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Collective Behavior of Multi-Agent Systems Under Digital Communication Network ⋆

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Abstract: This paper works on the collective behavior of multi-agent systems under digital networks. It is assumed that the agents are distributed on a plane and communicate through a digital network. The location coordinates of each agent are measured by some remote sensor and transmitted digitally to its neighbors. The topology of communication network is described by an undirected graph and control protocol is designed by a linear decentralized law. In our setting the whole dynamics of the multi-agent system is described by a hybrid system. We explore the relationship between the collective behavior of agents and the properties of agent and network. It is shown that the agents under digital communication network may display different collective behaviors: aggregation, divergence, and periodic oscillation, under different conditions. Examples show the effectiveness of our theoretical results.

1. INTRODUCTION

In recent years distributed coordination for multi-agent systems has emerged as a hot research area. This is mainly because of the demand of engineering applications such as cooperative control of unmanned air vehicles (UAVs), formation control, distributed sensor networks, attitude alignment of clusters of satellites, congestion control in communication networks, flocking of biological swarm, etc. [1] - [13]. One of interesting research topics is to find the conditions under which the dynamic agents in network achieve aggregation, or called consensus stability. *Consensus* control has been discussed systematically by Saber and Murray [5], where the *consensus* means to reach an agreement (or aggregation of agents) regarding a certain quantity of interest that depends on the initial states of all agents in network (or dynamical multi-agent system). In their work the dynamics of the agents is modeled by a simple scalar continuous-time integrator $\dot{x} = u$. Following the work of [5], Xie and Wang [13] studied the average-consensus problem where the agent is a point-mass located in a line, and its dynamics is described by the Newton's law $ma = F$. A linear consensus control protocol is established for solving such a consensus problem in their works.

In our paper the dynamics of all agents are identical and considered *Lyapunov* stable if the agents are control-free. The dynamics of the agent may describe the approximate behavior of an unmanned vehicle. Different from the framework given in [1, 5, 13], the agent communicates with its neighbors through a digital communication network in our setting. Thus, under a linear control protocol the agents in network are formulated as a hybrid decentralized networked control system (*NCS*).

We show that the dynamic multi-agent system under digital communication network possesses very different properties from that studied in [5, 13]. The agents under digital network may exhibit aggregation, divergence and periodic oscillation, under different conditions depending on such as dynamic behavior of agent, the sampling time period T, as well as the algebraic characterization of network graph.

The paper is organized as follows. Section 2 presents some properties on graph theory and describe the problem formulation. Section 3 provides main results of this paper and gives their proof. The simulation results are presented in section 4. Finally we conclude this paper in section 5.

2. PRELIMINARIES AND PROBLEM DESCRIPTION

By $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ we denote an undirected graph with an adjacency matrix $\mathcal{A} = [a_{ij}]$, where $\mathcal{V} = \{p_1, p_2, \cdots, p_M\}$ is the set of nodes, $\mathcal{E} \subseteq V \times V$ is the set of edges. The node indices belong to a finite index set $\underline{M} := \{1, 2, \cdots, M\}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (p_i, p_j)$ for some $i, j \in \underline{M}$. The adjacency elements a_{ij} are defined in the following way: $e_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} = 1$ and $e_{ij} \notin \mathcal{E} \Leftrightarrow a_{ij} = 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \underline{M}$. The set of neighbors of node p_i is denoted by $N_i = \{p_j \in \mathcal{V}; (p_i, p_j) \in \mathcal{E}\}$.

A path between a pair of distinct nodes, p_i and p_j, is meant by a sequence of distinct edges of $\mathcal{G}$ in the form $(p_i, p_{k_1}), (p_{k_1}, p_{k_2}), \cdots, (p_{k_l}, p_j)$. A graph is called *connected* if there exists a path between any two distinct nodes of the graph. The node set $\mathcal{V}$ consists of M identical

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continuous-time dynamic agents, which may represent unmanned vehicles wildly distributed in a plane. The communication among the agents is defined in the following way: if $e_{ij} \in \mathcal{E}$, then it implies that the agent p_i receives the message from p_j and $e_{ij} \notin \mathcal{E}$ implies there is no message from p_j to p_i.

The mathematical model of dynamic agent is described as follows. By $x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} \in R^2$ we denote the location coordinate of the *i-th* agent and $v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$ represents its velocity. By p_i we denote the *i-th* agent and the dynamical equation of agent p_i is described by

$$\begin{aligned} \dot{x}_i &= \begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \end{pmatrix} = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} \\ m_i \dot{v}_i &= \begin{pmatrix} \dot{v}_{i1} \\ \dot{v}_{i2} \end{pmatrix} = \rho \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} + \begin{pmatrix} u_{i1} \\ u_{i2} \end{pmatrix} \\ y_i &= F \begin{pmatrix} x_i \\ v_i \end{pmatrix} \end{aligned} \tag{1}$$

where y_i is the measured output of agent i from some remote sensor and transmitted by communication network to other agents. We assume the sensor can get the information of agents' locations, rather than their velocity. Thus, let $F = (I\ 0)$ and $y_i = x_i$.

The $u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \end{pmatrix}$ is the control input of p_i. For dynamics (1) ρ is the speed feedback gain and $\rho < 0$ implies that the dynamics of the vehicle are *Lyapunov* stable, i.e. the agent will gradually stop if there is no input control signal. The dynamical response of the agent is affected by the ρ. It is obvious that the larger ρ is (i.e. the ρ tends 0), the slower the dynamical response of the agent (1) is. Without the loss of generality, we assume that $m_i = 1$ for all $i \in \underline{M} := \{1, 2, \cdots, M\}$.

The sensors measure the state at time instants with constant interval $\{t_0, t_1, \cdots, t_k, \cdots\}$, i.e. $t_{k+1} - t_k = T; k \geq 0$ and transmit the data through communication network to its neighbors. Then we write $x_i(k) = x_i(kT)$, $v_i(k) = v_i(kT), k \geq 0$. A zero-order holder (*ZOH*) is used and we have $u_i(k) = u_i(t) = constant$ when $t \in [kT, (k+1)T)$, $k = 0, 1, 2, \cdots$. Denoting

$$\xi_i(k) = (x_i^\tau(k), v_i^\tau(k))^\tau, \quad i \in \underline{M},$$

the discretized version of (1) with sampling period T is

$$\xi_i(k+1) = A_d \xi_i(k) + B_d u_i(k) \tag{2}$$

where

$$A_d = \begin{bmatrix} I_2 & -\frac{1}{\rho}(1 - e^{\rho T}) I_2 \\ 0 & e^{\rho T} I_2 \end{bmatrix},$$

$$B_d = \begin{bmatrix} [-\frac{T}{\rho} + \frac{1}{\rho^2}(e^{\rho T} - 1)] I_2 \\ \frac{1}{\rho}(e^{\rho T} - 1) I_2 \end{bmatrix}$$

The control protocol for each agent in network is defined as follows.

$$\begin{aligned} u_i(k) &= \sum_{j \in N_i} a_{ij} F(\xi_j(k) - \xi_i(k)) \\ &= \sum_{j \in N_i} a_{ij} (x_j(k) - x_i(k)) \end{aligned} \tag{3}$$

where N_i is the set of neighbors of agent p_i and $F = [I \quad 0]$.

This paper works on the collective behaviors of multi-agent system described by (1), graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ under control protocol (3).

3. CONDITIONS OF CONSENSUS STABILITY FOR MULTI-AGENT SYSTEM

Denoting $\xi(k) = [\xi_1^\tau(k)\, \xi_2^\tau(k) \cdots \xi_M^\tau(k)]^\tau$, $\xi(k); k = 0, 1, 2 \cdots$, describe the collective behavior of the agents. The controlled dynamic agents in network are of the following form:

$$\xi(k+1) = \Omega \xi(k) \tag{4}$$

where

$$\Omega = I_M \otimes A_d - L \otimes B_d F \tag{5}$$

and L is the *Laplacian* associated with the graph $\mathcal{G}$.

Assume the M eigenvalues of the *Laplacian* L for the graph $\mathcal{G}$ are denoted as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_M$ (see [15]).

With initial condition $\xi(0)$ one has

$$\xi(k) = \Omega^k \xi(0)$$

Let J_1 be the Jordan form associated with L, there exists an orthogonal matrix W such that $W^\tau L W = W^{-1} L W = J_1$. It follows that

$$\begin{aligned} &(W^\tau \otimes I_2) \Omega (W \otimes I_2) \\ &= I_M \otimes A_d - J_1 \otimes B_d F \\ &= \text{diag}\{A_d, A_d - \lambda_2 B_d F, \cdots, A_d - \lambda_M B_d F\} \end{aligned}$$

Therefore, the behavior of the agents largely depends on the eigenvalues of $A_d - \lambda_i B_d F$, $i \in \underline{M}$.

Lemma 1. Given $\rho < 0$ and $T > 0$, the matrix A_d of (2) has only two eigenvalues equal to 1, the other two are less than 1. In other words, dynamics (2) is *Lyapunov* stable with control free.

Proof: The eigenvalues of the matrix A_d are

$$s_{11} = s_{12} = 1, s_{13} = s_{14} = \mathrm{e}^{\rho T} < 1.$$

We now discuss the eigenvalues of $A_d - \lambda_i B_d F$ for $i \in \{2, \cdots, M\}$. We write

$$\begin{aligned} &A_d - \lambda_i B_d F \\ &= \begin{bmatrix} 1 + \frac{\lambda_i T}{\rho} + \frac{\lambda_i}{\rho^2}(1 - e^{\rho T}) & \frac{1}{\rho}(e^{\rho T} - 1) \\ \lambda_i \frac{1}{\rho}(1 - e^{\rho T}) & e^{\rho T} \end{bmatrix} \otimes I_2 \end{aligned}$$

and denote

$$\overline{A}_i = \begin{bmatrix} 1 + \frac{\lambda_i T}{\rho} + \frac{\lambda_i}{\rho^2}(1 - e^{\rho T}) & \frac{1}{\rho}(e^{\rho T} - 1) \\ \lambda_i \frac{1}{\rho}(1 - e^{\rho T}) & e^{\rho T} \end{bmatrix}$$

Thus, we focus on discussing the eigenvalues of $\overline{A}_i$.

Consider the characteristic polynomial of $\overline{A}_i$,

$$\begin{aligned} f_i(s) &= det(sI - \overline{A}_i) \\ &= s^2 + a_{i1}s + a_{i0} \end{aligned} \tag{6}$$

where $i \in \{2, \cdots, M\}$ and

$$\begin{aligned} a_{i1} &= -[e^{\rho T} + 1 + \frac{\lambda_i}{\rho^2}(\rho T + 1 - e^{\rho T})] \\ a_{i0} &= e^{\rho T} + \frac{\lambda_i}{\rho^2}(1 + \rho T e^{\rho T} - e^{\rho T}) \end{aligned} \tag{7}$$

For any $\lambda_i > 0$ the eigenvalues of $A_d - \lambda_i B_d F$ are located inside the unit circle centered at the origin if and only if the following inequalities hold

$$\begin{cases} f_i(1) = 1 + a_{i1} + a_{i0} > 0 \\ f_i(-1) = 1 - a_{i1} + a_{i0} > 0 \\ |a_{i0}| < 1 \end{cases} \tag{8}$$

Using (7), one can obtain the equivalent inequalities as the following

$$\begin{cases} 1 - e^{\rho T} - \frac{\lambda_i}{\rho^2}(\rho T e^{\rho T} + 1 - e^{\rho T}) > 0 \\ 2(1 + e^{\rho T}) + \frac{\lambda_i}{\rho^2}(2 + \rho T + \rho T e^{\rho T} - 2e^{\rho T}) > 0 \end{cases} \tag{9}$$

It is well known that the matrix $A_d - \lambda_i B_d F$ is *Schur* stable if and only if (8) hold. Thus, the inequalities (9) are fundamental tools for us to study the behaviors of the agents.

3.1 How Sampling Period Affects Collective Behavior of Multi-Agents?

We discuss how the sampling period T affect the collective behavior of the multi-agent system?

Lemma 2. Given $\rho < 0$ and $\lambda > 0$, the first inequality in (9) holds if one of the following conditions holds.

(1) $\rho \le -\sqrt{\lambda}$;

(2) there exists an unique $T_1 > 0$ such that $T \in (0, T_1)$ where T_1 satisfies

$$1 - e^{\rho T_1} - \frac{\lambda}{\rho^2}(\rho T_1 e^{\rho T_1} + 1 - e^{\rho T_1}) = 0. \tag{10}$$

Proof: Denoting $p(T) = 1 - e^{\rho T} - \frac{\lambda}{\rho^2}(\rho T e^{\rho T} + 1 - e^{\rho T})$, one can obtain its derivative

$$p'(T) = -(\rho + \lambda T) \cdot e^{\rho T}.$$

If $T < \frac{-\rho}{\lambda}$, then $p(T)$ is an increasing function of $T > 0$. If $T > \frac{-\rho}{\lambda}$, then $p(T)$ is an decreasing function of $T > 0$. Consider

$$p(0) = 0, \quad p(+\infty) = \lim_{T \to +\infty} p(T) = 1 - \frac{\lambda}{\rho^2}.$$

Therefore, if $\rho \le -\sqrt{\lambda}$, which implies $p(+\infty) > 0$, then the first inequality in (9) holds for all $T \in (0, +\infty)$. Otherwise, there exists an unique $T_1 > \frac{-\rho}{\lambda}$ such that $p(T_1) = 0$ and the first inequality in (9) holds for all $T \in (0, T_1)$.

Lemma 3. Given $\rho < 0$ and $\lambda > 0$, there exists an unique $T_2 > 0$ such that the second inequality in (9) holds for all $T \in (0, T_2)$ where T_2 satisfies

$$2(1 + e^{\rho T_2}) + \frac{\lambda}{\rho^2}(2 + \rho T_2 + \rho T_2 e^{\rho T_2} - 2e^{\rho T_2}) = 0. \tag{11}$$

Proof: Denoting

$$q(T) = 2(1 + e^{\rho T}) + \frac{\lambda}{\rho^2}(2 + \rho T + \rho T e^{\rho T} - 2e^{\rho T}),$$

one can obtain

$$q'(T) = 2\rho e^{\rho T} + \frac{\lambda}{\rho}(1 + \rho T e^{\rho T} - e^{\rho T}).$$

Let $r(T) = 1 + \rho T e^{\rho T} - e^{\rho T}$, one can obtain $r'(T) = \rho^2 T e^{\rho T} > 0$ for all $T > 0$. Consider $r(0) = 0$, it is hold that $r(T) > 0$ for all $T > 0$. Thus, one can obtain $q'(T) = 2\rho e^{\rho T} + \frac{\lambda}{\rho} \cdot r(t) < 0$, which implies that $q(T)$ is an decreasing function of $T > 0$. Consider

$$q(0) = 4, \quad q(+\infty) = \lim_{T \to +\infty} q(T) = -\infty.$$

Therefore, there exists an unique T_2 such that $q(T_2) = 0$ and the second inequality in (9) holds for all $T \in (0, T_2)$.

From above analysis, we can get the following result. We assume that the dynamics (1) with $\rho < 0$ and the topology $\mathcal{G}$ of communication network are given. Thus, the M eigenvalues, $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_M$, of the *Laplacian* L for the graph $\mathcal{G}$ are fixed.

Proposition 4.

(1) If $\rho \le -\sqrt{\lambda_M}$, then there exists an unique $T_M > 0$ such that the inequalities in (9) holds for $i \in \underline{M}$ and $T \in (0, T_M)$, where $T_M > 0$ satisfies

$$2(1+e^{\rho T_M}) + \frac{\lambda_M}{\rho^2}(2+\rho T_M+\rho T_M e^{\rho T_M} - 2e^{\rho T_M}) = 0; \tag{12}$$

(2) If $\rho > -\sqrt{\lambda_M}$, then there exists an unique $\bar{T}_M = min\{T_M, \tilde{T}_M\} > 0$ such that the inequalities in (9) holds for all $T \in (0, \bar{T}_M)$ where $\tilde{T}_M$ satisfies

$$1 - e^{\rho \tilde{T}_M} - \frac{\lambda_M}{\rho^2}(\rho \tilde{T}_M e^{\rho \tilde{T}_M} + 1 - e^{\rho \tilde{T}_M}) = 0 \tag{13}$$

Then we have the following theorem.

Theorem 5. Under linear control protocol (3), the agents in network is described by (4)-(5). If the sampling period T satisfies

$$0 < T < T^* \tag{14}$$

where

$$T^* = \begin{cases} T_M \quad if \quad \rho \le -\sqrt{\lambda_M} \\ \bar{T}_M \quad if \ -\sqrt{\lambda_M} < \rho < 0 \end{cases} \tag{15}$$

λ_M denote the biggest eigenvalue of the *Laplacian* matrix L, T_M and $\bar{T}_M$ are defined in Proposition 4. Then it hold that

$$\lim_{k \to \infty} \Omega^k = \mathrm{w}_r \mathrm{w}_l^T \tag{16}$$

where

$$\begin{aligned} \mathrm{w}_r &= \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho^{-1}I_2 \\ 0 \end{bmatrix} \\ \mathrm{w}_l &= \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho I_2 \\ -I_2 \end{bmatrix} \end{aligned} \quad (17)$$

and $\mathrm{w}_l^\tau \mathrm{w}_r = I_2$.

Proof: From Proposition 4, all eigenvalues of $A_d - \lambda_i B_d F$, for $i \geq 1$, are not located outside the unit circle centered at origin when $0 < T < T^*$. There are $4M-2$ eigenvalues of Ω are located inside a unit circle and two eigenvalues are 1.

By (5) one has

$$\Omega \cdot \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho^{-1}I_2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho^{-1}I_2 \\ 0 \end{bmatrix}$$

Thus, $\mathrm{w}_r = \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho^{-1}I_2 \\ 0 \end{bmatrix}$ are the two right-eigenvectors of Ω with respect to eigenvalue $\lambda = 1$.

In a similar way,

$$\frac{1}{\sqrt{M}}\mathbf{1}_M^\tau \otimes [\rho I_2 \quad -I_2] \cdot \Omega = \frac{1}{\sqrt{M}}\mathbf{1}_M^\tau \otimes [\rho I_2 \quad -I_2],$$

It is obvious that $\mathrm{w}_l = \frac{1}{\sqrt{M}}\mathbf{1}_M \otimes \begin{bmatrix} \rho I_2 \\ -I_2 \end{bmatrix}$ are the two left-eigenvectors of Ω with respect to eigenvalue $\lambda = 1$.

Furthermore, as there are $4M-2$ eigenvalues of Ω locate inside a unit circle, one can check

$$\lim_{k\to\infty}\Omega^k = \mathrm{w}_r \mathrm{w}_l^\tau$$

It is also easy to check that $\mathrm{w}_l^\tau \mathrm{w}_r = I_2$.

Now we give the main result of our paper.

Theorem 6. Under linear control protocol (3) consider dynamic agents (1) in digital communication network described by $\mathcal{G}$.

(1) If the sampling period T satisfying $T < T^*$, where the T^* is defined in (15), then the agents will achieve consensus stability;

(2) If $T = T^*$, then the dynamic agents appear globally asymptotically stable periodic trajectories;

(3) If $T > T^*$, then the dynamic agents appear divergent trajectories.

Proof: If $0 < T < T^*$, by Lemma 5 we get

$$\xi(k) = \Omega^k \xi(0) \quad \text{and} \quad \lim_{k\to\infty}\Omega^k = \mathrm{w}_r \mathrm{w}_l^\tau$$

It follows that

$$\begin{aligned} &\lim_{k\to\infty}\xi(k) = \lim_{k\to\infty}\Omega^k \xi(0) \\ &= \mathrm{w}_r \mathrm{w}_l^\tau \xi(0) \\ &= \frac{1}{M}\mathbf{1}_M \otimes \begin{bmatrix} \rho^{-1}I_2 \\ 0 \end{bmatrix} \mathbf{1}_M^\tau \otimes [\rho I_2 \quad -I_2]\xi_j(0) \\ &= \frac{1}{M}\mathbf{1}_M \mathbf{1}_M^\tau \otimes \begin{bmatrix} I_2 & -\frac{1}{\rho}I_2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ v_1(0) \\ \vdots \\ x_M(0) \\ v_M(0) \end{bmatrix} \end{aligned}$$

Therefore, as $k \to \infty$,

$$x_{ij}(k) \to \frac{1}{M}\{\sum_{l=1}^{M}[x_{lj}(0) - \rho^{-1}v_{lj}(0)]\}$$

and it is obvious that

$$\lim_{k\to\infty} v_{ij}(k) = 0, \quad i \in \{1, 2, \cdots, M\}$$

This implies the agents in network globally asymptotically achieve the consensus stability.

When $T = T^*$, one can show that $A_d - \lambda_M B_d F$ has two eigenvalues located on the unit circle, and all other eigenvalues of $A_d - \lambda_j B_d F$, $j \in \{2, \ldots, M-1\}$ locate inside the unit circle. Thus all $A_d - \lambda_i B_d F$ $i \in \{2, \ldots, M-1\}$ are *Lyapunov* stable. By Lemma 1 the Ω has only four eigenvalues being 1 when $T = T^*$. According to the results of linear control theory, the agents tend to asymptotically periodic trajectories centered at same fixed point.

If $T > T^*$, then the system matrix Ω is not *Lyapunov* stable. Thus, the agents may have divergent trajectories.

3.2 How Feedback Gain ρ Affects on Collective Behavior of Multi-Agent Systems

Now we discuss how the negative feedback gain ρ affects on the collective behavior of the multi-agent systems when the sampling period T is fixed.

Lemma 7.

(1) Given $\lambda_i > 0$; $i \in$ $underlineM$ there exists an unique $\rho_i < 0$ such that the inequalities (9) hold for all $\rho \in (-\infty, \rho_i)$;

(2) All eigenvalues of $A_d - \lambda_i B_d F$ ($i \in$ $underlineM$) are *Schur* stable if and only if $\rho < \rho^* = \rho_M (< 0)$.

Proof: Denoting $g(\rho) = 1 - e^{\rho T} - \frac{\lambda_i}{\rho^2}(\rho T e^{\rho T} + 1 - e^{\rho T})$, one can obtain

$$g'(\rho) = -Te^{\rho T} + \frac{\lambda_i}{\rho^3}(2 + 2\rho T e^{\rho T} - 2e^{\rho T} - \rho^2 T^2 e^{\rho T}).$$

Let $h(\rho) = 2 + 2\rho T e^{\rho T} - 2e^{\rho T} - \rho^2 T^2 e^{\rho T}$, the derivation of $h(\rho)$ can be written as $h'(\rho) = -2\rho^2 T^3 e^{\rho T} < 0$, which implies that $h(\rho)$ is an decreasing function of $\rho < 0$. For given $T > 0$ and $\lambda_i > 0$, one easily get

$$\begin{aligned} h(0^-) &= \lim_{\rho\to 0^-} h(\rho) = 0, \\ h(-\infty) &= \lim_{\rho\to -\infty} h(\rho) = 2. \end{aligned}$$

Then $0 < h(\rho) < 2$ for all $\rho < 0$. Furthermore, one can obtain that $g'(\rho) < 0$ for all $\rho < 0$. We conclude that $g(\rho)$ is an decreasing function of $\rho < 0$. One also get

$$\begin{aligned} g(0^-) &= \lim_{\rho\to 0^-} g(\rho) = -\frac{\lambda_i T^2}{2}, \\ g(-\infty) &= \lim_{\rho\to -\infty} g(\rho) = 1. \end{aligned}$$

Therefore, there exists an unique $\rho_i < 0$ such that the inequality $g(\rho) > 0$ hold for all $\rho \in (-\infty, \rho_i)$, which implies that the first inequality of (9) hold for all $\rho \in (-\infty, \rho_i)$. When given the sampling period satisfying $T \in (0, T_M)$ where T_M defined in (12), it is obvious that the second inequality of (9) hold if $\rho < 0$. Therefore, For given $\lambda_i > 0$ and proper sampling period T there exists an unique $\rho_i < 0$ such that the inequalities (9) hold for all $\rho \in (-\infty, \rho_i)$. Moreover, all eigenvalues of $A_d - \lambda_i B_d F$ $(i \in \{2, 3, \cdots, M\})$ are *Schur* stable if and only if $\rho < \rho^* = \rho_M (< 0)$.

Theorem 8. Consider dynamic agents (1) in digital network $\mathcal{G}$ with sampling period $T \in (0, T_M)$, where T_M is defined in (12). Under linear control protocol (3), the agents in network $\mathcal{G}$ will achieve consensus stability if the ρ in the dynamical equation (1) satisfies

$$\rho < \rho^* < 0 \tag{18}$$

where ρ^* satisfies

$$1 - e^{\rho^* T} - \frac{\lambda_M}{(\rho^*)^2}(1 + \rho^* T e^{\rho^* T} - e^{\rho^* T}) = 0. \tag{19}$$

Moreover, if $\rho = \rho^*$, the dynamic agents appear globally asymptotically stable periodic trajectories.

If $\rho > \rho^*$, then the dynamic agents appear divergent trajectories.

Proof: Based on the Lemma 7, the proof of the theorem follows the similar line as that of Theorem 6, omitted to save the space.

4. SIMULATION

We study a simple example to show that our results are effective. The network of dynamic agents is described in Figure 1.

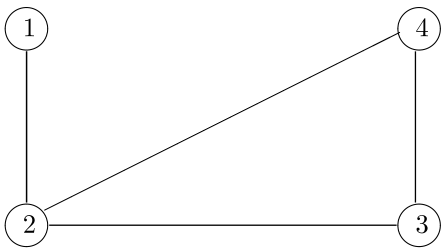

Fig.1. An undirected graph $\mathcal{G}$ with $M = 4$ nodes.

Its *Laplacian* is

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3, \lambda_4 = 4$.

First, we consider that the gain ρ is fixed and the sampling period T is varying. In order to implement the consensus stability we have to find the up bound T^* of the sampling period. Let

$$q(T) = 2(1 + e^{\rho T}) + \frac{\lambda_4}{\rho^2}(2 + \rho T + \rho T e^{\rho T} - 2e^{\rho T})$$
$$p(T) = 1 - e^{\rho T} - \frac{\lambda_4}{\rho^2}(\rho T e^{\rho T} + 1 - e^{\rho T})$$

One can obtain the different sampling periods T_M and $\tilde{T}_M$ (refer to Section III) using Maple computation.

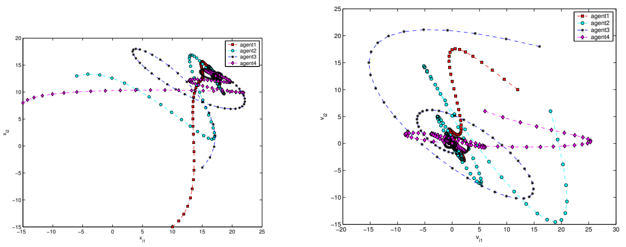

Fig. 1. State and Velocity trajectories of agents with $T = 0.1$,$\rho = -0.8$.

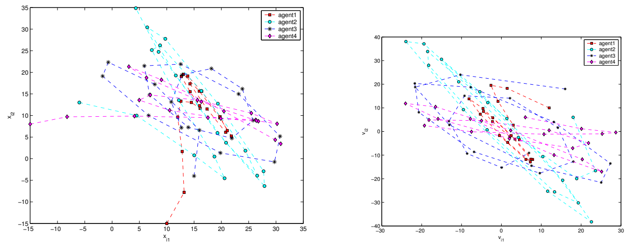

Fig. 2. State and Velocity trajectories of agents with $T = 0.5$, $\rho = -0.8$.

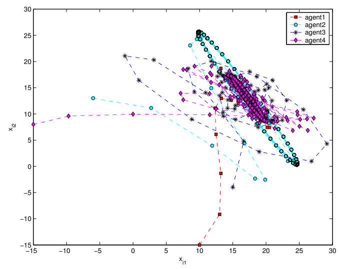

Fig. 3. State trajectories of agents with $T = 0.4239146850$, $\rho = -0.8$.

ρ	T_M	$\tilde{T}_M$	T^*
-0.8	2.130588974	0.4239146850	0.4239146850
-1.99	1.960174297	3.255052884	1.960174297
-2.5	2.040311827	arbitrary	2.040311827

We choose $\rho = -0.8$ and $T = 0.1 < T^*$, which satisfy (14). Figure 1 demonstrates that with initial conditions $x_1(0) = [10 \quad -15]^\tau$, $x_2(0) = [-6 \quad 13]^\tau$, $x_3(0) = [15 \quad -4]^\tau$, $x_4(0) = [-15 \quad ,8]^\tau$ and the initial velocities $v_1(0) = [12 \quad 10]^\tau$, $v_2(0) = [18 \quad 6]^\tau$, $v_3(0) = [16 \quad 18]^\tau$, $v_4(0) = [6 \quad 6]^\tau$, respectively, the state trajectories of agents aggregates to $x^* = [17.25 \quad 13]^\tau$, and velocites of agents tend to zero.

But if one chooses $\bar{T} = 0.5 > T^*$, $\rho = -0.8$ under the same initial conditions as before, Fig.2 shows the divergence of the agents.

Under the same initial condition and $\rho = -0.8$, the asymptotically stable periotic trajectories of states of the agents appear when $T = 0.4239146850 = T^*$, see Figure 3. Their velocity trajectories appear in periodic orbits, showing in Figure 4.

Next we study the behavior of dynamic agents under a fixed sampling period T and varying ρ. Without the loss of generality let $T = 1$, one gets $\rho^* = -1.513705800$. Under the same initial condition as above when $\rho = -2 <$

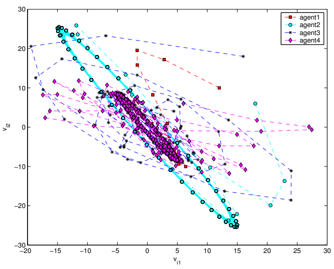

Fig. 4. Velocity trajectories of agents with $T = 0.4239146850$, $\rho = -0.8$.

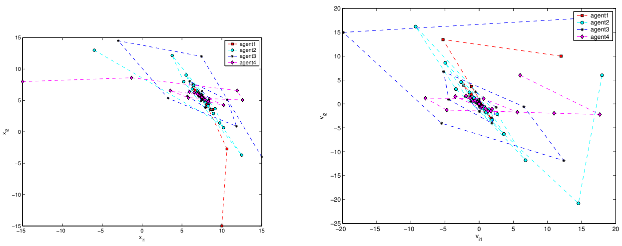

Fig. 5. State and Velocity trajectories of agents with $T = 1$, $\rho = -2$.

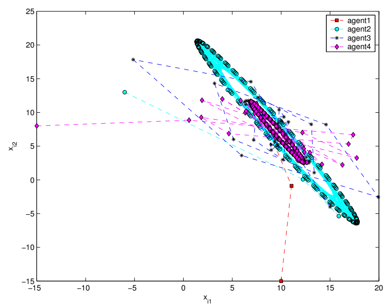

Fig. 6. State trajectories of agents with $T = 1$, $\rho = -1.513705800$.

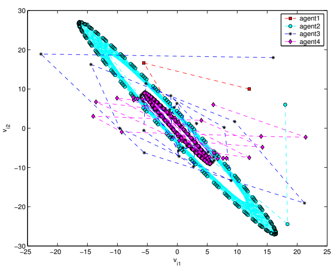

Fig. 7. Velocity trajectories of agents with $T = 1$, $\rho = -1.513705800$.

ρ^*, Figure 5 shows that the agents aggregate together, meantime their velocities tend to zero.

In Figure 6 the asymptotically stable periotic state trajectories of the agents appear when $\rho = -1.513705800$. Their velocity trajectories are shown in Figure 7.

5. CONCLUSION

This work shows that the dynamical agents in a digital communication network demonstrate quite different behaviors from that of time-continuous communication networks. The collective behaviors of agents depend on the sampling period and the dynamical property of dynamic agent. In order to implement consensus stability of the agents in such network the sampling period has to be smaller than certain value. The agents could have period trajectories centered at same point or divergent in a plane if the sampling period meets some conditions. Moreover, when the sampling period is fixed in an proper range, we discuss how the dynamic property of agents affects on collective behavior of agents.

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