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METHODS OF ELLIPSOIDAL ESTIMATION FOR LINEAR CONTROL SYSTEMS ⋆

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Abstract:
We present explicit formulas for ellipsoids bounding reachable sets for linear control dynamic systems with geometric bounds on control. We study both locally and globally optimal ellipsoidal estimates with regard to different optimality criteria. In particular, we solve some essentially nonlinear boundary problems related to the search for globally optimal ellipsoids with regard to the volume criterion. It is shown that by using the explicit formulas one can efficiently pass to limits in several asymptotic problems, including passing to the limit when the phase space dimension goes to infinity.

Keywords: reachable sets, linear control systems, ellipsoidal estimation

1. INTRODUCTION.

Reachable set is a central notion of the control theory. A good knowledge of these sets would allow us to solve easily basic optimization problems. For instance, the terminal functional optimization problem reduces to a problem of nonlinear programming. Unfortunately, usually this knowledge is missing for a simple reason: the reachable sets are complicated. In particular, their complete description requires infinite number of parameters, which is surely impractical. Therefore, it is natural to look for a simple substitute for reachable sets. Now it is well-known that, at least for linear control problem, one can efficiently find upper and lower ellipsoidal bounds for these sets. In what follows we'll discuss certain new properties of a class of upper ellipsoidal bounds. Still, up to section 6 we remind foundations of the ellipsoidal analysis as they can be found in, e.g., Chernousko, Ovseevich (2004). The new trends presented here include analytic and numerical study of globally volume-optimal ellipsoidal estimates, and asymptotics of ellipsoidal estimates as the number N of degrees of freedom of studied systems tends to infinity. Some of these developments are applicable to general linear systems as, e.g., numerical method for the search for globally volume-optimal ellipsoids, but others as, e.g., asymptotic theory as $N \to \infty$ are related to particular quite simple linear systems, and extension to a more general setup is out of reach at present. It should be stressed, that in spite of the simplicity of some linear systems considered, the study addresses rather refined and complicated analytical issues.

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2. EVOLUTIONAL ESTIMATORS.

We deal with a linear control system of the following form

$$\dot{x}(t) = Ax(t) + Bu(t) + c(t),\ x(s) \in M \subset \mathbf{R}^n,\quad u \in U(t) \subset \mathbf{R}^m,\ t \geq s, \tag{1}$$

where matrices A and B might depend on time.

The problem is to bound from above the family of reachable sets

$$D(t) = D(t, s, M) = \{x(t) \in \mathbf{R}^n : x(\cdot) \text{ — admissible trajectory}\}, \tag{2}$$

The reachable sets possess the following important evolutional property:

$$D(t) = D(t, \tau, D(\tau)) \quad \tau \in [s, t]. \tag{3}$$

Definition. A family $\Omega(t)$, $t \geq s$ of sets $\Omega(t) \subset V$ is called superreachable (superattainable) for system (1), if the inclusion holds

$$\Omega(t) \supset D(t, \tau; \Omega(\tau)),\ s \leq \tau \leq t \tag{4}$$

It is clear that the superreachable sets give upper bounds for reachable sets, provided that this is the case at the initial time instant. In other words, they are are evolutional estimators for reachable sets.

3. ELLIPSOIDAL ESTIMATION.

In what follows we are looking for ellipsoidal superreachable sets. An ellipsoid can be conveniently described by parameters a, Q

$$E(a, Q) = \left\{x \in V = \mathbf{R}^n : \left(Q^{-1}(x-a), x-a\right) \leq 1\right\},\quad E(Q) = E(0, Q), \tag{5}$$

where a is a vector, and Q is a positive definite symmetric matrix.

The control system (1) takes form

$$\dot{x} = Ax + Bu + c,\ u \in E(G),\ x(s) \in E(a_0, Q_0), \quad (6)$$

i.e., the bounds for control and initial position are ellipsoidal

The condition of superreachability (3) for ellipsoids $\mathcal{E}(t) = E(a(t), Q(t))$ takes the following analytic form Ovseevich (1997):

$$\begin{aligned} \dot{a} &= Aa + c, \\ \dot{Q} &\geq \{A, Q\} + \lambda Q + \lambda^{-1} H,\ H = BGB^*, \end{aligned} \quad (7)$$

$\{\alpha, \beta\} = \alpha\beta + \beta^*\alpha^*$, $\lambda > 0$, the matrix inequality means that the difference of the left- and right-hand sides is a positive definite symmetric matrix.

4. OPTIMAL ELLIPSOIDS

Suppose we are given a functional $\mathcal{E} \mapsto L(\mathcal{E})$ of ellipsoids, which depends smoothly on a, Q. A superreachable family of ellipsoids $\mathcal{E}(t)$ is

- locally optimal, if for any $\tau \geq s$
$$\left.\frac{d}{dt} L(\mathcal{E}(t))\right|_{t=\tau} \to \min, \quad (8)$$
- globally optimal, if
$$L(\mathcal{E}(T)) \to \min. \quad (9)$$

For linear control system (6) one can write down explicitly the equations of motion for optimal ellipsoids by optimizing wrt the parameter λ in relations (7) (see Chernousko (1994), Ovseevich (1997)).

5. EQUATIONS FOR OPTIMAL ELLIPSOIDS

Suppose that the criterion $L = L(Q)$ and $\partial L/\partial Q \geq 0$. Then the parameters of optimal ellipsoids satisfy the equations

$$\begin{aligned} \dot{a} &= Aa + c,\ a(s) = a_0 \\ \dot{Q} &= \{A, Q\} + \lambda Q + \lambda^{-1} H,\ Q(s) = Q_0, \\ \lambda &= \sqrt{\mathrm{Tr}(PH)/\mathrm{Tr}(PQ)}, \end{aligned} \quad (10)$$

- In the locally optimal case
$$P = \partial L/\partial Q. \quad (11)$$
so that (10) is an initial value problem
- In the globally optimal case
$$\dot{P} = -\{P, A\},\ P(T) = \partial L/\partial Q(Q(T)) \quad (12)$$
so that (10), (12) is a boundary value problem

Equations (12) are based on the Pontryagin maximum principle for the extremal problem (9).

6. A SPECIAL CLASS OF ESTIMATING ELLIPSOIDS.

The following notion was introduced by A.B. Kurzhansky and P. Varaiya:

Definition. A family of (upper) bounds $E(T)$ of reachable sets $D(T)$ is tight if it touches $D(T)$ for each T.

It turns out that globally optimal superreachable ellipsoids, which minimize the projection upon a fixed straight line are tight:

Theorem. Globally optimal ellipsoids wrt criterion $L(Q) = (Ql, l)$ are tight (Chernousko, Ovseevich (2003))

Thus, the notion of tight ellipsoids does not go beyond the class of optimal superreachable ellipsoids.

7. APPROXIMATION QUALITY.

We suggest to assess the approximation quality by using the Banach-Mazur distance between the ellipsoid and the reachable set. When we consider symmetric (wrt 0) convex bodies Ω_1, Ω_2 the Banach-Mazur distance between them is

$$d(\Omega_1, \Omega_2) = \log(t(\Omega_1, \Omega_2) t(\Omega_2, \Omega_1)), \quad (13)$$

where $t(\Omega_1, \Omega_2) = \inf\{t \geq 1; t\Omega_1 \supset \Omega_2\}$. In the language of support functions

$$h(\Omega_1, \Omega_2) = \sup_{|\xi|=1} \frac{H_{\Omega_2}(\xi)}{H_{\Omega_1}(\xi)}, \quad (14)$$

where $H_\Omega(\xi) = \sup_{x \in \Omega}(x, \xi)$ for any subset $\Omega \subset \mathbf{R}^n$.

8. GENERAL ASYMPTOTIC RESULTS

Suppose the system (6) is time-invariant and the Kalman controllability criterion holds. Then the reachable sets $D(T)$ are convex bodies for $T > 0$ (0 is initial time) and

(1) There exist affine transformations $C(T)$ such that there is a limit $\lim_{T \to \infty} C(T) D(T)$
(2) If $L(\mathcal{E}) = \mathrm{Tr}\, RQ$ ia a linear criterion, R being a fixed positive-definite symmetric matrix, $\mathcal{E}_t = E(a_t, Q_t)$ are globally optimal superreachable ellipsoids wrt the criterion $L(C(T)\mathcal{E}(T))$, then there exists a limit $\lim_{T \to \infty} \mathcal{E}(T)$
(3) The Banach-Mazur distance between ellipsoids $\mathcal{E}(T)$ and the reachable sets $D(T)$ remains bounded as $T \to \infty$
(4) Conjecturally, the same is true for the volume criterion

9. EXPLICIT FORMULAS FOR GLOBALLY OPTIMAL ELLIPSOIDS

The basic formula for globally optimal ellipsoid at the terminal instant T is as follows:

$$Q(T) = \left(\langle P(s), Q_0 \rangle^{1/2} + \int_s^T \langle P, H \rangle^{1/2} d\tau \right) \times \left(\frac{\rho(\Phi(T, s)) Q_0}{\langle P(s), Q_0 \rangle^{1/2}} + \int_s^T \frac{\rho(\Phi(T, \tau)) H(\tau)}{\langle P(\tau), H(\tau) \rangle^{1/2}} d\tau \right) = \Lambda(P(T)) \quad (15)$$

where $\Phi(t, s)$ is the fundamental matrix of $\dot{x} = Ax$,

$$\rho(A)B = ABA^*,\ \langle A, B \rangle = \mathrm{Tr}\, AB^*, \quad (16)$$

where A and B are square matrices of the same size.

For $Q_0 = 0$ we get

$$Q(T) = \int_s^T \langle P(\tau), H(\tau) \rangle^{1/2} d\tau \int_s^T \frac{\rho(\Phi(T, \tau)) H(\tau)}{\langle P(\tau), H(\tau) \rangle^{1/2}} d\tau. \quad (17)$$

The equation (15) reduces the two-point boundary problem for globally optimal ellipsoids to solving a transcendental matrix equation.

10. SOLUTION OF EQUATION (15)

Equations (15), (12) are equivalent to the search for a fixed point

$$P = \frac{\partial L}{\partial Q}(\Lambda(P)), \tag{18}$$

where P is a symmetric positive definite matrix. It is not at all clear in advance whether the fixed point is unique. In the case of volume optimization

$$P = \Lambda(P)^{-1} \tag{19}$$

one can show that this is indeed the case. Define the "Banach-Mazur distance in the projective space"

$$\begin{aligned} \rho(P_1, P_2) &= \log(t(P_1, P_2)t(P_2, P_2); \\ t(P_1, P_2) &= \inf\{t \geq 0 : tP_1 \geq P_2\}, \end{aligned} \tag{20}$$

$\rho(P_1, P_2)$ depends only on rays $\overline{P}_i = \{\lambda P_i\}$, $i = 1, 2$ and

$$\rho(\Lambda(P_1)^{-1}, \Lambda(P_2)^{-1}) \leq \frac{1}{2}\rho(P_1, P_2) \tag{21}$$

the contraction mapping principle proves that (18) has a unique solution = limit of iterations $P_{n+1} = \Lambda(P_n)^{-1}$ with an arbitrary initial point P_0

11. EXAMPLE: HARMONIC OSCILLATOR.

Consider the harmonic oscillator with unit frequency

$$\dot{x}_1 = x_2, \quad \dot{x_2} = -x_1 + u, \quad |u| \leq 1 \tag{22}$$

and zero initial condition. The globally and locally optimal ellipsoids wrt trace criterion coincide:

$$Q_{glob}(t) = Q_{loc}(t) = \begin{pmatrix} \frac{1}{2}\,t\,(t - \sin t\cos t) & \frac{1}{2}\,t\sin^2 t \\ \frac{1}{2}\,t\sin^2 t & \frac{1}{2}\,t\,(t + \sin t\cos t) \end{pmatrix} \tag{23}$$

The reachable set behaves like a disk of radius $(2/\pi)t$ as $t \to \infty$. The ellipsoidal estimate (23) also behaves like a disk of radius $t/\sqrt{2}$ as $t \to \infty$. The Banach-Mazur distance between the ellipsoid and the reachable set tends to $\log(\pi/2\sqrt{2}) = 0.1050\ldots$

12. SYSTEM OF OSCILLATORS.

Consider a system of pendulums with a common controlled suspension point:

$$\ddot{x}_k + \omega_k^2 x_k = u,\ |u| \leq 1,\ x_k(0) = \dot{x}_k(0) = 0, k = 1, \ldots, n. \tag{24}$$

The point is to estimate the difference between the optimal (wrt trace) ellipsoid $E(T)$ and the reachable set $D(T)$ for large T and n. Different ways of passing to double limit might lead to different problem statement and final answers. Let us first study the limit wrt T, and then wrt n. General asymptotic theory says that there exist a limit convex body $\mathcal{D} = \mathcal{D}_n = \lim_{T\to\infty} D(T)/T$ and the limit ellipsoid $\mathcal{E} = \mathcal{E}_n = \lim_{T\to\infty} E(T)/T$ as $T \to \infty$. Then, we compare $\mathcal{D}_n$ and $\mathcal{E}_n$ as $n \to \infty$.

Suppose the system (24) is nonresonant, i.e., there are no relations $\sum m_k\omega_k = 0$, where $0 \neq m = (m_1, \ldots, m_n) \in \mathbf{Z}$ is integral vector. Then, the limit Banach-Mazur distance $d_n = d(\mathcal{D}_n, \mathcal{E}_n)$ between the ellipsoid and reachable set depends only on dimension n: $d_n = -\log(\sqrt{2}s_n)$, where

$$s_n = \min_{|a|=1} s_n(a) = \min_{|a|=1} \fint \left|\sum_{k=1}^{n} a_k \sin\phi_k\right| d\phi, \tag{25}$$

where $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$, $|a| = \left(\sum a_k^2\right)^{1/2}$, and $\fint f(\phi)d\phi$ stands for the average of a multiply periodic function f. Since $0 < s_{n+1} \leq s_n$, there exists a limit $s_\infty = \lim_{n\to\infty} s_n$. A nontrivial result is that $s_\infty > 0$ which says in other words that the Banach-Mazur distance between the ellipsoid and reachable set remains bounded as $n \to \infty$.

Many evidences, both theoretical and numerical, witness in favor of the following Conjecture:

$$s_\infty = 2\pi^{-1}\int_0^\infty (1 - e^{-\frac{x^2}{4}})x^{-2}dx = 1/\sqrt{\pi}. \tag{26}$$

If this is true, then the limit Banach-Mazur distance between the limit ellipsoid and reachable set is $\lim_{n\to\infty} d_n = \log\sqrt{\frac{\pi}{2}} = 0.2258\ldots$

13. MASS POINT: THE VOLUME CRITERION.

One can explicitly describe superreachable ellipsoids for the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \leq 1,\ x(0) = 0 \tag{27}$$

which are globally optimal wrt volume. The boundary problem is reduced to determination of the momentum matrix at terminal instant:

$$P(T) = Q(T)^{-1},\ Q(T) = \frac{2u^2}{9\left(u^2 - \frac{1}{4}\right)}\begin{pmatrix} u^2T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{pmatrix}. \tag{28}$$

where $u = 0.5621535\ldots$ is a unique solution $> 1/2$ of the transcendental equation

$$3\left(u^2 - \frac{1}{4}\right)\log\left(\frac{2u-1}{2u+1}\right) + u = 0, \tag{29}$$

Curious results come out of comparison of optimal ellipsoids wrt different criteria, if one uses the volume (area) of the ellipsoids in normalized coordinates in order to assess the approximation quality. The area of ellipsoid (28) is

$$V_1 = \frac{2}{9}\pi u^2\left(u^2 - \frac{1}{4}\right)^{-1/2} = 0.8587\ldots$$

The area of ellipsoid which is globally optimal wrt to trace is

$$V_2 \approx 0.9068$$

The area of ellipsoid which is globally optimal wrt to projection to the axe x_2 is

$$V_3 \approx 0.9069$$

The area of ellipsoid which is locally optimal wrt to volume is

$$V_4 \approx 1.2489$$

The area of the reachable set is

$$V_D = 2/3 \approx 0.6667$$

14. EQUATION $X^{(N)} = U$.

The control system $x^{(n)} = u$, $|u| \leq 1$ corresponds to

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \; c = 0$$

The initial condition is zero, the optimality criterion is linear $L(Q) = \operatorname{Tr} CQ$, where

$$C(T) = \rho(F)I,$$

where F is the diagonal matrix, $F_{kk} = T^{-(n-k)}$. Vectors $\xi \in \mathbf{R}^n$ can be identified with polynomials

$$p = p_\xi(t) = \sum_{k=1}^{n} \xi_k \frac{t^{n-k}}{(n-k)!}$$

of degree $n-1$. The quadratic form $\langle Q(T)\xi, \xi\rangle$ for the globally optimal ellipsoid is given by

$$\langle Q(T)F\xi, F\xi\rangle = C_n \int_0^1 p(t)^2 j_n(t)^{-1/2} dt, \qquad (30)$$

where

$$j_n(t) = \sum_{k=0}^{n-1} \frac{t^{2k}}{(k!)^2}, \; C_n = \int_0^1 j_n(t)^{1/2} dt.$$

Note that $\sum_{k=0}^{\infty} \frac{t^{2k}}{(k!)^2} = J_0(2it)$, where J_0 is the Bessel function of order 0, $i = \sqrt{-1}$. The normalized reachable set $FD(T)$, when regarded as consisting of polynomials, has the L_1-norm on the interval $[0, 1]$ as the support function:

$$H_{FD(T)}(\xi) = \int_0^1 |p_\xi(t)| dt.$$

The problem of comparison of the reachable set $D(T)$ with the approximating ellipsoid reduces to the comparison of $L_2(d\mu)$-norm and $L_1(dt)$-norm in the space of polynomials of degree $(n-1)$, where $d\mu = C_n j_n(t)^{-1/2} dt$. The Banach-Mazur distance $d_n = d(D(T), E(Q(T)))$ between the reachable set $D(T)$ and the approximating ellipsoid and $E(Q(T))$ does not depend on T and is equal to

$$d(D(T), E(Q(T))) = \frac{1}{2} \log \sup_{p \neq 0} \frac{C_n \int_0^1 p(t)^2 j_n(t)^{-1/2} dt}{\left(\int_0^1 |p(t)| dt\right)^2}$$

where sup is taken over nonzero polynomials of degree $\leq (n-1)$. Unlike the case of many oscillators this distance goes to infinity as $n \to \infty$:

$$d_n \sim \log n \qquad (31)$$

The proof of (31) uses the theory of Legendre polynomials. According to the F. John theorem (John (1948)) the Banach-Mazur distance from any given symmetric convex body to the closest ellipsoid does not exceed $\frac{1}{2} \log n$. For large n the reachable set $D(T)$ is far from the globally optimal ellipsoid $E(Q(T))$, which, in turn, is far from being closest to $D(T)$. This is not surprising, because the body $D(T)$ does not look like an ellipsoid, e.g., its boundary has complicated singularities (Ovseevich (1998)).

15. CONCLUSION.

We present exact solutions to equations of optimal ellipsoids approximating reachable sets. It is shown, that contrary to the first impression it is often easier to find natural globally optimal ellipsoids than the locally optimal ones. In particular cases we explicitly indicate optimal ellipsoids which approximate the reachable sets in an asymptotically correct fashion. It is shown that the problem of assessment of the ellipsoidal approximation quality leads to interesting analytic problem, including averaging over multidimensional tori and the theory of the Legendre and Bessel functions.

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