2)s^2+K\tau(\tilde{q}_1 s^2+\tilde{q}_0 s)+ \\ &+K(\tilde{q}_1 s+\tilde{q}_0)+s \end{aligned} \tag{33}$$

This polynomial is robustly stable (Ackermann *et al.*, 1993), (Barmish, 1994), (Bhattacharyya *et al.*, 1995), i.e. the whole system is robustly stable. The real closed-loop control behaviour of the bulb temperature can be seen in Fig. 6. The control signal is depicted only in 25% of its true size because of better perspicuity of controlled variable.

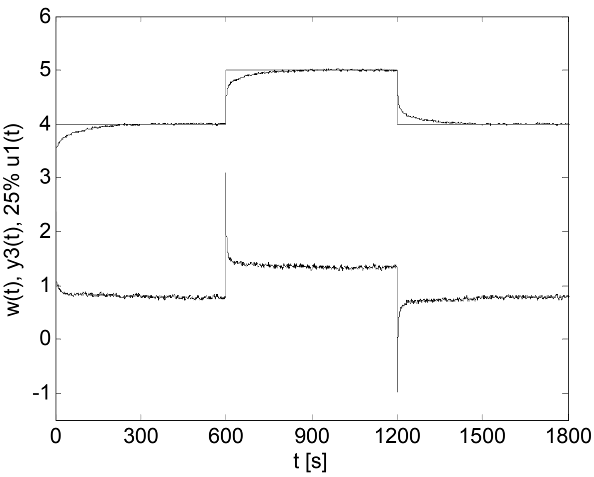

Fig. 6. Control of bulb temperature

The second loop consists of primary ventilator voltage $u2$ (control signal) and airflow speed measured by vane flowmeter $y7$ (controlled variable). The bulb voltage $u1$ and secondary ventilator voltage $u3$ have been set to 0 V.

Again, the plant is considered as system with parametric uncertainty. Now, its transfer function is:

$$G(s,K,T)=\frac{K}{(Ts+1)^2}=\frac{[0.3;\,1.2]}{([1;\,3]s+1)^2} \tag{34}$$

The nominal system equals to:

$$\frac{0.7}{(1.9s+1)^2} \approx \frac{0.7}{3.8s+1} = \frac{0.1842}{s+0.2632} = G_N(s) \qquad (35)$$

The chosen value $m = 0.2632$ (0% of overshoot for nominal plant) results in PI controller:

$$\frac{Q}{P} = \frac{\tilde{q}_1 s + \tilde{q}_0}{s} = \frac{1.4289s + 0.3761}{s} \qquad (36)$$

The system (34) and regulator (36) leads to the closed-loop characteristic polynomial:

$$\begin{aligned} p(s,K,T) &= (Ts+1)^2 s + K(\tilde{q}_1 s + \tilde{q}_0) = \\ &= T^2 s^3 + T2s^2 + K(\tilde{q}_1 s + \tilde{q}_0) + s \end{aligned} \qquad (37)$$

Analogously to the previous event, robust stability of (37) entails robust stability of the whole closed-loop system. It is practically confirmed by Fig. 7, which shows final control behaviour of the airflow speed in the end of the tunnel.

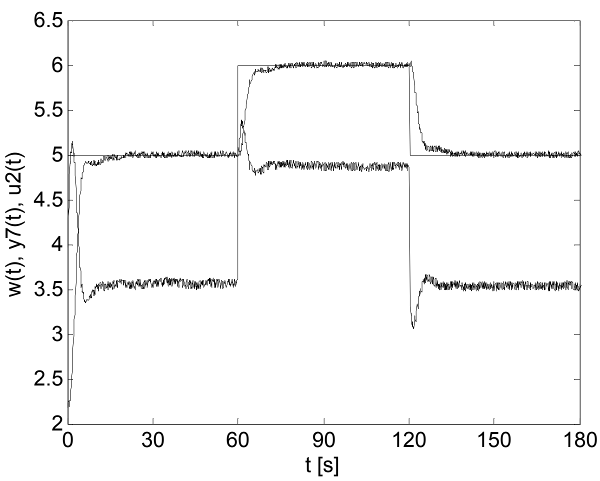

Fig. 7. Control of airflow speed

As can be seen, the proposed methodology of simple PI controllers tuning can be practically applied not only for first, but also for second (or higher) order uncertain systems thanks to the robustness of the control algorithms.

5. CONCLUSIONS

Controller design based on general solutions of diophantine equations in the specified ring is one of the elegant tools which algebra brings to control theory. Among other things, it gives a scalar positive parameter $m > 0$ which deeply influences control behavior. Generally, controller parameters are nonlinear function of this tuning knob. The principal goal of the contribution has been to study and analyze relations between this parameter and the desired properties of the closed-loop control system for the case of first order stable systems. The experimental analysis was performed through a set of simulations and the simple dependence between *m* and the size of undershoot or overshoot has been achieved. Then, tuning rules for PI controller parameters have been established. The results were verified during control of bulb temperature and airflow speed in laboratory model of hot-air tunnel. The controlled plants have been considered as second order systems affected by parametric uncertainty.

ACKNOWLEDGMENTS

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