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Bilaterally Flexible Lyapunov Inequalities for Nonlinear Small-Gain Method Covering iISS Systems ⋆

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Abstract: This paper develops a new tool to study stability of interconnection of integral input-to-state stable (iISS) systems. A sort of freedom is introduced into Lyapunov inequalities each system is to satisfy in addition to small-gain-type conditions. The purpose of this paper is to extend the technique of "flexible Lyapunov inequalities" developed previously for input-to-state stable(ISS) systems. The achievement is threefold. One is the employment of flexibility for both the systems connected with each other. The former technique only allows the flexibility to appear in one of the mutually connected systems. The second accomplishment is to cover iISS systems. The third is unification of the treatment of iISS and ISS systems. Establishment of stability is based on explicit construction of smooth Lyapunov functions.

1. INTRODUCTION

In the framework of dissipative theory(see Willems (1972)), one can derive dissipative properties of interconnected systems from dissipativity of individual subsystems (see, e.g., Hill & Moylan (1977)). One of useful dissipative properties is input-to-state stability(ISS) proposed in Sontag (1989). Its storage functions are ISS Lyapunov functions. The ISS small-gain theorem proposed by Jiang et al. (1994) and Teel (1996) deals with feedback interconnections of ISS systems and establishes their stability based on nonlinear gain when nonlinear loop gain is less than identity. The nonlinear gain is computed from dissipation inequalities of individual subsystems. The inequalities are sometimes referred to as Lyapunov inequalities or Hamilton-Jacobi inequalities. There are two ways to look at the inequalities.

- Given a supply rate, solve a dissipation inequality for a storage function.
- Given a storage function, modify the dissipation inequality maintaining eligible dissipation.

The latter fits the idea of Lyapunov redesign. The former view is essentially the direct approach to optimal control. In both the situations, when one applies the ISS small-gain theorem, obtaining a successful Lyapunov inequality conforming to a small-gain condition is not a straightforward task. This fact motivated the author to develop his own idea of flexible Lyapunov inequalities, which provides many Lyapunov inequalities with which a single small-gain-type condition can establish stability of an interconnection of ISS systems (see Ito (2003, 2005a)). The technique introduces flexibility in choosing supply rates.

This paper is the fundamental upgrade of flexible Lyapunov inequalities. The essential progress is threefold.

(1) Introduction of flexibility into both the systems connected with each other

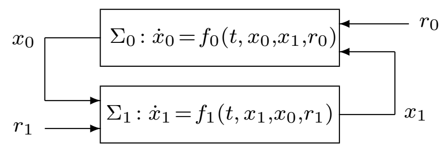

Fig. 1. Interconnected system Σ

(2) Covering integral input-to-state stability(iISS) property
(3) Complete unification of the treatment of iISS and ISS systems

All of the previous studies only allows the flexibility to be included in one of the mutually connected systems. The second point was partially achieved in Ito (2005b, 2006b). This paper removes all restrictive and artificial assumptions required there, which enables the third point. The class of iISS is broader than ISS. An ISS system is always iISS. The converse does not hold. The class of iISS systems encompasses more systems of practical importance than the ISS (see Angeli et al. (2000)).

In this paper, the interval $[0, \infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. Euclidean norm of a vector in $\mathbb{R}^n$ of dimension n is denoted by $|\cdot|$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ and written as $\gamma \in \mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ and written as $\gamma \in \mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty} \gamma(r) = \infty$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively.

2. INTERCONNECTED SYSTEM

Consider the nonlinear interconnected system Σ shown in Fig.1. Suppose that the subsystems are described by

$$\Sigma_0 : \ \dot{x}_0 = f_0(t, x_0, u_0, r_0) \tag{1}$$

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \tag{2}$$

These two systems are connected with each other through $u_0 = x_1$ and $u_1 = x_0$. Assume that $f_0(t, 0, 0, 0) = 0$

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and $f_1(t,0,0,0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$. We also assume that the functions f_0 and f_1 are piecewise continuous in t, and locally Lipschitz in the other arguments. The state vector of the interconnected system Σ is $x = [x_0^T, x_1^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. The exogenous signals $r_0 \in \mathbb{R}^{b_0}$ and $r_1 \in \mathbb{R}^{b_1}$ form a vector $r = [r_0^T, r_1^T]^T \in \mathbb{R}^b$. We will exploit dissipative property of each system instead of using f_i directly. When we investigate global asymptotic stability of the interconnection, we suppose that $r_i(t) \equiv 0$, $i = 1, 2$.

3. BILATERALLY FLEXIBLE LYAPUNOV INEQUALITIES

The main result is stated in the following theorem, which establishes stability of the interconnected system Σ based on Lyapunov inequalities of the subsystems. The functions $\hat{\lambda}_i$'s below provide flexibility in the Lyapunov inequalities.

Theorem 1. For $i=0,1$, consider the following functions:

$$\alpha_i, \sigma_i, \sigma_{ri} \in \mathcal{K} \tag{3}$$

$$\hat{\lambda}_i, \hat{\lambda}_{ri} : \mathbb{R}_+ \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{4}$$

$$\hat{\lambda}_i(s) > 0, \ \forall s \in (0, \infty) \tag{5}$$

$$V_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}, \ \mathbf{C}^1 \tag{6}$$

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \overline{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{7}$$

$$\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty \tag{8}$$

For each $i = 0, 1$, assume that

$$\frac{dV_i}{dt} \leq \hat{\lambda}_i(V_i(t,x_i))\left[-\alpha_i(|x_i|) + \sigma_i(|x_{1-i}|)\right] + \hat{\lambda}_{ri}(V_i(t,x_i))\sigma_{ri}(|r_i|) \tag{9}$$

holds along the trajectories of the system Σ_i for all $x \in \mathbb{R}^n$, $r_i \in \mathbb{R}^{b_i}$ and $t \in \mathbb{R}_+$. Suppose that there exist real numbers $c_0 > 1$ and $c_1 > 1$ such that

$$c_1\sigma_1 \circ \underline{\alpha}_0^{-1} \circ \overline{\alpha}_0 \circ \alpha_0^{-1} \circ c_0\sigma_0(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \tag{10}$$

is satisfied, then the following facts hold.

(a) If
$$\lim_{s\to\infty} \alpha_1(s) < \infty \ \Rightarrow \ \lim_{s\to\infty} \hat{\lambda}_1(s) < \infty \tag{11}$$
holds, the equilibrium $x = 0$ of Σ is uniformly globally asymptotically stable (UGAS).

(b) If (11),
$$\left.\begin{array}{l}\lim_{s\to\infty} \alpha_0(s) < \infty \\ \lim_{s\to\infty} \alpha_1(s) < \infty\end{array}\right\} \Rightarrow \ \lim_{s\to\infty} \hat{\lambda}_0(s) < \infty \tag{12}$$
$$\limsup_{s\to\infty} \frac{\hat{\lambda}_{ri}(s)}{\hat{\lambda}_i(s)} < \infty \tag{13}$$
and one of
[A1] $\lim_{s\to\infty} \alpha_0(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_0(s) < \infty$
[A2] $\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_0(s) < \infty$
are satisfied, the interconnected system Σ is iISS with respect to input r and state x.

(c) If (11), (12), (13) and
[A3] $\lim_{s\to\infty} \alpha_0(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty$
are satisfied, the interconnected system Σ is ISS with respect to input r and state x.

It is stressed that (10) requires

$$\lim_{s\to\infty} \alpha_0(s) = \infty \ \vee \ \infty > \lim_{s\to\infty} \alpha_0(s) > \lim_{s\to\infty} \sigma_0(s) \tag{14}$$

When we take $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$, i.e., no flexibilities, Theorem 1 can be viewed as a nonlinear small-gain theorem. The dissipation inequality (9) implies that each Σ_i is iISS with respect to input (x_{1-i}, r_i) and state x_i. In the case of *[A3]*, the two systems Σ_i, $i = 0, 1$ are ISS (see Sontag & Wang (1995)). If $\lim_{s\to\infty} \alpha_i(s) < \infty$, the system Σ_i is not necessarily ISS (see Sontag (1998); Angeli et al. (2000)). The choice $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$ in Theorem 1 includes the results of Ito (2006a) as special cases. In the case of *(c)*, Theorem 1 with $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$ reduces to the ISS small-gain theorem in Jiang et al. (1994) and Teel (1996). Indeed, the condition (10) associated with (9) for $\hat{\lambda}_0 = \hat{\lambda}_1 = 1$ is a nonlinear small-gain condition.

Remark 2. The above theorem cannot be explained by individual Lyapunov functions in the form of $\hat{V}_i = \int_0^{V_i} 1/\hat{\lambda}_i(s)ds$ since they are not guaranteed to be integrable and radially unbounded. It is also mentioned that the technique of changing supply functions proposed in Sontag & Teel (1995) is not applicable to iISS systems.

Remark 3. When

$$\lim_{s\to\infty} \alpha_1(s) = \infty \ \vee \ \infty > \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) \tag{15}$$

holds, the inequality

$$c_0\sigma_0 \circ \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1(s) \leq \alpha_0 \circ \overline{\alpha}_0^{-1} \circ \underline{\alpha}_0(s), \ \forall s \in \mathbb{R}_+ \tag{16}$$

held with a pair of $c_0, c_1 > 1$ is implied by the existence of another pair of $c_0, c_1 > 1$ achieving (10). Therefore, when (14) and (15) hold, the inequalities (10) and (16) are equivalent in the sense of the existence of $c_0, c_1 > 1$.

Remark 4. The primitive idea of flexible Lyapunov inequalities first appeared in Ito (2003) for ISS systems in a restrictive setting. It was extended to a general case of ISS systems by Ito (2005a). The result was, however, unilateral, i.e., it is the *(c)* case with $\hat{\lambda}_1 = 1$ in Theorem 1. The concept of unilateral flexible Lyapunov inequalities was first extended to iISS system in Ito (2005b) with technical assumptions and a small-gain-type condition which were more complex than (10). Ito (2006b) was able to simplify that condition into (10) without paying attention to construction of Lyapunov functions. It, however, only covers *[A2]* of Theorem 1 with $\hat{\lambda}_1 = 1$. In addition, Ito (2006b) imposes global smoothness and local analyticity.

Remark 5. Arcak et al. (2002) considers an time-invariant cascade interconnection in which an iISS system is driven by a globally asymptotically stable (GAS) system. Instead of constructing Lyapunov functions, they take a trajectory-based approach to prove GAS of the cascade under the assumption of trade-off between the convergence rate of the driving subsystem and the growth rate of iISS gain of the driven subsystem. This paper continues pursuing the stability problem of interconnections involving iISS subsystems in order to tackle feedback interconnections. However, the focus is not simply on the extention of their philosophy, but rather on the introduction of flexibilities $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ and the construction of Lyapunov functions for the whole system in the presence of external signals.

4. STATIC SYSTEMS

This section supposes that one of the subsystems in Σ of Fig.1 is static. For this purpose, replace (1) by

$$\Sigma_0: \ v_0 = h_0(t, u_0, r_0) \tag{17}$$

which is connected with Σ_1 through $u_0 = x_1 \in \mathbb{R}^{n_1}$ and $u_1 = v_0 \in \mathbb{R}^{n_0}$. Assume that $h_0(t, 0, 0) = 0$ holds for all $t \in [t_0, \infty)$, $t_0 \geq 0$. Piecewise continuity in t and locally Lipschitzness in (u_0, r_0) are also assumed for h_0. The state vector of the interconnected system Σ becomes $x = x_1 \in \mathbb{R}^n$, where $n = n_1$. The following theorem demonstrates that (10) can be relaxed slightly in the static case.

Theorem 6. Consider the following functions:

$$\alpha_i, \sigma_i, \sigma_{ri} \in \mathcal{K}, \ i = 0, 1 \tag{18}$$
$$\hat{\lambda}_0, \hat{\lambda}_{r0} : \mathbb{R}^{n_0} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{19}$$
$$\hat{\lambda}_0(v_0) > 0, \ \forall v_0 \in \mathbb{R}^{n_0} \setminus \{0\} \tag{20}$$
$$\hat{\lambda}_1, \hat{\lambda}_{r1} : \mathbb{R}_+ \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{21}$$
$$\hat{\lambda}_1(s) > 0, \ \forall s \in (0, \infty) \tag{22}$$
$$V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}, \ \mathbf{C}^1 \tag{23}$$
$$\underline{\alpha}_1(|x_1|) \leq V_1(t, x_1) \leq \overline{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, t \in \mathbb{R}_+ \tag{24}$$
$$\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty \tag{25}$$

Assume that the system Σ_0 satisfies

$$0 \leq \hat{\lambda}_0(v_0) \left[-\alpha_0(|v_0|) + \sigma_0(|x_1|)\right] + \hat{\lambda}_{r0}(v_0)\sigma_{r0}(|r_0|) \tag{26}$$

for all $x_1 \in \mathbb{R}^{n_1}$, $r_0 \in \mathbb{R}^{b_0}$ and $t \in \mathbb{R}_+$, and that

$$\frac{dV_1}{dt} \leq \hat{\lambda}_1(V_1(t, x_1)) \left[-\alpha_1(|x_1|) + \sigma_1(|u_1|)\right] + \hat{\lambda}_{r1}(V_1(t, x_1))\sigma_{r1}(|r_1|) \tag{27}$$

holds along the trajectories of the system Σ_1 for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_0}$, $r_1 \in \mathbb{R}^{b_1}$ and $t \in \mathbb{R}_+$. Suppose that there exist real numbers $c_0 > 1$ and $c_1 > 1$ such that

$$c_1\sigma_1 \circ \alpha_0^{-1} \circ c_0\sigma_0(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \tag{28}$$

is satisfied, then the following facts hold.

(a) If (11) holds, the equilibrium $x = 0$ of Σ is UGAS.

(b) If (11), (12),

$$\lim_{s\to\infty} \sup_{s \leq |v_0|} \frac{\hat{\lambda}_{r0}(v_0)}{\hat{\lambda}_0(v_0)} < \infty \tag{29}$$

$$\limsup_{s\to\infty} \frac{\hat{\lambda}_{r1}(s)}{\hat{\lambda}_1(s)} < \infty \tag{30}$$

and one of *[A1]* and *[A2]* are satisfied, the interconnected system Σ is iISS with respect to input r and state x_1.

(c) If (11), (12), (29), (30) and *[A3]* are satisfied, the interconnected system Σ is ISS with respect to input r and state x_1.

The right-hand side of (26) plays the role of a supply rate of Σ_0 although energy is never stored in static systems. Note that (28) again requires (14). The local Lipschitzness of $h_0(t, \cdot)$ guarantees the existence of a triplet $\{\alpha_0, \sigma_0, \sigma_{r0} \in \mathcal{K}\}$ fulfilling (14) whenever the static system Σ_0 satisfies (26). The following shows that the stability condition for the interconnected system can be simplified further when the external signal r_0 affecting the static system is absent.

Theorem 7. Consider functions satisfying

$$\alpha_0, \sigma_0, \alpha_1, \sigma_1, \sigma_{r1} \in \mathcal{K} \tag{31}$$
$$\hat{\lambda}_0 : \mathbb{R}^{n_0} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{32}$$
$$\hat{\lambda}_0(v_0) > 0, \ \forall v_0 \in \mathbb{R}^{n_0} \setminus \{0\} \tag{33}$$
$$\hat{\lambda}_1, \hat{\lambda}_{r1} : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{34}$$
$$\inf_{t \in \mathbb{R}_+} \hat{\lambda}_1(t, x_1) > 0, \quad \forall x_1 \in \mathbb{R}^{n_1} \setminus \{0\} \tag{35}$$

and (23), (24) and (25). Assume that Σ_0 satisfies

$$0 \leq \hat{\lambda}_0(v_0) \left[-\alpha_0(|v_0|) + \sigma_0|x_1|)\right] \tag{36}$$

for all $x_1 \in \mathbb{R}^{n_1}$ and $t \in \mathbb{R}_+$, and that

$$\frac{dV_1}{dt} \leq \hat{\lambda}_1(t, x_1) \left[-\alpha_1(|x_1|) + \sigma_1(|u_1|)\right] + \hat{\lambda}_{r1}(t, x_1)\sigma_{r1}(|r_1|) \tag{37}$$

holds along the trajectories of the system Σ_1 for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_0}$, $r_1 \in \mathbb{R}^{b_1}$ and $t \in \mathbb{R}_+$. Suppose that there exist real numbers $c_0 > 1$ and $c_1 > 1$ such that

$$c_1\sigma_1 \circ \alpha_0^{-1} \circ c_0\sigma_0(s) \leq \alpha_1(s), \ \forall s \in \mathbb{R}_+ \tag{38}$$

is satisfied, then the following facts hold.

(a) The equilibrium $x = 0$ of Σ is UGAS.

(b) If there exists $k \in (-\infty, 1]$ such that

$$\lim_{s\to\infty} \sup_{s \leq |x_1|, t \in \mathbb{R}_+} \frac{\hat{\lambda}_{r1}(t, x_1)}{[\underline{\alpha}_1(|x_1|)]^k} < \infty \tag{39}$$

is satisfied, the interconnected system Σ is iISS with respect to input r_1 and state x_1.

(c) If there exists $k \in (-\infty, 1]$ such that (39) and

$$\lim_{s\to\infty} \inf_{s \leq |x_1|, t \in \mathbb{R}_+} \frac{\hat{\lambda}_1(t, x_1)\alpha_1(|x_1|)}{[\bar{\alpha}_1(|x_1|)]^k} = \infty \tag{40}$$

are satisfied, the interconnected system Σ is ISS with respect to input r_1 and state x_1.

It is worth mentioning that the flexibility in the inequality (36) of the static system Σ_0, has no effect, i.e, (36) implies $0 \leq -\alpha_0(|v_0|) + \sigma_0(|x_1|)$.

5. PROOFS AND LYAPUNOV FUNCTIONS

5.1 Proof of Theorem 1

First, suppose that $\limsup_{s\to\infty} \hat{\lambda}_i(s) = 0$ and define

$$B_i(s) = \int_0^s \frac{1}{A_i(t)} dt, \quad A_i(s) = \begin{cases} \hat{\lambda}_i(T), & s \in [0, T) \\ \hat{\lambda}_i(s), & s \in [T, \infty) \end{cases}$$

$$W_i(t, x_i) = B_i \circ V_i(t, x_i), \ \underline{\beta}_i(s) = B_i \circ \underline{\alpha}_i(s), \ \overline{\beta}_i(s) = B_i \circ \overline{\alpha}_i(s)$$

for some $T > 0$. We can transform $\limsup_{s\to\infty} \hat{\lambda}_i(s) = 0$ into

$$\limsup_{s\to\infty} \hat{\lambda}_i(s) > 0, \quad i = 0, 1 \tag{41}$$

via the following substitution.

$$V_i \to W_i, \quad \underline{\alpha}_i \to \underline{\beta}_i, \quad \overline{\alpha}_i \to \overline{\beta}_i$$
$$\hat{\lambda}_i(s) \to \frac{\hat{\lambda}\circ B^{-1}(s)}{A_i\circ B_i^{-1}(s)}, \quad \hat{\lambda}_{ri}(s) \to \frac{\hat{\lambda}_{ri}\circ B^{-1}(s)}{A_i\circ B_i^{-1}(s)}$$

Note that (13) remains the same under this operation. The rest assumes (41). Suppose that

$$\lim_{s\to\infty}\alpha_0(s)=\lim_{s\to\infty}\sigma_0(s)=\infty \wedge \infty>\lim_{s\to\infty}\alpha_1(s)>\lim_{s\to\infty}\sigma_1(s) \quad (42)$$

does not hold. In the case of

$$\lim_{s\to\infty}\alpha_0(s)=\infty \quad \wedge \quad \lim_{s\to\infty}\alpha_1(s)=\infty \quad (43)$$

and the case of

$$\infty=\lim_{s\to\infty}\alpha_0(s)>\lim_{s\to\infty}\sigma_0(s) \wedge \lim_{s\to\infty}\alpha_1(s)>\lim_{s\to\infty}\sigma_1(s) \ , \quad (44)$$

there exist $\hat{\sigma}_1 \in \mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1 > 1$ such that

$$\hat{c}_1\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}\circ\bar{\alpha}_0\circ\alpha_0^{-1}\circ\hat{c}_0\circ\sigma_0(s) \le \hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s), \quad \forall s\in\mathbb{R}_+ \quad (45)$$
$$\sigma_1(s)\le\hat{\sigma}_1(s), \quad \hat{\alpha}_1(s)\le\alpha_1(s), \ \forall s\in\mathbb{R}_+ \quad (46)$$
$$\lim_{s\to\infty}\hat{\sigma}_1(s)\ge\lim_{s\to\infty}\hat{\alpha}_1(s) \quad (47)$$

are satisfied with

$$\hat{\alpha}_1=\alpha_1 \quad (48)$$

In the case of

$$\lim_{s\to\infty}\alpha_0(s)<\infty \ , \quad (49)$$

there exist $\hat{\alpha}_1 \in \mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1 > 1$ such that (45), (46), and (47) are satisfied with

$$\hat{\sigma}_1=\sigma_1 \quad (50)$$

If none of (43), (44) and (49) holds, the inequalities (45), (46) and (47) are fulfilled with

$$\hat{\alpha}_1=\alpha_1, \quad \hat{\sigma}_1=\sigma_1, \quad \hat{c}_i=c_i, \ i=0,1 \ . \quad (51)$$

Pick real numbers $\tau_1, \phi \ge 0$ satisfying

$$1<\tau_1<\hat{c}_1, \quad (\tau_1/\hat{c}_1)^\phi \le (\tau_1-1)(\hat{c}_0-1) \quad (52)$$

Define $\hat{\zeta}_0, \hat{\zeta}_1 \in \mathcal{K}$ as

$$\hat{\zeta}_0(s)=\frac{\hat{c}_0}{(\hat{c}_0-1)}\sqrt{\frac{\hat{c}_1}{\tau_1}}\left[\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]^{\phi+1} \quad (53)$$
$$\hat{\zeta}_1(s)=\left[\alpha_0\circ\bar{\alpha}_0^{-1}\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]^\phi \quad (54)$$

For each $i = 0, 1$, we can always select a continuous function $F_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F_i(s)>0, \quad \forall s\in(0,\infty)$$
$$F_i(s)\hat{\lambda}_i(s),\ F_i(s)\hat{\zeta}_i(s) : \text{non-decreasing on } \mathbb{R}_+$$
$$\lim_{s\to\infty}\hat{\alpha}_1(s)<\infty \ \Rightarrow \ \lim_{s\to\infty}F_i(s)<\infty \quad (55)$$

hold. Here, (41) is used for (55). Let U_i, $i = 0, 1$, denote

$$U_0(s)=\left[F_0\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}(s)\right]\left[\hat{\lambda}_0\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}(s)\right], \ s\in[0,\hat{\sigma}_1(\infty))$$
$$U_1(s)=\begin{cases}\left[F_1\circ\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}(\tau_1 s)\right]\left[\hat{\lambda}_1\circ\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}(\tau_1 s)\right] & , s\in[0,\frac{\hat{\alpha}_1(\infty)}{\tau_1}) \\ F_1(\infty)\hat{\lambda}_1(\infty) & , s\in[\frac{\hat{\alpha}_1(\infty)}{\tau_1},\hat{\sigma}_1(\infty))\end{cases}$$

The properties (11) and (55) for $i = 1$ together with (46), (48) and (51) make ensure that U_1 is well-defined. Note that $\hat{\sigma}_1(\infty) \ge (1/\tau_1)\hat{\alpha}_1(\infty)$ holds since (47). Define

$$\nu(s)=U_0(s)U_1(s) \quad : s\in[0,\hat{\sigma}_1(\infty))\to\mathbb{R}_+$$

which is non-decreasing. Let $\lambda_0, \lambda_1, \lambda_M \in \mathcal{K}$ be given by

$$\lambda_0(s)=\frac{\hat{c}_0}{(\hat{c}_0-1)}\sqrt{\frac{\hat{c}_1}{\tau_1}}\left[\nu\circ\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]\left[\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]^{\phi+1} \quad (56)$$
$$\lambda_1(s)=\left[\alpha_0\circ\bar{\alpha}_0^{-1}\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right] \times\left[\nu\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]^\phi \quad (57)$$
$$\lambda_{M.0}(s)=F_0(s)\hat{\zeta}_0(s)\left[U_1\circ\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]$$
$$\lambda_{M.1}(s)=F_1(s)\hat{\zeta}_1(s)\left[U_0\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]$$

The pair of (10) and (52) yields

$$\left[\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}\circ\bar{\alpha}_0\circ\alpha_0^{-1}\circ\hat{c}_0\sigma_0(s)\right]^{\phi+1} \le \frac{1}{\hat{c}_1\tau_1^\phi}(\tau_1-1)(\hat{c}_0-1)\left[\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right]^{\phi+1} \quad (58)$$

When (12) holds, it follows from (46), (47), (48), (51), (55) and (11) that

$$\alpha_0(\infty)<\infty\wedge\hat{\sigma}_1(\infty)<\infty \Rightarrow \lambda_0(\infty)<\infty\wedge\lambda_1(\infty)<\infty \quad (59)$$

By virtue of (13), we can pick $C_i > 0$ so that $C_i < \lim_{s\to\infty}\hat{\lambda}_i(s)/\hat{\lambda}_{ri}(s)$ holds. Define $\tilde{\lambda}_{ri}, \tilde{\sigma}_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ by $\tilde{\lambda}_{ri} = C_i\hat{\lambda}_{ri}$ and $\tilde{\sigma}_{ri} = \sigma_{ri}/C_i$. Then, we have

$$\lambda_{M.i}(s)\hat{\lambda}_i(s)+D_i\ge\lambda_{M.i}(s)\tilde{\lambda}_{ri}(s), \quad \forall s\in\mathbb{R}_+ \quad (60)$$

for some $R_i > 0$, where $D_i = \max_{s\in[0,R_i]}\lambda_{M.i}(s)\tilde{\lambda}_{ri}(s)$. It can be verified that the inequality (10) together with (58) and (59) guarantees the existence of $\alpha_{cl}\in\mathcal{K}$ satisfying

$$\lambda_{M.0}(V_0(t,x_0))\hat{\lambda}_0(V_0(t,x_0))\left[-\alpha_0(|x_0|)+\sigma_0(|x_1|)+\tilde{\sigma}_{r0}(|r_0|)\right]+\lambda_{M.1}(V_1(t,x_1))\hat{\lambda}_1(V_1(t,x_1))\left[-\hat{\alpha}_1(|x_1|)+\hat{\sigma}_1(|x_0|)+\tilde{\sigma}_{r1}(|r_1|)\right]\le-\alpha_{cl}(|x|)+\sigma_{cl}(|r|) \quad (61)$$

We have $\sigma_{cl} \equiv 0$ in the absence of r. In the presence of r, we have $\sigma_{cl}\in\mathcal{K}$ if one of *[A1]* and *[A2]* is true. In the case of *[A3]*, we can verify that $\alpha_{cl}\in\mathcal{K}_\infty$. Now, define

$$V_{cl}(t,x)=\int_0^{V_0(t,x_0)}\lambda_{M.0}(s)ds+\int_0^{V_1(t,x_1)}\lambda_{M.1}(s)ds \quad (62)$$

The property (7) and $\lambda_{M.0}, \lambda_{M.1} \in \mathcal{K}$ imply that there exist $\underline{\alpha}_{cl}, \overline{\alpha}_{cl} \in \mathcal{K}_\infty$ such that $\underline{\alpha}_{cl}(|x|) \le V_{cl}(t,x) \le \overline{\alpha}_{cl}(|x|)$ holds. Due to (9), (60) and (61), the property

$$\frac{dV_{cl}}{dt}\le-\alpha_{cl}(|x|)+\sigma_{cl}(|r|)+\sum_{i=0}^{1}D_i\tilde{\sigma}_{ri}(|r_i|)$$

holds along Σ for all $x \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ and $t \in \mathbb{R}_+$. The functions σ_{cl} and $\tilde{\sigma}_{ri}$ disappear in the case of UGAS. In the case of (42), there exist $\hat{\sigma}_1 \in \mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1 > 1$ such that (45), (46) and $\lim_{s\to\infty}\hat{c}_1\hat{\sigma}_1(s) = \lim_{s\to\infty}\hat{\alpha}_1(s)$ are satisfied with (48). Define

$$L=\lim_{s\to\infty}\hat{\sigma}_1(s), \quad \tau_1(s)=(\underline{\tau}_1+\frac{\hat{c}_1-\underline{\tau}_1}{L}s)s, \quad 1<\underline{\tau}_1<\hat{c}_1$$

$$Q(t) = \frac{1}{\tau_1^{-1}(\hat{c}_1 t) - t} \times \max\left\{ \left(\frac{\hat{c}_1}{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)} - 1 \right), \left(\frac{\hat{c}_1}{\underline{\tau}_1} - 1 \right) \right\}$$

$$\psi(s) = e^{G(s)}, \quad G(s) = \int_{L/2}^{s} Q(t)dt, \ s \in [0, L)$$

The function ψ is continuous, increasing and bounded on $[0, L)$. It is verified that ψ satisfies

$$\left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0 \sigma_0(s)\right] \leq \frac{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)}{\hat{c}_1} \left[\psi \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] \quad (63)$$

Replace (53), (54), (56) and (57) by

$$\hat{\zeta}_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)} \sqrt{\frac{\hat{c}_1}{\underline{\tau}_1}} \left[\hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right]$$

$$\hat{\zeta}_1(s) = \left[\alpha_0 \circ \bar{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \hat{\sigma}_1^{-1} \circ \frac{1}{\tau_1} \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \left[\psi \circ \frac{1}{\tau_1} \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]$$

$$\lambda_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)} \sqrt{\frac{\hat{c}_1}{\tau_1}} \left[\hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \times \left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right]$$

$$\lambda_1(s) = \left[\alpha_0 \circ \bar{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \hat{\sigma}_1^{-1} \circ \frac{1}{\tau_1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \times \left[\nu \circ \frac{1}{\tau_1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \left[\psi \circ \frac{1}{\tau_1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]$$

respectively. Then, using (10) and (63), we obtain (61) for some $\alpha_{cl} \in \mathcal{K}$ and $r(t) \equiv 0$. Finally, it is verified that (42) does not hold if one of *[A1]*, *[A2]* and *[A3]* holds.

5.2 Proof of Theorem 6

Using (20), decompose $\hat{\lambda}_0$ as $\hat{\lambda}_0(v_0) = \hat{\lambda}_{A0}(|v_0|)\hat{\lambda}_{B0}(v_0)$, where $\hat{\lambda}_{B0}(v_0) > 0$, $\forall v_0 \in \mathbb{R}^{n_0}$ and $\lim_{s\to\infty} \hat{\lambda}_{A0}(s) < \infty$. By virtue of (29), the inequality (26) ensures that

$$0 \leq \hat{\lambda}_{A0}(|v_0|)\left[-\alpha_0(|v_0|) + \sigma_0(|x_1|)\right] + \hat{\lambda}_{Ar0}(|v_0|)\sigma_{r0}(|r_0|)$$

holds for some $\mathbf{C}^0$ function $\hat{\lambda}_{Ar0} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\limsup_{s\to\infty} \hat{\lambda}_{Ar0}(s)/\hat{\lambda}_{A0}(s) < \infty$. The iISS and ISS Lyapunov functions are given by

$$V_{cl}(t, x) = \int_0^{V_1(t,x_1)} \lambda_{M.1}(s)ds \quad (64)$$

5.3 Proof of Theorem 7

Combining (20) and (36) with (38) we obtain

$$-\alpha_1(|x_1|) + \sigma_1(|v_0|) \leq -\delta_1\alpha_1(|x_1|)$$

where $\delta_1 = (1 - 1/c_1) > 0$. Let $\lambda_{M.1} : \mathbb{R}_+ \to \mathbb{R}_+$ be any function fulfilling $\lambda_{M.1}(s) > 0$, $\forall s \in (0, \infty)$ and $0 < \lim_{s\to\infty} s^k \lambda_{M.1}(s) < \infty$. The assumptions (39) and (40) guarantee

$$\lim_{s\to\infty} \sup_{s \leq |x_1|, t\in\mathbb{R}_+} \lambda_{M.1}(V_1(t, x_1))\hat{\lambda}_{r1}(t, x_1) < \infty$$

$$\lim_{s\to\infty} \inf_{s \leq |x_1|, t\in\mathbb{R}_+} \lambda_{M.1}(V_1(t, x_1))\hat{\lambda}_1(t, x_1)\alpha_1(|x_1|) = \infty$$

A Lyapunov function proving UGAS, iISS and ISS is (64).

6. ILLUSTRATIVE EXAMPLES

In order to illustrate Theorem 1, consider the following feedback interconnection defined on $\mathbb{R}_+^2$.

$$\Sigma_0 : \dot{x}_0 = -\left(\frac{x_0}{x_0+5}\right)\left(\frac{6x_0}{x_0+6}\right) + \left(\frac{x_0}{x_0+5}\right)\left(\frac{x_1}{x_1+3}\right) + r_0^2, \quad x_0(0) \in \mathbb{R}_+ \quad (65)$$

$$\Sigma_1 : \dot{x}_1 = -4\left(\frac{x_1}{x_1+1}\right)\left(\frac{x_1}{x_1+3}\right)^2 + \left(\frac{x_1}{x_1+1}\right)\left(\frac{6x_0}{x_0+6}\right)^2 + \left(\frac{x_1}{x_1+2}\right) r_1, \quad x_1(0) \in \mathbb{R}_+ \quad (66)$$

The system Σ_1 is not ISS, and it is only iISS. Let

$$V_0 = x_0, \ \alpha_0(s) = \frac{6s}{s+6}, \ \sigma_0(s) = \frac{s}{s+3}$$

$$\hat{\lambda}_0(s) = \frac{s}{s+5}, \ \hat{\lambda}_{r0}(s) = 1, \ \sigma_{r0}(s) = s^2$$

$$V_1 = x_1, \ \alpha_1(s) = 4\left(\frac{s}{s+3}\right)^2, \ \sigma_1(s) = \left(\frac{6s}{s+6}\right)^2$$

$$\hat{\lambda}_1(s) = \frac{s}{s+1}, \ \hat{\lambda}_{r1}(s) = \frac{s}{s+2}, \ \sigma_{r1}(s) = s$$

The properties (3)-(6), (11), (12), (13), (14) and *[A2]* are fulfilled. It follows from $\bar{\alpha}_i = \underline{\alpha}_i = s$, $i = 1, 2$, that (10) is

$$c_1\left(\frac{c_0 s}{s+3}\right)^2 \leq 4\left(\frac{s}{s+3}\right)^2, \ \forall s \in \mathbb{R}_+$$

Since there exist $c_1, c_0 > 1$ such that this inequality holds, Theorem 1 guarantees the interconnection of (65) and (66) to be iISS with respect to r and x. The proof of Theorem 1 yields an iISS Lyapunov function $V_{cl}(x)$ of the interconnection given by (62) with

$$\lambda_{M.0}(s) = \frac{4\sqrt{3} + 2\sqrt{6}}{3}\left(\frac{6s}{s+6}\right)^{14} U_1(\sigma_1(s))$$

$$\lambda_{M.1}(s) = \left(\frac{8}{3}\right)^{\frac{13}{2}}\left(\frac{s}{s+3}\right)^{13}\frac{6\sqrt{6}s}{(45+\sqrt{6})s + 135}$$

$$U_1(s) = \begin{cases} \dfrac{9s + 6\sqrt{6s}}{6s + 6\sqrt{6s} + 8}, & s \in [0, 8/3) \\ 1, & s \in [8/3, \infty) \end{cases}$$

Note that we cannot use the iISS small-gain theorem in Ito (2006a) and Ito and Jiang (2008) for this example due to the presence of $x_0/(x_0 + 5)$ and $x_1/(x_1 + 1)$. Theorem 1 allows us to ignore $x_0/(x_0 + 5)$ and $x_1/(x_1 + 1)$ when we resort to the small-gain argument.

Another example is the following:

$$\Sigma_0 : \dot{x}_0 = 12x_0^2\left(\frac{x_1}{x_1+1}\right)^2 + x_0 v_0 + r_0, \ x_0(0) \in \mathbb{R}_+ \quad (67)$$

$$\Sigma_1 : \dot{x}_1 = -\frac{3x_1}{x_1+1} + x_0 + r_1^2, \quad x_1(0) \in \mathbb{R}_+ \quad (68)$$

The state $x = [x_0, x_1]^T$ evolves on $\mathbb{R}_+^2$. The vector $r = [r_0, r_1]^T$ is disturbance. The purpose is to design a scalar input $v_0(t)$ to make the whole interconnected system iISS with respect to input r to state x. Since Σ_1 is not ISS, the control $v_0(t)$ should render Σ_0 stable strongly enough to compensate the shortage of stability. Let

$$V_1 = x_1,\ \alpha_1(s) = \frac{3s}{s+1},\ \sigma_1(s) = s,\ \sigma_{r1}(s) = s^2$$
$$\hat{\lambda}_0(s) = 1,\ \hat{\lambda}_{r0}(s) = 1$$

For the choice of $V_0 = x_0$, we obtain

$$\dot{V}_0 = \frac{4x_0^2}{3}\left\{\frac{3v_0}{4x_0} + \left(\frac{3x_1}{x_1+1}\right)^2\right\} + r_0$$

Define

$$\alpha_0(s) = \frac{3v_0(s)}{4s},\ \sigma_0(s) = \left(\frac{3s}{s+1}\right)^2,\ \sigma_{r2}(s) = s$$
$$\hat{\lambda}_0(s) = \frac{4}{3}s^2,\ \hat{\lambda}_{r0}(s) = 1$$

where $\alpha_0 \in \mathcal{K}_\infty$ has yet to be determined. These functions satisfy (3)-(6), (11), (12), (13) and (14), *[A1]*. Using $\bar{\alpha}_i = \underline{\alpha}_i = s$, $i = 1, 2$, we obtain (10) as

$$\alpha_0^{-1}\left(c_0\left(\frac{3s}{s+1}\right)^2\right) \le \frac{3s}{c_1(s+1)},\ \forall s \in \mathbb{R}_+$$

This inequality holds for some $c_0, c_1 > 1$ if and only if $\alpha_0(s) > s^2$ is satisfied for all $s \in (0, \infty)$. Hence, Theorem 1 guarantees that $v_0(x_0) = -kx_0^3$ with any $k > 4/3$ renders the interconnected system iISS. An important feature of this paper is that we obtain an iISS Lyapunov function explicitly as $V_{cl}(x)$ in (62), where

$$\lambda_{M.0}(s) = \frac{k+3}{k-1}\sqrt{\frac{\hat{c}_1}{\tau_1}}\, s^{\phi+1},\ \lambda_{M.1}(s) = \frac{4k}{3}\left(\frac{3s}{\tau_1(s+1)}\right)^{\phi+4}$$
$$\hat{c}_1 = 2\sqrt{\frac{k}{k+3}},\quad \tau_1 = 2\sqrt{\frac{k}{2(k+3)} + \frac{1}{8}}$$
$$L = \sqrt{\frac{1}{2} + \frac{1}{8}\sqrt{\frac{k+3}{k}}},\quad \phi = \max\left\{\frac{\log\frac{(\tau_1-1)(k-1)}{4}}{\log L}, 0\right\}$$

It is stressed that Lyapunov functions V_{cl} of the two examples cannot be derived from Ito (2006b). The examples are not covered by Ito (2005b) either.

7. CONCLUSIONS

This paper introduces a flexible Lyapunov formulation into the small-gain methodology for stability analysis of interconnected systems. Namely, the technique of bilaterally flexible Lyapunov inequalities is proposed, which can be considered as a thoroughly nonlinear counterpart of the popular scaling technique in linear robust control. The bilaterality and treating iISS and ISS systems equally are new in the literature. Examples have shown that the flexibility is useful in exploiting and coping with nonlinearity in stability analysis and feedback design. The Lyapunov function of the interconnected system is expressed explicitly in terms of a "smooth" nonlinear combination of given Lyapunov functions of individual subsystems. This paper has focused on the construction a continuously differentiable Lyapunov function since such a Lyapunov function is directly amenable to a large variety of techniques for further analysis and design of control systems. This contrasts with the max-type construction leading only to a Lipschitz continuous function, which requires methods of non-smooth analysis or additional mathematical process of smoothing(see Jiang et al. (1996)). Finally, it is worth mentioning that $c_i\sigma_i$ in (10), (28) and (38) can be replaced by $(\mathbf{Id} + \rho_i) \circ \sigma_i$ where ρ_i belongs to class $\mathcal{K}_\infty$ in the ISS case. It can be relaxed further when iISS or UGAS is targeted. This paper employs the simplest case $s + \rho_i(s) = c_i s$ for brevity of the presentation of Lyapunov functions. For the generalization, the reader can consult Ito and Jiang (2008) devoted to the iISS small-gain theorem without flexible parameters $\{\hat{\lambda}_i,\ \hat{\lambda}_{ri}\}$.

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