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Decentralized boundary control of irrigation canal networks via a strict Lyapunov method

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Abstract: A decentralized boundary control problem for irrigation canal networks is considered in this paper. The control scheme is based on a strict Lyapunov method introduced in Coron et al. (2007). A sufficient condition is presented to guarantee the closed-loop system to be locally convergent to a desired set point, which extends the results in Coron et al. (2007) for the single-pool case to a decentralized fashion for the multi-pool case. By eliminating the redundant variables, the derived condition involves certain contractive condition and discrete-time Lyapunov inequality with variables in a diagonal structure. This provides an easier way to check the existence of the solution. An application to a two-pool canal with overflow spillways is presented to demonstrate the proposed approach.

1. INTRODUCTION

This paper considers the problem of decentralized boundary control for irrigation canal networks. The control problem of irrigation canals has been of increasing interest since 1990s; e.g., see Malaterre et al. (1998); Mareels et al. (2005); Cantoni et al. (2007) and the references therein. The considered irrigation canal is composed of several pools separated by hydraulic gates. Each pool is described by hyperbolic partial differential equations (PDEs), the so-called Saint-Venant equations. In recent years boundary control problems have attracted much attention (de Halleux et al., 2003; Bastin et al., 2005; Prieur et al., 2005; Santos and Toure, 2005; Litrico and Fromion, 2006; Coron et al., 2007). This is motivated by the fact that such systems are usually regulated through the gates located at the boundaries of the pools to maintain the water levels and discharges at their operation points.

A strict Lyaponov approach to the problem of boundary control of single-pool canals is presented in Coron et al. (2007). This approach can enable the trajectories of the state variables converge to a desired set point. This is achieved by introducing a Lyapunov function suitable to the underlining nonlinear PDEs and choosing appropriate boundary control strategy, such that the time derivative of the chosen Lyapunov function is strict negative. In general, the canals in a irrigation networks can be represented as strings of pools separated by regulating gates. The water level and velocity in a canal are then regulated by controlling the gate opening. This motivates us to apply the idea of Coron et al. (2007) to the decentralized control of multi-pool canals.

In this paper, we consider a multi-pool canal network; e.g., see de Halleux et al. (2003). The proposed control scheme extends the results in Coron et al. (2007) to a decentralized fashion. A sufficient condition for the existence of a decentralized state feedback controller is presented such that the corresponding closed loop system is locally convergent to a desired set point. This sufficient condition involves choosing appropriate boundary control, such that the closed-loop boundary conditions satisfy certain contractive condition and discrete-time Lyapunov inequality with variables in a diagonal form. Note that the results in Coron et al. (2007) involve some redundant variables; c.f. Proposition 3. Finding a solution satisfying such condition is more complicated. In this paper, instead, we eliminate those redundant variables, leading to an easier way to check whether a solution to the condition exists; c.f. Lemma 4.

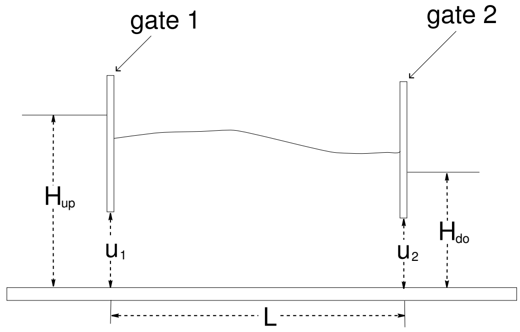

Fig. 1. Single-pool canal with underflow gates

The organization of this paper is as follows. In Section 2 we describe the system model and introduce Riemann invariants approach. Section 3 shows the main results of this paper, presenting a sufficient condition for the closed-loop system to be locally convergent. We illustrate the method by an application to a two-pool canal with overflow spillways presented in Section 4. Conclusions and future works are given in Section 5.

2. MODELING OF A SINGLE-POOL CANAL

In this section we give a brief review on the model of a single-pool open canal controlled by two hydraulic gates (de Halleux et al., 2003). Some typical gates are underflow gate, overflow gate, over-shot gate etc.. The single-pool canal with underflow gates are shown in Fig. 1, where the upstream and downstream water level denoted by H_{up} and H_{do} are supposed to be constant and $H_{up} > H_{do}$. The canal under consideration satisfies the following assumptions (Coron et al., 2007; de Halleux et al., 2003): the canal is one-dimensional and horizontal; the canal is prismatic with a constant rectangular cross section and a unit width; the friction effects are neglected. Under these assumptions, as is well known, the dynamics of open channel are governed by the so-called Saint-Venant equations:

$$\frac{\partial}{\partial t}\begin{bmatrix} H \\ V \end{bmatrix} + A(H,V)\frac{\partial}{\partial x}\begin{bmatrix} H \\ V \end{bmatrix} = 0, \qquad (1)$$

for $(x,t) \in (0,L) \times (0,+\infty)$, where x is one-dimensional space variable, t is time variable, $V = V(x,t)$ is the water velocity, $H = H(x,t)$ is the water depth, g is the gravity constant, and $A(H,V) = \begin{bmatrix} V & H \\ g & V \end{bmatrix}$.

The boundary conditions obtained by a standard discharge relationship at each gate are

$$\hat{f}_1(V(0,t), H(0,t), u_1) = 0, \qquad (2a)$$

$$\hat{f}_2(V(L,t), H(L,t), u_2) = 0. \qquad (2b)$$

Under constant gate openings $\bar{u}_1, \bar{u}_2$, constant steady state solutions $(\bar{H}, \bar{V})$ of (1) can be derived from (2).

The system (1) under consideration is *strictly hyperbolic*, that is, the eigenvalues of $A(H,V)$

$$\lambda_1 = \hat{c}(H,V) = v + \sqrt{gH}, \quad \lambda_2 = -\hat{d}(H,V) = v - \sqrt{gH} \qquad (3)$$

satisfy $\hat{c}(H,V) > 0, \hat{d}(H,V) > 0$. This allows us to rewrite the system (1) under *Riemann invariants*. Indeed, consider the following bijection change of coordinates,

$$\begin{aligned} a &= V - \bar{V} + 2(\sqrt{gH} - \sqrt{g\bar{H}}), \\ b &= V - \bar{V} - 2(\sqrt{gH} - \sqrt{g\bar{H}}). \end{aligned} \qquad (4)$$

Then the system (1) can be diagonalized with the new coordinates (a,b) as

$$\frac{\partial}{\partial t}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c(a,b) & 0 \\ 0 & -d(a,b) \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} a \\ b \end{bmatrix} = 0. \qquad (5)$$

Here $c(a,b)$, $d(a,b)$ are the equivalent forms of $\hat{c}(H,V)$, $\hat{d}(H,V)$ in terms of (a,b) coordinates:

$$\begin{aligned} c(a,b) &= \frac{3}{4}a + \frac{1}{4}b + \bar{V} + \sqrt{g\bar{H}}, \\ d(a,b) &= -(\frac{3}{4}a + \frac{1}{4}b + \bar{V} - \sqrt{g\bar{H}}). \end{aligned}$$

The equivalent forms of boundary conditions (2) in terms of (a,b) are then expressed as

$$\check{f}_1(a(0,t), b(0,t), u_1) = 0, \qquad (6a)$$

$$\check{f}_2(a(L,t), b(L,t), u_2) = 0. \qquad (6b)$$

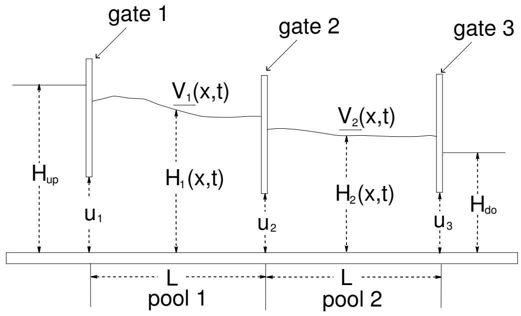

Fig. 2. Two-pool canal with underflow gates

The new (a,b)-coordinate system provides a convenient way for the controller design, see for example de Halleux et al. (2003), Coron et al. (2007) and Aamo et al. (2006). In the next section we will work on the system of the form (5) and (6) to carry out decentralized control for irrigation canals.

3. DECENTRALIZED BOUNDARY CONTROL OF MULTI-POOL IRRIGATION CANALS

In this section we consider an open canal consisting of $n(n > 1)$ pools, extending the results of Coron et al. (2007) for the single-pool case. For the sake of simplicity, the control design is carried out only for $n = 2$; other cases can be derived similarly.

The two-pool canal, as shown in Fig. 2, can be described in the (a,b) coordinate system,

$$\frac{\partial}{\partial t}\begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} c_i(a_i,b_i) & 0 \\ 0 & -d_i(a_i,b_i) \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = 0, \quad i = 1,2, \qquad (7)$$

with $c_i(a_i,b_i) > 0, d_i(a_i,b_i) > 0$ and the boundary conditions

$$\check{f}_1(a_1(0,t), b_1(0,t), u_1) = 0, \qquad (8a)$$

$$\check{f}_2(a_1(L,t), b_1(L,t), a_2(0,t), b_2(0,t), u_2) = 0, \qquad (8b)$$

$$\check{f}_3(a_1(L,t), b_1(L,t), a_2(0,t), b_2(0,t)) = 0, \qquad (8c)$$

$$\check{f}_4(a_2(L,t), b_2(L,t), u_2) = 0, \qquad (8d)$$

in which (8a), (8b) and (8d) are obtained by a standard discharge relationship at each gate, and (8c) by the flow conservation law.

Define

$$\xi_+(x,t) = \begin{bmatrix} a_1(x,t) \\ a_2(x,t) \end{bmatrix}, \qquad \xi_-(x,t) = \begin{bmatrix} b_1(x,t) \\ b_2(x,t) \end{bmatrix},$$

$$\begin{aligned} a_{i,0}(t) &= a_i(0,t), & a_{i,L}(t) &= a_i(L,t), \\ b_{i,0}(t) &= b_i(0,t), & b_{i,L}(t) &= b_i(L,t), \\ a_i^{\#}(x) &= a_i(x,0), & b_i^{\#}(x) &= b_i(x,0). \end{aligned}$$

We aim to design a decentralized boundary control scheme, namely which only uses local information at the boundary points for the feedback,

$$\begin{aligned} u_1 &= u_1(a_{1,0}, b_{1,0}), \\ u_2 &= u_2(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0}), \\ u_3 &= u_3(a_{2,L}, b_{2,L}), \end{aligned}$$

such that the system is stabilized at a desired steady state. Without loss of generality, we take $\bar{\xi} = (\bar{\xi}_+^T, \bar{\xi}_-^T)^T = 0$ to be the steady state. The corresponding boundary conditions for the closed-loop system are given by,

$$\begin{aligned} f_1(a_{1,0}, b_{1,0}) &= 0, \\ f_2(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0}) &= 0, \\ f_3(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0}) &= 0, \\ f_4(a_{2,L}, b_{2,L}) &= 0. \end{aligned} \tag{9}$$

In what follows, we will present a sufficient condition for the closed-loop system to be stable under such boundary conditions. Following Coron et al. (2007), the sufficient condition is derived along two steps. First, the stability of the linearized model is analyzed by introducing an appropriate Lyapunov function. Then, this Lyapunov function is extended for the nonlinear case.

3.1 Linear case

Similar to Coron et al. (2007), we first consider a simpler case, the linear approximation of (7) around the origin,

$$\frac{\partial}{\partial t}\begin{bmatrix} a_i \\ b_i \end{bmatrix} + \begin{bmatrix} \bar{c}_i & 0 \\ 0 & -\bar{d}_i \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} a_i \\ b_i \end{bmatrix} = 0, \quad i = 1, 2. \tag{10}$$

with boundary conditions

$$\begin{bmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{bmatrix} = K \begin{bmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{bmatrix}. \tag{11}$$

Here $\bar{c}_i > 0, \bar{d}_i > 0$.

Based on the analysis in Coron et al. (2007), the following Lyapunov function is considered for the cascade system (10):

$$U(t) = U_1(t) + U_2(t)$$

with

$$U_1(t) = \sum_{i=1}^{2} \frac{A_i}{\bar{c}_i} \int_0^L a_i^2(x,t) e^{-(\mu/\bar{c}_i)x} dx,$$
$$U_2(t) = \sum_{i=1}^{2} \frac{B_i}{\bar{d}_i} \int_0^L b_i^2(x,t) e^{+(\mu/\bar{d}_i)x} dx,$$

where A_i, B_i, μ are positive constant numbers to be determined.

Proposition 1. Given a system (10) with boundary conditions (11), if there exist positive constants A_1, A_2, B_1, B_2, μ satisfying

$$\begin{aligned} &K^T \operatorname{diag}(B_1 e^{\mu L/\bar{d}_1},\ B_2 e^{\mu L/\bar{d}_2},\ A_1,\ A_2) K \\ &\quad - \operatorname{diag}(B_1,\ B_2,\ A_1 e^{-\mu L/\bar{c}_1},\ A_2 e^{-\mu L/\bar{c}_2}) < 0, \end{aligned} \tag{12}$$

then $\dot{U}(t) \le -\mu U(t)$ along the trajectories of the system (10).

Proof.

$$\begin{aligned} &\dot{U}(t) \\ &= -\mu U(t) - \left[\sum_{i=1}^{2} A_i a_i^2(x,t) e^{-\mu x/\bar{c}_i} \right]_0^L \\ &\quad + \left[\sum_{i=1}^{2} B_i b_i^2(x,t) e^{+\mu x/\bar{d}_i} \right]_0^L \\ &= -\mu U(t) \\ &\quad + \begin{bmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{bmatrix}^T \operatorname{diag}(B_1 e^{\frac{\mu L}{\bar{d}_1}}, B_2 e^{\frac{\mu L}{\bar{d}_2}}, A_1, A_2) \begin{bmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{bmatrix} \\ &\quad - \begin{bmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{bmatrix}^T \operatorname{diag}(B_1, B_2, A_1 e^{-\frac{\mu L}{\bar{c}_1}}, A_2 e^{-\frac{\mu L}{\bar{c}_2}}) \begin{bmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{bmatrix} \\ &= -\mu U(t) \\ &\quad + \begin{bmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{bmatrix}^T \big[K^T \operatorname{diag}(B_1 e^{\mu L/\bar{d}_1}, B_2 e^{\mu L/\bar{d}_2}, A_1, A_2) K \\ &\quad - \operatorname{diag}(B_1, B_2, A_1 e^{-\mu L/\bar{c}_1}, A_2 e^{-\mu L/\bar{c}_2}) \big] \begin{bmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{bmatrix} \\ &\le -\mu U(t). \end{aligned}$$

The above result guarantees that the solution of the linear system (10) converges in $L^2(0,L)$-norm. We will use this property to prove the convergence for the nonlinear system (7) in the next section. Note that the condition (12) can be further simplified, which is postponed to the next section; c.f. Lemma 4.

3.2 Nonlinear case

To analyze the convergence of the general nonlinear system (7), an extended Lyapunov function was constructed in Coron et al. (2007), which involves not only the state variable, but their spatial first and second order partial derivatives. In what follows, this approach is extended to the cascade network of the form (7).

To this end, we first define some variables,

$$\begin{aligned} v_i(x,t) &= \partial_x a_i(x,t), \qquad & w_i(x,t) &= \partial_x b_i(x,t), \\ q_i(x,t) &= \partial_x v_i(x,t), \qquad & r_i(x,t) &= \partial_x w_i(x,t). \end{aligned}$$

Consider the following extended Lyapunov function,

$$S(t) = U(t) + V(t) + W(t), \tag{13}$$

where

$$\begin{aligned} V(t) &= V_1(t) + V_2(t), \\ V_1(t) &= \sum_{i=1}^{2} \bar{c}_i A_i \int_0^L v_i^2(x,t) e^{-(\mu/\bar{c}_i)x} dx, \\ V_2(t) &= \sum_{i=1}^{2} \bar{d}_i B_i \int_0^L w_i^2(x,t) e^{+(\mu/\bar{d}_i)x} dx, \\ W(t) &= W_1(t) + W_2(t), \\ W_1(t) &= \sum_{i=1}^{2} \bar{c}_i^3 A_i \int_0^L q_i^2(x,t) e^{-(\mu/\bar{c}_i)x} dx, \\ W_2(t) &= \sum_{i=1}^{2} \bar{d}_i^3 B_i \int_0^L r_i^2(x,t) e^{+(\mu/\bar{d}_i)x} dx. \end{aligned}$$

Define

$$\mathbf{f}(\xi_-(0,t), \xi_-(L,t), \xi_+(0,t), \xi_+(L,t)) = (f_1, f_2, f_3, f_4)^T,$$

where f_i is defined in (9). Note that here $\mathbf{f}(0) = 0$. As in de Halleux et al. (2003), the following assumption is essential to the convergence analysis of the nonlinear system (7).

Assumption 2. $\mathbf{f}$ is continuously differentiable and satisfies $det\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}(0) \neq 0$, where $\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}$ denotes the Jacobian of $\mathbf{f}$ with respect to $[\xi_-(L,t)^T, \xi_+(0,t)^T]^T$.

Under the above assumption, in the neighborhood of origin, by the Implicit Function Theorem (see e.g. Spivak, 1965), the boundary conditions (9) can be rewritten as

$$\begin{bmatrix} \xi_-(L,t) \\ \xi_+(0,t) \end{bmatrix} = \mathbf{g}\begin{pmatrix} \xi_-(0,t) \\ \xi_+(L,t) \end{pmatrix} \tag{14}$$

for some continuously differentiable function $\mathbf{g}$. Note that Eq. (14) is a general form of (11), which explicitly describes the relation between *incoming invariants* $\xi_-(0,t), \xi_+(L,t)$ and *outgoing invariants* $\xi_-(L,t), \xi_+(0,t)$; see de Halleux et al. (2003) for more details. We can then apply the result of Proposition 1 to obtain the following for the nonlinear system (7).

Proposition 3. Given a system in the form of (7) with boundary conditions (9), under Assumption 2, if the positive constants A_1, A_2, B_1, B_2, μ satisfy

$$\begin{aligned} &\nabla\mathbf{g}(0)^T \operatorname{diag}(B_1 e^{\mu L/\bar{d}_1},\ B_2 e^{\mu L/\bar{d}_2},\ A_1,\ A_2)\nabla\mathbf{g}(0) \\ &\qquad - \operatorname{diag}(B_1,\ B_2,\ A_1 e^{-\mu L/\bar{c}_1},\ A_2 e^{-\mu L/\bar{c}_2}) < 0, \end{aligned} \tag{15}$$

where $\nabla\mathbf{g}$ denotes the Jacobian of $\mathbf{g}$ with respect to the vector $[\xi_-(0,t)^T,\ \xi_+(L,t)^T]^T$, there exist positive constants λ_0 and δ_0 such that if $S(t) < \delta_0$, then $\dot{S}(t) \leq -\lambda_0 S(t)$ along the trajectories of the closed-loop system (7) and (9).

Proof. Using the result of Proposition 1, it follows the routine similar to the proof of Lemma 4 in Coron et al. (2007).

Lemma 4. The condition (15) in the positive real variables A_1, A_2, B_1, B_2, μ is feasible if and only if the following condition

$$\begin{aligned} &\nabla\mathbf{g}(0)^T \operatorname{diag}(\hat{B}_1,\ \hat{B}_2,\ \hat{A}_1,\ \hat{A}_2)\nabla\mathbf{g}(0) \\ &\qquad\qquad - \operatorname{diag}(\hat{B}_1,\ \hat{B}_2,\ \hat{A}_1,\ \hat{A}_2) < 0 \end{aligned} \tag{16}$$

in the positive real variables $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$ is feasible.

Proof. The sufficiency part can be easily verified by continuity. As for the necessity part, supposing that the condition (15) holds, we have

$$\begin{aligned} &\nabla\mathbf{g}(0)^T \operatorname{diag}(B_1,\ B_2,\ A_1 e^{-\mu L/\bar{c}_1},\ A_2 e^{-\mu L/\bar{c}_2})\nabla\mathbf{g}(0) \\ &\quad - \operatorname{diag}(B_1,\ B_2,\ A_1 e^{-\mu L/\bar{c}_1},\ A_2 e^{-\mu L/\bar{c}_2}) \\ &< \nabla\mathbf{g}(0)^T \operatorname{diag}(B_1 e^{\mu L/\bar{d}_1},\ B_2 e^{\mu L/\bar{d}_2},\ A_1,\ A_2)\nabla\mathbf{g}(0) \\ &\quad - \operatorname{diag}(B_1,\ B_2,\ A_1 e^{-\mu L/\bar{c}_1},\ A_2 e^{-\mu L/\bar{c}_2}) \\ &< 0. \end{aligned}$$

Then $\hat{B}_1 = B_1$, $\hat{B}_2 = B_2$, $\hat{A}_1 = A_1 e^{-\mu L/\bar{c}_1}$, $\hat{A}_2 = A_2 e^{-\mu L/\bar{c}_2}$ verify (16). This completes the proof.

The above lemma provides a simplified version of the condition (15) by eliminating the redundant variable μ. The new condition (16), as we can see, has the form of discrete-time Lyapunov inequality. It is straightforward from (16) to obtain the following result.

Lemma 5. A necessary condition for (16) to hold is that $\rho(\nabla\mathbf{g}(0)) < 1$, where $\rho(\cdot)$ denote the spectral radius.

Now we are in the position to state the main result of this paper.

Theorem 6. Under Assumption 2, if $\rho(\nabla\mathbf{g}(0)) < 1$, and the positive constants $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$ satisfy (16), there exist positive real constants M, δ, λ such that, for any initial condition $(a_1^\#(x), a_2^\#(x), b_1^\#(x), b_2^\#(x))$ in $H^2(0,L)^4$ [1] satisfying the compatibility conditions

$$\begin{bmatrix} b_1^\#(L) \\ b_2^\#(L) \\ a_1^\#(0) \\ a_2^\#(0) \end{bmatrix} = \mathbf{g}\begin{pmatrix} b_1^\#(0) \\ b_2^\#(0) \\ a_1^\#(L) \\ a_2^\#(L) \end{pmatrix},$$

$$\operatorname{diag}\begin{pmatrix} d_1(a_1^\#(L), b_1^\#(L)) \\ d_2(a_2^\#(L), b_2^\#(L)) \\ -c_1(a_1^\#(0), b_1^\#(0)) \\ -c_2(a_2^\#(0), b_2^\#(0)) \end{pmatrix} \frac{\partial}{\partial x}\begin{bmatrix} b_1^\#(L) \\ b_2^\#(L) \\ a_1^\#(0) \\ a_2^\#(0) \end{bmatrix}$$

$$= \nabla\mathbf{g}\begin{pmatrix} b_1^\#(0) \\ b_2^\#(0) \\ a_1^\#(L) \\ a_2^\#(L) \end{pmatrix} \operatorname{diag}\begin{pmatrix} d_1(a_1^\#(0), b_1^\#(0)) \\ d_2(a_2^\#(0), b_2^\#(0)) \\ -c_1(a_1^\#(L), b_1^\#(L)) \\ -c_2(a_2^\#(L), b_2^\#(L)) \end{pmatrix} \frac{\partial}{\partial x}\begin{bmatrix} b_1^\#(0) \\ b_2^\#(0) \\ a_1^\#(L) \\ a_2^\#(L) \end{bmatrix},$$

and such that

$$\sum_{i=1}^{2} (|a_i^\#(x)|_{H^2(0,L)} + |b_i^\#(x)|_{H^2(0,L)}) < \delta,$$

the closed-loop system (7) with boundary conditions (9) has a unique solution which is continuous from $[0, +\infty)$ into $H^2(0,L)^4$ and satisfies

$$\begin{aligned} &\sum_{i=1}^{2} (|a_i(x,t)|_{H^2(0,L)} + |b_i(x,t)|_{H^2(0,L)}) \\ &< M\sum_{i=1}^{2} (|a_i^\#(x)|_{H^2(0,L)} + |b_i^\#(x)|_{H^2(0,L)})e^{-\lambda t}. \end{aligned}$$

Proof. On the basis of Lemma 4-5 and Proposition 3, the proof is essentially the same as that of Theorem 1 of Coron et al. (2007), thus omitted here.

Remark 7. If $n = 1$, Theorem 6 recovers the results of Coron et al. (2007) for the single-pool case. Similar to de Halleux et al. (2003), the stability condition involves the contraction of $\nabla\mathbf{g}(0) = -(\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}(0))^{-1} \nabla_{[\xi_-(0,t),\xi_+(L,t)]}\mathbf{f}(0)$. This is due to the discrete-time nature of the (a,b) coordinate system (7) at the boundary points. Intuitively, $g(\cdot)$ describes the relation between the outgoing invariants $\xi_-(L,t)$, $\xi_+(0,t)$ and the incoming invariants $\xi_-(0,t)$, $\xi_+(L,t)$, and the characteristic curves of $a(x,t)$, $b(x,t)$ link the outgoing invariants and the incoming invariants together, at different time instances, to form a discrete-time-like system at the boundary points. The discrete-time Lyapunov inequality in (16) justifies this connection. Note that the internal mechanism for the single-pool case, driven by the characteristic curves and the boundary controls, is depicted mathematically in de Halleux et al. (2003, Section 3.2).

[1] Here $H^2(0,L)$ denotes the Sobolev space defined on $[0,L]$, see Taylor (1996, Chapter 4) for a complete description and related operations.

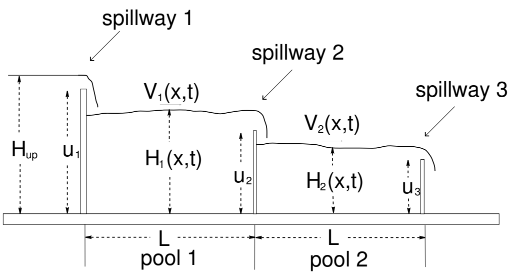

Fig. 3. Two-pool canal with overflow spillways

In general, it is hard to justify the contraction of $\nabla\mathbf{g}(0)$. One possible approach to reduce the computation of $\nabla\mathbf{g}(0)$ is proposed in de Halleux et al. (2004), which exploits the structure of the *invariant graph* to break some of its connectivity.

Note that the condition in the above theorem differs from the one presented in de Halleux et al. (2003) which requires that

$$\rho(abs(\nabla\mathbf{g}(0))) < 1;$$

here $abs(\cdot)$ means, for $A = [a_{ij}] \in \mathbf{R}^{m\times n}$, $abs(A) = [|a_{ij}|] \in \mathbf{R}^{m\times n}$. There is no clear indication showing which one is tighter compared to the other. Either one could be feasible while the other is infeasible.

We now summarize the overall control design procedure.

Procedure 8. (Control design procedure)

(1) Choose suitable closed-loop boundary conditions (9) such that $\rho(\nabla\mathbf{g}(0)) < 1$.
(2) Check the feasibility of (16). If not feasible, go back to Step 1.
(3) Derive the decentralized controller $u_1(a_{1,0}, b_{1,0})$, $u_2(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0})$, $u_3(a_{2,L}, b_{2,L})$ from the boundary conditions (8) and (9).

4. APPLICATION TO A TWO-POOL CANAL WITH OVERFLOW SPILLWAYS

In this section we illustrate the method presented in Section 3 by an application to a two-pool canal with overflow spillways depicted in Fig. 3; see Coron et al. (2007) for the single-pool case.

The open-loop boundary conditions are

$$H_{1,0}V_{1,0} = C_0(H_{up} - u_1)^{3/2}, \tag{17a}$$
$$H_{1,L}V_{1,L} = C_0(H_{1,L} - u_2)^{3/2}, \tag{17b}$$
$$H_{1,L}V_{1,L} = H_{2,0}V_{2,0}, \tag{17c}$$
$$H_{2,L}V_{2,L} = C_0(H_{2,L} - u_3)^{3/2}, \tag{17d}$$

where C_0 is the characteristic constant of the spillways, and for $i = 1, 2$,

$$H_{i,0}(t) = H_i(0,t), \qquad H_{i,L}(t) = H_i(L,t),$$
$$V_{i,0}(t) = V_i(0,t), \qquad V_{i,L}(t) = V_i(L,t).$$

For constant spillway positions $\bar{u}_1, \bar{u}_2$ and $\bar{u}_3$, the steady state solutions are given by

$$\bar{H}_1 = H_{up} - \bar{u}_1 + \bar{u}_2,$$
$$\bar{H}_2 = H_{up} - \bar{u}_1 + \bar{u}_3,$$
$$\bar{V}_1 = \frac{C_0(H_{up} - \bar{u}_1)^{3/2}}{H_{up} - \bar{u}_1 + \bar{u}_2},$$
$$\bar{V}_2 = \frac{C_0(H_{up} - \bar{u}_1)^{3/2}}{H_{up} - \bar{u}_1 + \bar{u}_3}.$$

The corresponding closed-loop boundary conditions (9) are chosen to have the following form:

$$f_1(a_{1,0}, b_{1,0}) = a_{1,0} + k_1 b_{1,0}, \tag{18a}$$
$$f_2(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0}) = b_{1,L} + k_2 a_{1,L} + k_3 b_{2,0} + k_4 a_{2,0}, \tag{18b}$$
$$\begin{aligned} f_3(a_{1,L}, b_{1,L}, a_{2,0}, b_{2,0}) &= (a_{1,L} + b_{1,L} + 2\bar{V}_1)(a_{1,L} - b_{1,L} + 4\sqrt{g\bar{H}_1})^2 \\ &\quad - (a_{2,0} + b_{2,0} + 2\bar{V}_2)(a_{2,0} - b_{2,0} + 4\sqrt{g\bar{H}_2})^2, \end{aligned} \tag{18c}$$
$$f_4(a_{2,L}, b_{2,L}) = b_{2,L} + k_5 a_{2,L}, \tag{18d}$$

where (18c) is the (a,b) coordinate expression of (17c), and $k_i, i = 1, \cdots, 5$ are constants to be selected such that the closed-loop system is stable in the sense of Theorem 6.

For the stability condition in Theorem 6, $\nabla\mathbf{g}(0)$ can be computed by $\nabla\mathbf{g}(0) = -(\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}(0))^{-1} * \nabla_{[\xi_-(0,t),\xi_+(L,t)]}\mathbf{f}(0)$ in which

$$\nabla_{[\xi_-(L,t),\xi_+(0,t)]}\mathbf{f}(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & k_4 \\ 16\sqrt{g\bar{H}_1}(\sqrt{g\bar{H}_1} - \bar{V}_1) & 0 & 0 & -16\sqrt{g\bar{H}_2}(\sqrt{g\bar{H}_2} + \bar{V}_2) \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\nabla_{[\xi_-(0,t),\xi_+(L,t)]}\mathbf{f}(0) = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_3 & k_2 & 0 \\ 0 & -16\sqrt{g\bar{H}_2}(\sqrt{g\bar{H}_2} - \bar{V}_2) & 16\sqrt{g\bar{H}_1}(\sqrt{g\bar{H}_1} + \bar{V}_1) & 0 \\ 0 & 0 & 0 & k_5 \end{bmatrix}.$$

Combing (4), (17) and (18), the boundary control strategy is as follows:

$$u_1 = H_{up} - \sqrt[3]{\frac{H_{1,0}^2}{C_0^2}\left(\bar{V}_1 - 2\sqrt{g}\lambda_1(\sqrt{H_{1,0}} - \sqrt{\bar{H}_1})\right)^2},$$

$$u_2 = H_{1,L} - \sqrt[3]{\frac{H_{1,L}^2 H_{2,0}^2}{C_0^2(H_{2,0} + \lambda_3 H_{1,L})^2}} \times \sqrt[3]{\begin{pmatrix} \bar{V}_1 + \lambda_3\bar{V}_2 + 2\sqrt{g}\lambda_2(\sqrt{H_{1,L}} - \sqrt{\bar{H}_1}) \\ +2\sqrt{g}\lambda_4(\sqrt{H_{2,0}} - \sqrt{\bar{H}_2}) \end{pmatrix}^2},$$

$$u_3 = H_{2,L} - \sqrt[3]{\frac{H_{2,L}^2}{C_0^2}\left(\bar{V}_2 + 2\sqrt{g}\lambda_5(\sqrt{H_{2,L}} - \sqrt{\bar{H}_2})\right)^2},$$

where

$$\lambda_1 = \frac{1-k_1}{1+k_1}, \lambda_2 = \frac{1-k_2}{1+k_2},$$
$$\lambda_3 = \frac{k_3+k_4}{1+k_2}, \lambda_4 = \frac{k_3-k_4}{1+k_2}, \lambda_5 = \frac{1-k_5}{1+k_5}.$$

As seen above, only water level at the local boundary is fed back to the local control. The decentralized feature makes this algorithm easier to implement in practical systems.

5. CONCLUSIONS AND FUTURE WORKS

In this paper, we consider a decentralized control design for irrigation canals based on a strict Lyapunov method introduced in Coron et al. (2007). The obtained results extend the boundary control strategy for a single-pool canal in Coron et al. (2007) to a decentralized control design for a multi-pool canal. The proposed boundary control design uses only the states measured at each gate. It is shown that, by selecting appropriate boundary control, such that the closed-loop boundary conditions satisfy certain contractive condition and discrete-time Lyapunov inequality with diagonal variables, the local convergence to a desired set point can be guaranteed.

Our future work will focus on the following issues:

- Explore possible ways to overcome the difficulties in the control design caused by the spectral-radius-related stability condition.
- Instead of the feasibility problem considered in this paper, try to solve the optimization problem involving, for example, the decay exponential μ, or some suitable cost function.
- Extend the control algorithm to robust analysis, for example, studying the uncertain model with offtake at each gate.
- Investigate how the errors propagate over the network and how to drive the errors as small as possible.

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