Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

State-feedback H_∞ Control for Nonlinear Stochastic Systems with Markovian Jumps in Infinite Time Horizon ⋆

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Abstract: This paper discusses the H_∞ control problem for a class of nonlinear stochastic systems with Markovian jumps subjected to both state- and disturbance-dependent noise. We establish the equivalent conditions among Hamilton-Jacobi inequality (HJI), Hamilton-Jacobi equality (HJE), the dissipative inequality and $\mathcal{L}_2$-gain property for this class of systems. As to the infinite time horizon case, we synthesize a worst case-based state-feedback H_∞ controller.

1. INTRODUCTION

In recent years, stochastic H_∞ control systems, such as Markovian jump systems, H_∞ Gaussian control design, and Itô differential systems (governed by the Itô equation) have received a great deal of attention, see Bjork [1980], Doyle [1989] and Hinrichsen [1998]. Zhang [2006] and Berman [2005] both developed the H_∞ theory from the dissipation point of view for stochastic nonlinear systems. In particular, the nonlinear counterparts of the stochastic Lur'e equations proposed by Zhang [2006] can be applied to solve many stochastic control problems. Dissipation theory is attractive because the general notion of dissipativity comprises a family of system properties. One or another property can be specified by defining the so-called supply rate. The readers may refer to Willems [1972], Schaft [1992], Polushin [2000] and Andrievskii [2006] for more details about the tendencies in the development of this theory for deterministic systems.

Jump system has been studied since the pioneering work on quadratic control of linear jump systems in 1960s. It switches from one mode to another in a random way, and the switching between the modes is governed by a Markov process with discrete and finite state space. Noticeable achievements have been made in the last three decades on controller design, filtering and stability analysis of linear jump systems, see Ghaoui [1996], Shi [1997], Li [2003], Caines [1995], Dragan [2002] and Mao [1999]. In particular, the problems of designing state feedback controllers for stochastic linear jump systems to achieve stochastic stability (Ghaoui [1996]), and a prescribed H_∞ performance (Shi [1997]) or guaranteed cost control (Li [2003]) have been well studied. However, to the best of the authors' knowledge, there are still little count of publications dealing with dissipativity of nonlinear stochastic systems with Markovian jumps, which merits our study on this topic.

This paper will discuss the H_∞ control problem for nonlinear stochastic systems with Markovian jumps, which extends the result of Schaft [1992] to nonlinear Itô differential systems with Markovian jumps. In section 2, by applying the generalized stochastic Lur'e equations, we establish the equivalent conditions for HJI, HJE, the dissipative inequality and $\mathcal{L}_2$-gain property for this class of systems. Section 3 synthesizes a worst case-based state-feedback H_∞ controller as to the infinite time horizon case. An example is given to illustrate the use of our method. Section 4 concludes this paper.

For convenience, we adopt the following notation:

A': the transpose of the corresponding matrix A;
$A \geq 0 (A > 0)$: the positive semi-definite (positive-definite) matrix;
I: the identity matrix;
R^n: n-dimensional Euclidean space;
$\mathcal{C}^2(U)$: the class of functions $V(x)$ twice continuously differentiable with respect to $x \in U$;
$\mathcal{L}^2_{\mathcal{F}}(R^+, R^{n_y})$: space of non-anticipative stochastic processes $y(t) \in R^{n_y}$ with respect to an increasing σ-algebra $\mathcal{F}_t (t \geq 0)$, satisfying $\|y\|^2_{\mathcal{L}^2(R^+)} := E\int_0^\infty |y(t)|^2 dt < \infty$.

2. STOCHASTIC DISSIPATIVE THEORY

In this section, we discuss the dissipative theory for a class of stochastic systems with Markovian jumps. The relationship among HJE, HJI, the dissipative inequality and $\mathcal{L}_2$-gain property is established.

First of all, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space where there lives a standard one-dimensional Brownian motion $w(t)$ on $[0, +\infty)$ (with $w(0) = 0$) and a Markov chain $r_t \in \{1, 2, \ldots, N\}$ with the generator $\Pi = (\lambda_{ij})$, and $\mathcal{F}_t = \sigma\{w(s), r_s | 0 \leq s \leq t\}$. The Brownian motion is assumed to be one dimensional only for simplicity; there is no essential difference for the multi-dimensional case. In addition, the processes r_t and $w(t)$ are assumed to be independent throughout this paper.

Consider the nonlinear stochastic system with Markovian jumps defined by

⋆ This work was supported by the National Natural Science Foundation of China(No.60174001).

$$\begin{cases} dx(t) = (f(x,r_t) + k(x,r_t)d)\,dt + (h(x,r_t) \\ \qquad +l(x,r_t)d)\,dw, \quad f(0,r_t)=0,\ h(0,r_t)=0, \\ z = m(x,r_t),\ m(0,r_t)=0, \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $x_0 \in R^n$ is the initial state, $d(t) \in R^m$ is the control input, also can be regarded as the exogenous disturbance, $w(t)$ is a one-dimensional Brownian motion and $z(t) \in R^{n_z}$ is the regulated output. Assume that $d(t)$ is an adapted and measurable process with respect to $\mathcal{F}_t$. $\{r_t, t \geq 0\}$ is the continuous-time Markov process taking values in a finite set $\varphi = \{1, 2, \ldots, N\}$ and describes the evolution of the mode at time t. For every $i \in \varphi$, $f(x,i)$, $k(x,i)$, $h(x,i)$, $l(x,i)$ and $m(x,i)$ are uniformly continuous and Lipschitz satisfying a linear growth condition, which guarantees that system (1) has a unique strong solution (Yong and Zhou [1999]).

The Markov process $\{r_t, t \geq 0\}$ takes values in the finite set φ, which represents the switching between the different modes and its dynamics is described by the following transition probabilities:

$$P[r_{t+h} = j | r_t = i] = \begin{cases} \lambda_{ij}h + o(h) \\ 1 + \lambda_{ii}h + o(h) \end{cases},$$

where λ_{ij} is the transition rate from mode i to j with $\lambda_{ij} \geq 0$ when $i \neq j$ and $\lambda_{ii} = -\sum_{j=1, j\neq i}^{N} \lambda_{ij}$ and $o(h)$ is such that $\lim_{h\to 0} o(h)/h = 0$.

According to the terminology introduced by Willems [1972], the function $W(\cdot,\cdot) : R^{n_d} \times R^{n_z} \to R$ related with system (1) will be called the supply rate on $[s, \infty]$, if for any $d \in \mathcal{L}^2_{\mathcal{F}}([s,T], R^{n_d})$, and x satisfying (1), the inequality

$$E\int_s^T |W(d,z)|\,dt < \infty, \quad \forall T \geq s \geq 0, \quad i \in \varphi$$

is satisfied.

Definition 1. System (1) with supply rate W is said to be dissipative on $[s, \infty]$, $s \geq 0$, if there exists a set of nonnegative continuous functions $V(x,i) : R^n \to R^+$, called the storage functions, such that for all $t \geq s \geq 0$, $x(s) \in R^n$, $i \in \varphi$,

$$E\{V(x(t), r_t) - V(x(s), r_s) | r_s = i\} \leq E\int_s^t W(d(\tau), z(\tau))\,d\tau. \tag{2}$$

(2) can be regarded as the dissipative inequality for stochastic systems with Markovian jumps.

Here, we assume that the set of storage functions $V(x,i)$, $i \in \varphi$, if exists, belongs to $C^2(R^n)$.

Let the supply rate be given by $W(u,z) = z'Qz + 2z'Sd + d'Rd$. In this case, the dissipative conditions are provided by the following lemma, which can be regarded as a parallel generalization of Theorem 1 of Zhang [2006].

Lemma 1 (Generalized stochastic Lur'e equations). A necessary and sufficient condition for system (1) to be dissipative on $[s, \infty]$ with respect to a supply rate $W(d,z)$ is that there exists $V_s(x,i) \in C^2(R^n) : R^n \to R^+$, for every $i \in \varphi$, $V_s(0,i) = 0$, $\tilde{l}_i : R^n \to R^q$, and $\tilde{w}_i : R^n \to R^{q\times n_d}$ for some integer $q > 0$, such that

$$m_i'Qm_i - \frac{\partial V_s'(x,i)}{\partial x} f_i - \frac{1}{2} h_i' \frac{\partial^2 V_s(x,i)}{\partial x^2} h_i + \sum_{j=1}^{N} \lambda_{ij} V_s(x,j) = \tilde{l}_i'\tilde{l}_i, \tag{3}$$

$$R - \frac{1}{2} l_i' \frac{\partial^2 V_s(x,i)}{\partial x^2} l_i = \tilde{w}_i'\tilde{w}_i, \tag{4}$$

$$2S'm_i - k_i' \frac{\partial V_s(x,i)}{\partial x} - l_i' \frac{\partial^2 V_s(x,i)}{\partial x^2} h_i = 2\tilde{w}_i'\tilde{l}_i. \tag{5}$$

Proof. See the appendix.

Definition 2 ($\mathcal{L}_2$-gain property). System (1) is said to have the $\mathcal{L}_2$-gain property, if $\|z\|^2_{\mathcal{L}^2[0,T]} \leq \gamma^2 \|d\|^2_{\mathcal{L}^2[0,T]}$ is satisfied for given scalar $\gamma > 0$ and all $T \geq 0$.

The following theorem establishes the equivalent conditions among HJE, HJI, the dissipative inequality and $\mathcal{L}_2$-gain property.

Theorem 1. The following conditions are equivalent:

(i) For system (1), given any scalar $\gamma > 0$, there exists a set of nonnegative solutions $V(x,i) \in C^2(R^n) : R^n \to R^+$, with $V(0,i) = 0$ and $i \in \varphi$, to the HJE

$$\begin{aligned} H_\infty(V(x,i)) :=& \frac{\partial V'(x,i)}{\partial x} f_i + \frac{1}{2} m_i'm_i + \frac{1}{2} h_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i \\ &+ \sum_{j=1}^{N} \lambda_{ij} V_s(x,j) + \frac{1}{2}\Big(\frac{\partial V'(x,i)}{\partial x} k_i + h_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i\Big) \\ &\times \Big(\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i\Big)^{-1} \Big(k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i\Big) \\ =& 0, \\ &\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i > 0. \end{aligned} \tag{6}$$

(ii) For system (1), given $\gamma > 0$, there exists a set of nonnegative solutions $V(x,i) \in C^2(R^n) : R^n \to R^+$, with $V(0,i) = 0$ and $i \in \varphi$, to the HJI

$$H_\infty(V(x,i)) \leq 0, \quad \gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i > 0. \tag{7}$$

(iii) There exists a set of nonnegative solutions $V(x,i) \in C^2(R^n) : R^n \to R^+$, with $V(0,i) = 0$ and $i \in \varphi$, for any $t \geq s \geq 0$, and $x(s) \in R^n$, satisfying the following dissipative inequality

$$E\{V(x(t), r_t) - V(x(s), r_s) | r_s = i\} \leq E\int_s^t W(d,z)\,d\tau \tag{8}$$

with the supply rate $W(d,z) = \frac{1}{2}(\gamma^2 d'd - z'z)$, $\forall (d,z) \subset R^{n_d} \times R^{n_z}$.

(iv) For system (1), given $\gamma > 0$, $x_0 = 0$, the following holds for all $T \geq 0$ and $d \in \mathcal{L}^2_{\mathcal{F}}([0,T], R^{n_d})$

$$\|z\|^2_{\mathcal{L}^2[0,T]} \leq \gamma^2 \|d\|^2_{\mathcal{L}^2[0,T]}. \tag{9}$$

Proof.

$(i) \to (ii)$: Obviously, (ii) holds for every solution $V(x,i)$ satisfying condition (i).

$(ii) \to (iii)$: Assume that there exists a set of nonnegative solutions $V(x,i) \in C^2(R^n) : R^n \to R^+$, with $V(0,i) = 0$, for every $i \in \varphi$, satisfying condition (ii), then by Generalized Itô's formula (Bjork [1980]), integrating the equality from s to t, taking expectation of both sides and recalling that $E\int_s^t \frac{\partial V'(x,i)}{\partial x}(h_i+l_i d)dw(t) = 0$, we have that for any $0 < s < t$, $r_s = i$,

$$\begin{aligned}
&\{EV(x(t),r_t) - V(x(s),r_s)|r_s = i\}\\
&= E\int_s^t [\frac{\partial V'(x,i)}{\partial x}(f_i + k_i d) + \sum_{j=1}^N \lambda_{ij} V(x,j)\\
&\qquad + \frac{1}{2}(h_i + l_i d)' \frac{\partial^2 V(x,i)}{\partial x^2}(h_i + l_i d)]\, d\tau\\
&= \frac{1}{2}E\int_s^t (2H_\infty(V(x,i)) - \|d - (\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)^{-1}\\
&\qquad \cdot (k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i)\|^2_{\gamma,l_i,V(x,i)}\\
&\qquad - z'z + \gamma^2 d'd)\, d\tau,
\end{aligned}$$

where $\|\mathcal{Z}(x)\|^2_{\gamma,l_i,V(x,i)} := \mathcal{Z}'(x)(\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)\mathcal{Z}(x)$. Then, by condition (ii) and $\|\cdot\|^2_{\gamma,l_i,V(x,i)} \geq 0$, we get

$$E\{V(x(t),r_t) - V(x(s),r_s)|r_s = i\} \leq E\int_s^t \frac{1}{2}(\gamma^2 d'd - z'z)\, d\tau.$$

$(iii) \to (iv)$: Assume that there exists a set of storage functions $V(x,i)$ satisfying condition (iii), then, for any given $T \geq 0$, integrating (8) from 0 to T, we obtain

$$EV(x(T),r_T) - V(0,r_0) \leq E\int_0^T \frac{1}{2}(\gamma^2 d'd - z'z)\, dt,$$

which follows that

$$E\int_0^T z'z\, dt \leq E\int_0^T \gamma^2 d'd\, dt - 2EV(x(T),r_T).$$

So (iv) is obtained.

$(iv) \to (i)$: For any $T \geq 0$, $i \in \varphi$, we know that there always exists a set of positive definite functions $q(x,i)$ with $q(0,i) = 0$, such that

$$\|z\|^2_{\mathcal{L}^2[0,T]} \leq \gamma^2 \|d\|^2_{\mathcal{L}^2[0,T]} + q(x,i).$$

For system (1), we define the function $V_{a,0}$ as

$$V_{a,0}(x,i) = -\inf_{T\geq 0,\, x(0)=x} \left\{ E\int_0^T (\frac{1}{2}\gamma^2 d'd - \frac{1}{2}z'z)\, dt \right\}.$$

Obviously, we have

$$0 \leq V_{a,0}(x,i) \leq \frac{1}{2}q(x,i), \quad V_{a,0}(0,i) = 0.$$

By the proof of Lemma 1, we know that $V_{a,0}$ is finite, which implies that it is itself a storage function. Take $R = \frac{1}{2}\gamma^2 I$, $S = 0$, $Q = -\frac{1}{2}I$, $V_s(x,i) = V_{a,0}(x,i)$, and $\tilde{\omega} = \frac{\sqrt{2}}{2}(\gamma^2 I - l_i' \frac{\partial^2 V_{a,0}(x,i)}{\partial x^2} l_i)^{\frac{1}{2}}$. Then, by Lemma 1, we obtain HJE (6). The proof of Theorem 1 is completed.

Theorem 1 gives $\mathcal{L}^2$ input-output stable conditions for system (1). However, if system (1) satisfies the above conditions, it does not mean that the homogeneous system (1) with $d \equiv 0$ is stable.

Definition 3. System (1) is said to be stale in probability, if for any $\varepsilon > 0$, $\lim_{x\to 0} P(\sup_{t\geq 0}\|x(t)\| > \varepsilon) = 0$.

Sufficient conditions of stochastic stability for system (1) will be given in the following corollary. For simplicity, in what follows we consider only globally asymptotically stable in probability (Has'minskii [1980]), which requires that the system is stable in probability, and $P(\lim_{t\to\infty} x(t) = 0) = 1$ for any initial state $x \in R^n$. However, we note that other kinds of stability request, such as exponentially mean square stability, can be discussed by following the lines of Berman [2005].

Corollary 1. Assume that there exists a set of positive functions $V(x,i) \in C^2(R^n)$ with $V(0,i) = 0$, such that $\inf_{t>0} V(x,i) \to \infty$ as $\|x\| \to \infty$, and $V(x,i)$ satisfy the HJI (7), for all $x \in R^n$, $i \in \varphi$. Then, the homogeneous system (1) ($d \equiv 0$) is globally asymptotically stable in probability.

Proof. Note that for $d = 0$, HJI (7) reduces to

$$\begin{aligned}
H_{\infty d=0}(V(x,i)) &:= \frac{\partial V'(x,i)}{\partial x} f_i + \frac{1}{2}m_i' m_i + \frac{1}{2}h_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i\\
&\quad + \sum_{j=1}^N \lambda_{ij} V(x,j) \leq 0, \quad \forall x \in R^n,\, i \in \varphi.
\end{aligned}$$

Thus, we have $\mathcal{L}_{d=0}V(x,i) \leq -\frac{1}{2}m_i' m_i \leq 0$, where $\mathcal{L}$ is the infinitesimal generator of system (1). It implies that

$$\Theta_i = \{x : \mathcal{L}_{d=0}V(x,i) = 0\} \subset \{x : m(x,i) = 0\} = \{0\}.$$

Then, for every $i \in \varphi$, the corresponding $V(x,i)$, applying the LaSalle's invariance principle (Kushner [1972]), the asymptotic stability of system is proved.

Remark 1. Assume there exists a set of solutions $V(x,i)$ to HJI (7), a sufficient condition for $V(x,i)$ to be positive is the zero state observability of the system with respect to z. We refer the reader to Definition 3.2 and the proof of Corollary 3.1 of Zhang [2006].

3. STOCHASTIC STATE-FEEDBACK H_∞ CONTROL

In this section, we consider the state feedback H_∞ control problem in infinite time horizon case for the following controlled nonlinear stochastic systems with Markovian jumps

$$\begin{cases} dx(t) = (f(x,r_t) + k(x,r_t)d + g(x,r_t)u)\, dt\\ \quad +(h(x,r_t) + l(x,r_t)d)\, dw,\ f(0,r_t) = 0,\\ z = \begin{bmatrix} m(x,r_t)\\ u \end{bmatrix},\ m(0,r_t) = 0,\ h(0,r_t) = 0, \end{cases} \tag{10}$$

where $d(t)$ stands for the exogenous disturbance, which is an adapted process with respect to $\mathcal{F}_t$. Under very mild conditions (Yong and Zhou [1999]), (10) has a unique strong solution $x(t)$.

Definition 4 (nonlinear state feedback H_∞ control). Given $\gamma > 0$, the closed-loop system is said to have an H_∞ control if for every $i \in \varphi$, there exists an admissible control $u^*_\infty(x,i)$, such that for any $d \neq 0 \in \mathcal{L}^2_{\mathcal{F}}(R^+, R^{n_d})$, when $x(0) = 0$, the following inequality holds:

$$\|z\|^2_{\mathcal{L}_{R^+}} \leq \gamma^2 \|d\|^2_{\mathcal{L}_{R^+}}. \tag{11}$$

Equation (11) is equivalent to $\|\tilde{\mathcal{L}}_{zd}\|_\infty \leq \gamma$, where the perturbation operator $\tilde{\mathcal{L}}_{zd}$ is defined by $\tilde{\mathcal{L}}_{zd} : \mathcal{L}^2_{\mathcal{F}}(R^+, R^{n_d}) \to \mathcal{L}^2_{\mathcal{F}}(R^+, R^{n_z})$ as

$$\tilde{\mathcal{L}}_{zd} = z(x(t),i),\ t \geq 0,\ d \neq 0 \in \mathcal{L}^2_{\mathcal{F}}(R^+, R^{n_d}),\ x(0) = 0,$$

$$\begin{aligned}\|\tilde{\mathcal{L}}_{zd}\|_\infty &= \sup \frac{\|z\|_{\mathcal{L}^2_{R^+}}}{\|d\|_{\mathcal{L}^2_{R^+}}} \\ &= \sup \frac{\{E\int_0^\infty (\|m(x,i)\|^2 + \|u^*_\infty(x,i)\|^2)dt\}^{\frac{1}{2}}}{\{E\int_0^\infty \|d\|^2 dt\}^{\frac{1}{2}}}.\end{aligned}$$

Theorem 2. Assume that there exists a set of nonnegative functions $V(x,i) \in C^2(R^n)$ with $V(0,i) = 0$ and $i \in \varphi$, satisfying HJE

$$\begin{aligned}H^2_\infty(V(x,i)) &:= \frac{\partial V'(x,i)}{\partial x} f_i + \frac{1}{2} m_i' m_i + \frac{1}{2} h_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i \\ &+ \sum_{j=1}^N \lambda_{ij} V_s(x,j) - \frac{1}{2} \frac{\partial V'(x,i)}{\partial x} g_i g_i' \frac{\partial V(x,i)}{\partial x} \\ &+ \frac{1}{2}(\frac{\partial V'(x,i)}{\partial x} k_i + h_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)(\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)^{-1} \\ &\times (k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i) = 0,\end{aligned} \tag{12}$$

then $u^*_\infty(x,i) = -g_i' \frac{\partial V(x,i)}{\partial x}$ is an H_∞ control for system (10). Besides, if $V(x,i)$ is also positive, and satisfies $\inf_{t>0} V(x,i) \to \infty$ as $\|x\| \to \infty$, for every $i \in \varphi$, then the closed-loop system (with $d \equiv 0$, $u = u^*_\infty$)

$$dx = (f_i + g_i u^*_\infty)dt + h_i d\, dw \tag{13}$$

is globally asymptotically stable in probability.

Proof. By Generalized Itô's formula,

$$\begin{aligned}dV(x,i) &= [\frac{\partial V'(x,i)}{\partial x}(f_i + g_i u + k_i d) + \sum_{j=1}^N \lambda_{ij} V(x,j) \\ &+ \frac{1}{2}(h_i + l_i d)' \frac{\partial^2 V(x,i)}{\partial x^2}(h_i + l_i d)]\, dt \\ &+ \frac{\partial V'(x,i)}{\partial x}(h_i + l_i d)\, dw(t).\end{aligned} \tag{14}$$

For any $T > 0$, integrating (14) from 0 to T, completing the right-hand side to squares, we have

$$\begin{aligned}&E\{V(x(T), r_T) - V(0, r_0) | r_0 = i\} \\ &= E\int_0^T [\frac{\partial V'(x,i)}{\partial x}(f_i + g_i u + k_i d) + \sum_{j=1}^N \lambda_{ij} V(x,j) \\ &\quad + \frac{1}{2}(h_i + l_i d)' \frac{\partial^2 V(x,i)}{\partial x^2}(h_i + l_i d)]\, dt \\ &= \frac{1}{2} E\int_0^T (\|u + g_i' \frac{\partial V(x,i)}{\partial x}\|^2 - \|d - (\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} \\ &\quad \times l_i)^{-1}(k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i)\|^2_{\gamma, l_i, V(x,i)} \\ &\quad + 2H^2_\infty(V(x,i)) - \|z\|^2 + \gamma^2 \|d\|^2)\, dt \\ &= \frac{1}{2} E\int_0^T (\|u + g_i' \frac{\partial V(x,i)}{\partial x}\|^2 - \|d - (\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} \\ &\quad \times l_i)^{-1}(k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i)\|^2_{\gamma, l_i, V(x,i)})dt \\ &\quad - \|z\|^2 + \gamma^2 \|d\|^2.\end{aligned} \tag{15}$$

Obviously, when $u = u^*_\infty$, (15) leads to

$$\begin{aligned}E\int_0^T \|z\|^2 dt &= -E\int_0^T \|d - (\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)^{-1} \\ &\quad \times (k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i)\|^2_{\gamma, l_i, V(x,i)}\, dt \\ &\quad - 2EV(x(T), r_T) + 2V(0, r_0) + \gamma^2 E\int_0^T \|d\|^2\, dt.\end{aligned}$$

Let $T \to \infty$, then $\|\tilde{\mathcal{L}}_{zd}\|_\infty \leq \gamma$ follows because of $V(x,i) \geq 0$ and $V(0,i) = 0$. The rest is proved by Corollary 1 and is omitted. The proof of Theorem 2 is completed.

Remark 2. In (15), we define

$$d^*_\infty = (\gamma^2 I - l_i' \frac{\partial^2 V(x,i)}{\partial x^2} l_i)^{-1}(k_i' \frac{\partial V(x,i)}{\partial x} + l_i' \frac{\partial^2 V(x,i)}{\partial x^2} h_i)$$

as the disturbance in the worst case. Besides, we can also see that Theorem 2 still holds if HJE (12) is replaced by Hamilton-Jacobi inequality $H^2_\infty(V(x,i)) \leq 0$.

For the linear stochastic system with Markovian jumps

$$\begin{aligned}dx(t) &= (A(r_t)x + B_1(r_t)u + B_2(r_t)d)\, dt \\ &\quad + (C(r_t) + D(r_t)d)\, dw, \\ z &= \begin{bmatrix} M(r_t)x \\ u \end{bmatrix},\end{aligned}$$

let $V(x,i) = \frac{1}{2}x'P(i)x$, then Theorem 2 leads to the following corollary.

Corollary 2. For every $i \in \varphi$, suppose there exists a set of solutions $P(i) \geq 0$ to the generalized algebraic Riccati Equations (GAREs)

$$P_i A_i + A_i' P_i + C_i' P_i C_i + (P_i B_{2i} + C_i' P_i D_i)$$

$$\times (\gamma^2 I - D_i' P_i D_i)^{-1} (B_{2i}' P_i + D_i' P_i C_i)$$
$$- P_i B_{1i} B_{1i}' P_i + M_i' M_i + \sum_{j=1}^{N} \lambda_{ij} P_j = 0, \quad (16)$$
$$\gamma^2 I - D_i' P_i D_i > 0$$

for some $\gamma > 0$, then $u_\infty^*(x,i) = \tilde{k}_i x = -B_i' P_i x$ is an H_∞ control, which makes the closed loop system satisfy $\|\tilde{\mathcal{L}}_{zd}\|_\infty \le \gamma$. Additionally, if $P(i) > 0$ holds for every $i \in \varphi$, then the homogeneous closed-loop system (with $d \equiv 0$)

$$dx(t) = (A(r_t)x + B_1(r_t)u_\infty^*)\,dt + C(r_t)\,dw \quad (17)$$

is globally asymptotically stable in probability.

In what follows, we give an example for a two-mode jump nonlinear system based on the approach developed in the previous sections.

Example 1. Let $w(t)$ be a one-dimensional Brownian motion. Let $r(t)$ be a Markov chain taking values in $\varphi = \{1,2\}$ with the generator $\Pi = (\lambda_{ij}) = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$. Assume that $w(t)$ and $r(t)$ are independent. Consider a two-dimensional stochastic system with Markovian jumps of the form (10) with $l_1 = l_2 = 0$,

$$f_1 = \begin{bmatrix} x_1^3 + x_1 x_2^2 - 5x_1 - 4x_2 \\ x_2^3 + 8x_1^2 x_2 - 2x_1 - 3x_2 \end{bmatrix}, \; g_1 = \begin{bmatrix} 3x_1 \\ 2x_2 \end{bmatrix},$$
$$k_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \; h_1 = \begin{bmatrix} x_1 x_2 \\ x_1^2 \end{bmatrix}, \; m_1 = 2x_1 x_2,$$
$$f_2 = \begin{bmatrix} -3x_1^3 + 2x_2 - x_1 \\ -5x_2^3 + x_1 - 2x_2 \end{bmatrix}, \; g_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
$$k_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; h_2 = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}, \; m_2 = 2(x_1^2 - x_2^2).$$

Assume the disturbance attenuation $\gamma = 1$ for the H_∞ control designed of the above nonlinear stochastic jump system. Then by Theorem 2 and Remark 2, we need to solve the following HJIs for H_∞ control:

$$H_\infty^2(V(x,1)) = \frac{\partial V_1'}{\partial x} \begin{bmatrix} x_1^3 + x_1 x_2^2 - 5x_1 - 4x_2 \\ x_2^3 + 8x_1^2 x_2 - 2x_1 - 3x_2 \end{bmatrix}$$
$$+ \frac{1}{2}(2x_1x_2)(2x_1x_2) + \frac{1}{2} \left[\, x_1 x_2 \;\; x_1^2 \,\right] \frac{\partial^2 V_1}{\partial x^2} \begin{bmatrix} x_1 x_2 \\ x_1^2 \end{bmatrix}$$
$$+ \lambda_{11} V_1 + \lambda_{12} V_2 - \frac{1}{2} \frac{\partial V_1'}{\partial x} \begin{bmatrix} 3x_1 \\ 2x_2 \end{bmatrix} \left[\, 3x_1 \;\; 2x_2 \,\right] \frac{\partial V_1}{\partial x}$$
$$+ \frac{1}{2} \frac{\partial V_1'}{\partial x} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left[\, 2 \;\; 1 \,\right] \frac{\partial V_1}{\partial x} \le 0,$$
$$H_\infty^2(V(x,2)) = \frac{\partial V_2'}{\partial x} \begin{bmatrix} -3x_1^3 + 2x_2 - x_1 \\ -5x_2^3 + x_1 - 2x_2 \end{bmatrix}$$
$$+ \frac{1}{2} \left[\, x_1 + x_2 \;\; x_1 - x_2 \,\right] \frac{\partial^2 V_2}{\partial x^2} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$
$$+ \frac{1}{2}(2x_1^2 - 2x_2^2)(2x_1^2 - 2x_2^2) + \lambda_{21} V_1 + \lambda_{22} V_2$$
$$- \frac{1}{2} \frac{\partial V_2'}{\partial x} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left[\, 2 \;\; 1 \,\right] \frac{\partial V_2}{\partial x}$$
$$+ \frac{1}{2} \frac{\partial V_2'}{\partial x} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \left[\, x_1 \;\; x_2 \,\right] \frac{\partial V_2}{\partial x} \le 0.$$

We choose the solution as $V(x,1) = x_1^2 + 2x_2^2$, $V(x,2) = x_1^2 + x_2^2$. Then, it follows

$$\frac{\partial V_1}{\partial x} = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} \\ \frac{\partial V_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix}, \quad \frac{\partial V_2}{\partial x} = \begin{bmatrix} \frac{\partial V_2}{\partial x_1} \\ \frac{\partial V_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix},$$
$$\frac{\partial^2 V_1}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 V_1}{\partial x_1^2} & \frac{\partial^2 V_1}{\partial x_1 x_2} \\ \frac{\partial^2 V_1}{\partial x_2 x_1} & \frac{\partial^2 V_1}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix},$$
$$\frac{\partial^2 V_2}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 V_2}{\partial x_1^2} & \frac{\partial^2 V_2}{\partial x_1 x_2} \\ \frac{\partial^2 V_2}{\partial x_2 x_1} & \frac{\partial^2 V_2}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

So, we obtain that

$$H_\infty^2(V(x,1)) = -14x_1^4 - 11x_1^2x_2^2 - 28x_2^4 - 5x_2^2 - 2x_1^2 \le 0,$$
$$H_\infty^2(V(x,2)) = -2x_1^4 - 6x_2^4 - (x_1 + x_2)^2 - 7x_1^2 - x_2^2 \le 0.$$

Obviously, $V(x,1)$, $V(x,2)$ satisfy the above HJIs: $H_\infty^2(V(x,1))$ and $H_\infty^2(V(x,2))$, respectively. Then, the H_∞ control is achieved by choosing

$$u_\infty^*(x,1) = -g_1' \frac{\partial V(x,1)}{\partial x} = -6x_1^2 - 8x_2^2,$$
$$u_\infty^*(x,2) = -g_2' \frac{\partial V(x,2)}{\partial x} = -4x_1 - 2x_2.$$

4. CONCLUSION

In this paper, we have dealt with the H_∞ control problem for a class of nonlinear stochastic systems with Markovian jumps. The equivalent conditions among HJI, HJE, the dissipative inequality and $\mathcal{L}_2$-gain property for this class of systems have been established via the generalized stochastic Lur'e equation. A worst case-based state-feedback H_∞ controller has been synthesized as to the infinite horizon case.

Appendix A. PROOF OF LEMMA 1

At first, we define an available storage function with supply rate W on $[s,\infty)$, $s \ge 0$, as

$$V_a(x,i) = -\inf_{t \ge s,\, d \in \mathcal{L}_{\mathcal{F}}^2([s,t],R^{n_d})} \left\{ E \int_s^t W(d(\tau), z(\tau))\,d\tau \right\}$$
$$= \sup_{t \ge s,\, d \in \mathcal{L}_{\mathcal{F}}^2([s,t],R^{n_d})} \left\{ -E \int_s^t W(d(\tau), z(\tau))\,d\tau \right\}.$$

By Definition 1, system (1) is dissipative on $[s,\infty)$, which implies that

$$E\{V(x(t),r_t) - V(x(s),r_s) | r_s = i\} \le E \int_s^t W(d(\tau), z(\tau))\,d\tau$$

for any $x(s) \in R^n$. So,

$$V(x(s), r_s) \geq -E\int_s^t W(d(\tau), z(\tau))d\tau + EV(x(t), r_t)$$
$$\geq -E\int_s^t W(d(\tau), z(\tau))\, d\tau,$$

which yields $V(x(s), r_s) \geq sup\left\{-E\int_s^t W(d(\tau), z(\tau))\, d\tau\right\}$. Therefore, $V_a(x, i)$ is finite. Moreover, $V_a(x, i)$ is itself a possible storage function, for every $i \in \varphi$, following the lines of the proof of Theorem 1 (Willems [1972]).
Hence, for every $i \in \varphi$, $V_a(x, i)$ satisfies

$$E\{V_a(x(t), r_t) - V_a(x(s), r_s)|r_s = i\}$$
$$\leq E\int_s^t W(d(\tau), z(\tau))\, d\tau,$$

which yields
$$\frac{E\int_s^t W(d(\tau), z(\tau))d\tau}{t-s} - \frac{EV_a(x(t), r_t) - V_a(x(s), r_s)}{t-s} \geq 0.$$
For every $V_a(x(t), i)$, $i \in \varphi$, $r_s = i$, applying Generalized Itô's formula (Bjork [1980]), we know that

$$EV_a(x(t), r_t) = V_a(x(s), r_s) + E\int_s^t [\frac{\partial V_a'(x, i)}{\partial x}(f_i + k_i d)$$
$$+ \frac{1}{2}(h_i + l_i d)'\frac{\partial^2 V_a(x, i)}{\partial x^2}(h_i + l_i d)$$
$$+ \sum_{j=1}^N \lambda_{ij} V_a(x, j)]\, d\tau.$$

Combining with the above inequality and letting $t \downarrow s$, it follows that

$$J(x, d) = m_i' Q m_i + 2m_i' S d + d' R d - \frac{\partial V_a'}{\partial x}(f_i + k_i d)$$
$$- \frac{1}{2}(h_i + l_i d)'\frac{\partial^2 V_a}{\partial x^2}(h_i + l_i d)$$
$$- \sum_{j=1}^N \lambda_{ij} V_a(x, j) \geq 0.$$

We notice that $J(x, d)$ is quadratic in d, so there exist $\tilde{l}_i : R^n \to R^q$, and $\tilde{w}_i : R^n \to R^{q\times n_d}$ (not necessarily unique), such that $J(x, d) = (\tilde{l}_i(x) + \tilde{w}_i(x)d)'(\tilde{l}_i(x) + \tilde{w}_i(x)d)$. By comparing the coefficients of the same powers of d, we deduce (3), (4), and (5).

For any $x(s) \in R^n$, the necessity is achieved by using Definition 1 and noting that

$$E\int_s^t W(d(\tau), z(\tau))d\tau = EV_s(x(t), r_t) - V_s(x(s), r_s)$$
$$+E\int_s^t (\tilde{l}_i(x) + \tilde{w}_i(x)d)'(\tilde{l}_i(x) + \tilde{w}_i(x)d)\, d\tau$$
$$\geq EV_s(x(t), r_t) - V_s(x(s), r_s).$$

The proof of Lemma 1 is completed.

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