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A Period-Specific Realization of Linear Continuous-Time Systems

Ichijo Hodaka * **Ichiro Jikuya** **

* *University of Miyazaki, 1-1, Gakuen Kibanadai-nishi, Miyazaki, 889-2192 Japan (tel: +81-985-58-7352; e-mail: hij@ieee.org).*
** *Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8603 Japan (tel: +81-52-789-3289; e-mail: jikuya@nuae.nagoya-u.ac.jp).*

Abstract: It is a well-known fact that a weighting pattern matrix is realizable as a linear periodic system if and only if the matrix is separable and periodic. This fact, however, can not cope with a reasonable question when a weighting pattern matrix can be realized as a linear periodic system with a *specific* period of time. This paper answers the question by constructing two types of period-specific realizations. Moreover, this paper describes in detail how the lowest dimension of period-specific realizations is identified.

1. INTRODUCTION

This paper is concerned with a classical realization problem, an inverse problem of constructing a state-space representation from a given weighting pattern matrix which represents the input-output relation of a linear system, especially with a realization problem for linear *periodic* and *continuous-time* systems. This type of periodic realization problem was initiated by Silverman[1, 2] and further developed by [3], or discussed in somewhat different situation in [4]. A periodic realization problem for discrete-time systems was investigated in [5]. A realization problem of time-varying linear systems in infinite dimensional spaces was treated in [6].

For the realization problem for linear periodic continuous-time systems, Silverman exhibited a necessary and sufficient condition for a weighting pattern matrix to be realized by a linear periodic system[1, 2]. To be precise, it was proven for the necessity that a weighting pattern matrix inherits a period from a given linear periodic system, the source of the weighting pattern matrix, and for the sufficiency that there exists a linear periodic realization with *twice* the period of a given weighting pattern matrix. This suggests that the sufficiency part may accept an excessively wide class of linear periodic systems as candidates for periodic realization. This motivates us to initiate the question when a given weighting pattern matrix can be realized by a linear periodic system with a *specific* period. It should be strongly recognized, as clarified in Section 3, that in the period-specific realization problem, there may be no solution among linear periodic systems whose dimensions are equal to the order of a weighting pattern matrix, an object of realization, in contrast with realization problems for general linear time-varying systems or linear time-invariant systems; it is well-known that the linear time-varying/time-invariant realization problems can be solved with the dimension equal to the order of weighting pattern matrices. This situation makes it difficult to find a solution to the present problem, since it requires that we should scan possible realization candidates with dimensions larger than the order of weighting pattern matrices.

The first result of this paper answers the period-specific realization problem, by employing a linear periodic realization candidate with a redundant dimensional constant A-matrix. The second result of this paper shows how the redundancy of dimension can be reduced by introducing a linear periodic realization candidate with a non-constant A-matrix.

2. REALIZATION THEORY – A BRIEF REVIEW

2.1 Realization of general linear systems

Let us make a formal statement of the realization problem for general linear systems and summarize known results for the realization problem without proofs. Detailed discussions about these results can be found in the references[7, 8, 9].

Consider an n-dimensional linear time-varying control system with a state x, an input u and an output y in the form

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x, \tag{1}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times m}$ are supposed to be continuous in t and real matrix-valued. Since the output under the initial condition $x(t_0) = 0$ is expressed by

$$y(t) = \int_{t_0}^{t} C(t)\Phi_A(t,p)B(p)u(p)dp,$$

the input-output relation of the system (1) is completely determined by a *weighting pattern matrix* $W(t,p) := C(t)\Phi_A(t,p)B(p)$ of the system (1) ($\Phi_A(t,p)$ denotes the transition matrix of $\dot{x} = A(t)x$ throughout this paper).

A realization problem is an inverse problem of obtaining a system as (1) or equivalently a set of real matrix-valued functions $(A(t), B(t), C(t))$ from a given weighting pattern matrix. Our interest is to find these coefficient matrices $(A(t), B(t), C(t))$ as real matrices, since physical systems are usually approximated as linear systems with real coefficients. We will exclude any complex coefficient matrix from candidates of solution to the realization problem throughout this paper.

Definition 1. A matrix-valued function $W(t,p)$ with two variables t and p is said to be *realizable* (by a linear time-varying system) if there exist continuous real matrix-valued functions $A(t)$, $B(t)$, and $C(t)$ such that

$$W(t,p) = C(t)\Phi_A(t,p)B(p), \quad \forall t, \forall p \in \mathbb{R}. \qquad (2)$$

Then, the linear time-varying system in (1), or equivalently, the triplet $(A(t),\ B(t),\ C(t))$ is called a *realization* of $W(t,p)$.

The following fact is most fundamental in realization theory.

Theorem 1. $W(t,p)$ is realizable if and only if it is *separable*, that is,

$$W(t,p) = L(t)R(p), \quad \forall t, \forall p \in \mathbb{R} \qquad (3)$$

for some continuous real matrix-valued functions $L(t)$ and $R(t)$.

The separations of the form (3) have in general redundancy. For redundancy we mean linear dependence about functions $L(t)$ and $R(t)$ in (3). Removing the redundancy from a separation, we have an essential separation in the following.

Theorem 2. For any separable $W(t,p) \in \mathbb{R}^{r\times m}$, there exist continuous real matrix-valued functions $L_0(t) \in \mathbb{R}^{r\times n_0}$ and $R_0(t) \in \mathbb{R}^{n_0\times m}$ whose columns and rows are respectively linearly independent over $\mathbb{R}$ such that

$$W(t,p) = L_0(t)R_0(p), \quad \forall t, \forall p \in \mathbb{R}. \qquad (4)$$

This special type of expression for $W(t,p)$ is called a *globally reduced form* of $W(t,p)$.

A set of globally reduced forms of weighting pattern matrix can be characterized in the following.

Proposition 3. Let $W(t,p) = L_0(t)R_0(p)$ be a globally reduced form. Any other globally reduced form $W(t,p) = \tilde{L}_0(t)\tilde{R}_0(p)$ is parametrized by

$$\tilde{L}_0(t) = L_0(t)S, \quad \tilde{R}_0(t) = S^{-1}R_0(t) \quad (t \in \mathbb{R}) \qquad (5)$$

where S is an arbitrary nonsingular real matrix.

The fact above implies that all $L_0(t)$'s and $R_0(t)$'s resulting in globally reduced forms of a weighting pattern matrix have the same sizes. Therefore the integer n_0 in Theorem 2 is uniquely determined by $W(t,p)$ and is called the *order* of $W(t,p)$. The order is well defined for any separable (realizable) matrix-valued function.

The order of a weighting pattern matrix is closely related to a minimal dimension of possible realizations, as follows.

Proposition 4. Any separable $W(t,p)$ with the order n_0 has a realization with the dimension n_0, but no realization with a dimension less than n_0.

2.2 Realization of linear periodic systems – A procedure by Silverman

Let us here recall a Silverman's result for realization of linear periodic systems. In conclusion, Silverman proved the following[1, 2].

Theorem 5. $W(t,p)$ has a periodic realization if and only if it is separable and periodic.

We will rebuild the proof of Theorem 5 with some inspection in the below.

To prove the necessity, suppose that $W(t,p)$ has a T-periodic realization $(A(t),\ B(t),\ C(t))$ with some $T > 0$. Separability of $W(t,p)$ is obvious in view of Theorem 1. Since the functions $\Phi_A(t,p)$, $B(t)$ and $C(t)$ have the common period T, $W(t,p) = C(t)\Phi_A(t,p)B(p)$ has also the period T, i.e. $W(t+T, p+T) = W(t,p)$, $\forall t, \forall p \in \mathbb{R}$. Thus we can conclude that $W(t,p)$ is separable and T-periodic.

To prove the sufficiency, we shall construct a periodic realization from a given separable and periodic $W(t,p)$ with a period T. Notice that our objective here is a periodic realization with some period possibly independent of the period T of $W(t,p)$. This freedom of period will achieve a successful construction.

Since $W(t,p)$ is separable, we have a globally reduced form (4) by Theorem 1 and 2. Observe a trivial factorization

$$W(t,p) = L_0(t)R_0(p) = L_0(t)e^{-A_0 t}\cdot e^{A_0(t-p)}\cdot e^{A_0 p}R_0(p),$$

$\forall t, \forall p \in \mathbb{R}$ with any $A_0 \in \mathbb{R}^{n_0\times n_0}$. This implies that the triplet

$$A(t) := A_0,\ B(t) := e^{A_0 t}R_0(t),\ C(t) := L_0(t)e^{-A_0 t} \qquad (6)$$

is a (possibly non-periodic) realization of $W(t,p)$ for any $A_0 \in \mathbb{R}^{n_0\times n_0}$. We can expect that (6) is periodic for an appropriate A_0. In order to generate such an A_0, we note:

Lemma 6. [1] Let $W(t,p)$ be separable with the order $n_0 > 0$ and a globally reduced form $W(t,p) = L_0(t)R_0(p)$ be given. If $W(t,p)$ is T-periodic, then there exists a nonsingular real matrix $Q \in \mathbb{R}^{n_0\times n_0}$ such that

$$L_0(t+T) = L_0(t)Q,\ R_0(t+T) = Q^{-1}R_0(t),\ \forall t \in \mathbb{R}. \qquad (7)$$

Taking account of (7), we immediately see that $B(t)$ and $C(t)$ in (6) have the period T, provided that $e^{A_0 T} = Q$. Thus, a T-periodic realization of $W(t,p)$ is attainable if the matrix equation $e^{A_0 T} = Q$ has a solution $A_0 \in \mathbb{R}^{n_0\times n_0}$.

The substantial difficulty in this step is that there may be no real matrix solution A_0 to the matrix equation $e^{A_0 T} = Q$, even though Q is nonsingular and real.

Proposition 7. [10, 11] Let a square real matrix Q be arbitrarily given. The matrix equation $e^X = Q$ has a solution X as a real matrix, i.e., Q has a real logarithm if and only if Q is nonsingular and has an even number of Jordan blocks of each size for every negative real eigenvalue.

If it emerges that Q in the equation (7) has no real logarithm, the realization (6) by $e^{A_0 T} = Q$ is not functional, because it has a non-real A-matrix. To avoid this difficulty, we direct our attention to the following fact.

Proposition 8. [11] Given any nonsingular real matrix Q, there exists a real matrix X such that $e^X = Q^2$.

In light of this fact, we always find a real matrix A_0 such that $e^{2A_0 T} = Q^2$ because of nonsingularity of Q and take (6) as a candidate of periodic realization. The realization (6) is then actually $2T$-periodic, since we have, from (7),

$$L_0(t+2T) = L_0(t)Q^2, \quad R_0(t+2T) = Q^{-2}R_0(t), \quad \forall t \in \mathbb{R}.$$

Hereinbefore, we have proven that Theorem 5 is valid. Notice that for the sufficiency part we have constructed a $2T$-periodic realization for a separable and T-periodic $W(t,p)$, which was the realization procedure by Silverman.

3. MAIN RESULTS

This section introduces and solves a new type of realization problem, a linear periodic realization problem with a specific period of time. This problem contrasts sharply with the situation in Theorem 5 by Silverman where the period of realizations is not specified beforehand. We will prove in conclusion that there exists a periodic realization whose period is equal to a given weighting pattern matrix, by constructing two types of period-specific realizations; one is a periodic realization with a constant A-matrix, and the other with a non-constant periodically time-varying A-matrix. In particular, it will be clarified that the latter can be minimal with respect to the dimension of period-specific realizations.

3.1 Period-specific realization with constant A-matrices

Although the realization procedure by Silverman works well as demonstrated in Section 2.2, it always yields a $2T$-periodic realization for a T-periodic weighting pattern matrix, even though the weighting pattern matrix has a T-periodic realization. This means that the realization procedure by Silverman serves to construct a periodic realization, by overestimating candidates for periodic realization, since the set of all $2T$-periodic functions strictly contains the set of all T-periodic functions. This urges us to explore the question when a given weighting pattern matrix has a periodic realization with a *specific* period.

The first result of this paper answers the question even if the nonsingular real matrix Q in (7) has no real logarithm. This means that for any nonsingular real Q we need not suppose $2T$-periodic realization candidates, unlike the realization procedure by Silverman.

Theorem 9. $W(t,p)$ has a T-periodic realization if and only if it is separable and T-periodic.

Proof . The proof for the necessity part has already been shown in Section 2.2. To prove the sufficiency, we suppose that $W(t,p)$ is separable and T-periodic. We have a globally reduced form $W(t,p) = L_0(t)R_0(p)$ with an order n_0, by Theorem 2, and then, a nonsingular real matrix $Q \in \mathbb{R}^{n_0 \times n_0}$ satisfying (7), by Lemma 6. Note that Q is guaranteed to be nonsingular and real, and may have some negative real eigenvalues. The rest of realization procedure is alternative; we will consider the case when Q has a real logarithm, that is, when Q satisfies the condition in Proposition 7, and the other case.

Consider first the case when Q satisfies the condition in Proposition 7. In this case, we are able to obtain a real matrix $A_0 \in \mathbb{R}^{n_0 \times n_0}$ such that $e^{A_0 T} = Q$. With this A_0, we take a candidate of realization by (6) which is indeed a T-periodic realization of $W(t,p)$ as mentioned in Section 2.2.

Now we consider the case when Q does not satisfy the condition in Proposition 7. It is obvious that there exists a nonsingular real matrix $Q_\mu \in \mathbb{R}^{\mu \times \mu}$ such that an augmented matrix $\hat{Q} = \mathrm{diag}[Q, Q_\mu] \in \mathbb{R}^{(n_0+\mu)\times(n_0+\mu)}$ satisfies the condition in Proposition 7. Hence, we are able to find a real matrix $\hat{A} \in \mathbb{R}^{(n_0+\mu)\times(n_0+\mu)}$ such that $e^{\hat{A}T} = \hat{Q}$. Now we define

$$A(t) := \hat{A}, \quad B(t) := e^{\hat{A}t}\hat{R}(t), \quad C(t) := \hat{L}(t)e^{-\hat{A}t} \qquad (8)$$

as a candidate triplet of T-periodic realization of $W(t,p)$, where

$$\hat{L}(t) := \left[\, L_0(t)\ 0_{r\times\mu} \,\right], \quad \hat{R}(t) := \begin{bmatrix} R_0(t) \\ 0_{\mu\times m} \end{bmatrix}.$$

Since $\Phi_A(t,p) = e^{\hat{A}(t-p)}$,

$$\begin{aligned} C(t)\Phi_A(t,p)B(p) &= \hat{L}(t)e^{-\hat{A}t}e^{\hat{A}(t-p)}e^{\hat{A}p}\hat{R}(p) \\ &= \hat{L}(t)\hat{R}(p) = L_0(t)R_0(p) = W(t,p), \quad \forall t,\ \forall p \in \mathbb{R}. \end{aligned}$$

Therefore, (8) is certainly a realization of $W(t,p)$. Since $\hat{Q}$ is block-diagonal nonsingular, (7) immediately implies that

$$\hat{L}(t+T) = \hat{L}(t)\hat{Q}, \quad \hat{R}(t+T) = \hat{Q}^{-1}\hat{R}(t), \quad \forall t \in \mathbb{R}.$$

This, together with the equality $e^{\hat{A}T} = \hat{Q}$, shows that $B(t)$ and $C(t)$ in (8) are both T-periodic. We have thus proven Theorem 9.

Notice here that the statements in Theorem 5 and 9 differ only in that the former uses the term “periodic” and the latter “T-periodic”. Hence Theorem 9 provides a stricter and more informative statement than Theorem 5.

3.2 Minimal and period-specific realization

In Section 3.1, we have observed that Theorem 9 successfully resolves the period-specific realization problem. Here, we point out that the resultant realizations by Theorem 9 may be wasteful; the dimension of the realizations may be reducible, because A-matrices of realizations given in the proof of Theorem 9 are selected from among constant real matrices while they can be periodically time-varying. This motivates us to investigate whether or not the dimension of realizations by Theorem 9 could be reduced with a fixed period, by supposing non-constant A-matrices. In fact, we can show a “reducible” example in the following.

Example 1. Consider a real valued

$$W(t,p) = \cos\pi t \sin\pi p - e^{t-p}\sin\pi t\cos\pi p \qquad (9)$$

which has a globally reduced form $W(t,p) = L_0(t)R_0(p)$ with

$$L_0(t) = \left[\, \cos\pi t\ \ -e^t\sin\pi t \,\right], \quad R_0(t) = \begin{bmatrix} \sin\pi t \\ e^{-t}\cos\pi t \end{bmatrix}.$$

We can readily verify that a nonsingular real matrix

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & -e \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad (10)$$

with $T = 1$ solves the equation (7). Applying the procedure in the proof of Theorem 9 to Q, we obtain an augmented matrix $\hat{Q} = \mathrm{diag}[Q, Q]$ which has a real logarithm

$$\hat{A} = \begin{bmatrix} 0 & 0 & -\pi & 0 \\ 0 & 1 & 0 & -\pi \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

We then obtain an 1-periodic realization (8) written as

$$A(t) = \hat{A}, \quad B(t) = \frac{1}{2}\begin{bmatrix} \sin 2\pi t \\ 1+\cos 2\pi t \\ 1-\cos 2\pi t \\ \sin 2\pi t \end{bmatrix},$$

$$C(t) = \frac{1}{2}\left[\, 1+\cos 2\pi t \;\; -\sin 2\pi t \;\; \sin 2\pi t \;\; -1+\cos 2\pi t \,\right].$$

Although reducibility of dimension is not apparent in view of these coefficient matrices above, we can verify that the triplet

$$A(t) = \frac{1}{2}\begin{bmatrix} 1-\cos 2\pi t & -2\pi - \sin 2\pi t \\ 2\pi - \sin 2\pi t & 1+\cos 2\pi t \end{bmatrix}, \; B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C(t) = [\, 1 \;\; 0 \,] \quad (11)$$

gives another 1-periodic, lower-dimensional realization of $W(t,p)$. A full verification will be given in Example 2.

This example, thus, shows that exploiting non-constant periodic A-matrices enables to reduce the dimension of periodic realization preserving the period.

The second objective is to find a *period-specific minimal realization*, a period-specific realization with the lowest dimension. For this end, we begin by coding the connection between a weighting pattern matrix and "Q", the key matrix.

Proposition 10. Under the situation of Lemma 6, Q in (7) is unique.

Although Q in (7) is uniquely determined by a pair ($L_0(t)$, $R_0(t)$) resulting in a globally reduced form, it of course depends on the choice of $L_0(t)$'s and $R_0(t)$'s. Recall here that all the possible choice of $L_0(t)$'s and $R_0(t)$'s are parametrized as (5), by Proposition 3. Therefore, with a Q resulting from a globally reduced pair ($L_0(t)$, $R_0(t)$), each $\tilde{Q}$ resulting from every globally reduced pair ($\tilde{L}_0(t)$, $\tilde{R}_0(t)$) constitutes a set

$$\{S^{-1}QS : \text{nonsingular } S \in \mathbb{R}^{n_0 \times n_0}\}.$$

In particular, we notice that $\det Q = \det \tilde{Q}$. This means that $\det Q$ is independent of choice of $L_0(t)$'s and $R_0(t)$'s, and therefore, it is uniquely determined for a separable and periodic $W(t,p)$ and its period T. This justifies that we call the sign of $\det Q$ of some/any Q *the sign q corresponding to T of* $W(t,p)$: we let $q = 1$ if $\det Q > 0$ and $q = -1$ if $\det Q < 0$. The phrase "corresponding to T" may not be omitted. If a separable $W(t,p)$ has a period T, then it must have periods $2T$, $3T$, ... This implies that $W(t,p)$ has each sign q, q^2, q^3, ... corresponding to each period T, $2T$, $3T$, ..., because we have, from (7), for any positive integer k,

$$L_0(t+kT) = L_0(t)Q^k, \quad R_0(t+kT) = Q^{-k}R_0(t), \quad \forall t \in \mathbb{R}.$$

Therefore, the existence of negative q's urges us to define the sign of weighting pattern matrices with their periods.

With the notion of sign corresponding to a period of $W(t,p)$, we exhibit the second result of this paper.

Theorem 11. For a given separable and T-periodic $W(t,p)$ (assume $W(t,p) \not\equiv 0$), the minimal dimension $\hat{n}_0$ of all the possible T-periodic realizations of $W(t,p)$ is given by

$$\hat{n}_0 = \begin{cases} n_0 & (\text{if } q > 0) \\ n_0 + 1 & (\text{if } q < 0) \end{cases}$$

where n_0 and q respectively denote the order and the sign corresponding to the period T of $W(t,p)$.

Before we proceed to the proof of Theorem 11, we show a key lemma, a generalization of matrix exponential function.

Lemma 12. For a given real number $T > 0$ and a given nonsingular real matrix $Q \in \mathbb{R}^{n\times n}$ with $\det Q > 0$, there exists a C^∞ real matrix-valued function $\Xi(t) \in \mathbb{R}^{n\times n}$ such that

$$\Xi(t+T) = \Xi(t)Q, \quad \det \Xi(t) \neq 0, \quad \forall t \in \mathbb{R}. \qquad (12)$$

Proof . Consider first the case when Q has no negative real eigenvalue. Then we have a real matrix $A_0 \in \mathbb{R}^{n\times n}$ satisfying $e^{A_0 T} = Q$ by Proposition 7. With this A_0 we define $\Xi(t) := e^{A_0 t}$, a C^∞ real matrix-valued function. It is evident by noting properties of matrix exponential that $\Xi(t)$ satisfies (12).

Now consider the case when Q has at least one negative real eigenvalues. Note that Q can be block-diagonalized as

$$Q = U^{-1}\begin{bmatrix} Q_- & 0 \\ 0 & Q_+ \end{bmatrix} U,$$

where U is a nonsingular real matrix, the eigenvalues of $Q_- \in \mathbb{R}^{\nu\times\nu}$ are all negative real, and $Q_+ \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ has no negative real eigenvalue. This block-diagonalization can be done by performing the real Jordan canonical form, for example. We see that $\det Q_+ > 0$, because the eigenvalues of the real matrix Q_+ consist of positive real numbers and conjugate pairs of complex numbers. Noting $\det Q = \det Q_- \det Q_+$ and $\det Q > 0$, we have that $\det Q_- > 0$ because $\det Q_+ > 0$. Therefore, the product of all eigenvalues of Q_- is positive, which implies that ν, the size of Q_-, is even, because all the eigenvalues are negative real. This enables us to introduce a real matrix

$$H = \operatorname{diag}\left[\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\pi}{T}, \ldots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\pi}{T}\right] \in \mathbb{R}^{\nu\times\nu}.$$

Meanwhile, the matrices $-Q_-$ and Q_+ are nonsingular real and both of them have no negative real eigenvalue. We then find real matrices $A_- \in \mathbb{R}^{\nu\times\nu}$ and $A_+ \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ such that $-Q_- = e^{A_- T}$ and $Q_+ = e^{A_+ T}$, by Proposition 7. With the matrices H, A_-, A_+, and U defined above, we now define a C^∞ real matrix-valued function

$$\Xi(t) := \operatorname{diag}[e^{Ht}e^{A_- t}, e^{A_+ t}]\, U \in \mathbb{R}^{n\times n}.$$

It is straightforward that $\Xi(t)$ defined above meets (12). This completes the proof.

We are now in the position to complete the proof of Theorem 11. Note that the following gives another constructive proof for the sufficiency part of Theorem 9.

Proof . (Proof of Theorem 11). Suppose that a separable and T-periodic $W(t,p)$ is given. Consider first the case when $q > 0$. Then, we have a real $Q \in \mathbb{R}^{n_0\times n_0}$ with $\det Q > 0$ satisfying (7) for a globally reduced form $W(t,p) = L_0(t)R_0(p)$. Now apply Lemma 12 to Q. Then we obtain a C^∞ real matrix-valued function $\Xi(t) \in \mathbb{R}^{n_0\times n_0}$ which satisfies (12). At this stage, we are able to synthesize a triplet of matrix-valued functions

$$A(t) := \dot{\Xi}(t)\Xi(t)^{-1}, B(t) := \Xi(t)R_0(t),$$
$$C(t) := L_0(t)\Xi(t)^{-1} \quad (13)$$

as a candidate for the desired realization. Clearly, all these functions are continuous in t (in particular, $A(t)$ is C^∞)

and real matrix-valued. Since $\Xi(t)$ is a real fundamental matrix of $\dot{x} = A(t)x$ by the definition of $A(t)$ in (13), $\Phi_A(t,p) = \Xi(t)\Xi(p)^{-1}$. Together with the definitions of $B(t)$ and $C(t)$ in (13) we have

$$\begin{aligned} W(t,p) = L_0(t)R_0(p) &= C(t)\Xi(t)\cdot\Xi(p)^{-1}B(p) \\ &= C(t)\Phi_A(t,p)B(p), \quad \forall t,\ \forall p \in \mathbb{R}, \end{aligned}$$

which exactly means that the triplet of (13) is an n_0-dimensional realization of $W(t,p)$. Moreover, combining (7), (12) and (13), we immediately verify that the functions in (13) have the common period T. Therefore, (13) is indeed an n_0-dimensional T-periodic realization of $W(t,p)$.

Consider next the case when $q < 0$. In this case, we have a real $Q \in \mathbb{R}^{n_0\times n_0}$ with $\det Q < 0$ satisfying (7) for a globally reduced form $W(t,p) = L_0(t)R_0(p)$. Since $\det Q < 0$, we can not directly apply Lemma 12 to Q. Instead, we introduce an augmented matrix

$$\hat{Q} := \begin{bmatrix} Q & 0 \\ 0 & -\delta \end{bmatrix} \in \mathbb{R}^{(n_0+1)\times(n_0+1)}$$

with an arbitrarily fixed positive real number δ. We can then apply Lemma 12 to $\hat{Q}$, because $\det\hat{Q} > 0$. Thus we have a C^∞ real matrix-valued function $\hat{\Xi}(t) \in \mathbb{R}^{(n_0+1)\times(n_0+1)}$ which satisfies

$$\hat{\Xi}(t+T) = \hat{\Xi}(t)\hat{Q}, \quad \det\hat{\Xi}(t) \neq 0, \quad \forall t \in \mathbb{R} \tag{14}$$

instead of (12). With this $\hat{\Xi}(t)$, we synthesize a triplet of matrix-valued functions

$$\hat{A}(t) := \dot{\hat{\Xi}}(t)\hat{\Xi}(t)^{-1},\ \hat{B}(t) := \hat{\Xi}(t)\hat{R}(t),\ \hat{C}(t) := \hat{L}(t)\hat{\Xi}(t)^{-1} \tag{15}$$

as a candidate for the desired realization, where

$$\hat{L}(t) := [\, L_0(t)\ 0_{r\times 1}\,], \quad \hat{R}(t) := \begin{bmatrix} R_0(t) \\ 0_{1\times m} \end{bmatrix}.$$

These functions are continuous in t ($\hat{A}(t)$ is C^∞) and real matrix-valued. Noting $\Phi_{\hat{A}}(t,p) = \hat{\Xi}(t)\hat{\Xi}(p)^{-1}$ and the definitions of $\hat{B}(t)$ and $\hat{C}(t)$ in (15), we have

$$\begin{aligned} W(t,p) &= L_0(t)R_0(p) = \hat{L}(t)\hat{R}(t) \\ &= \hat{C}(t)\hat{\Xi}(t)\cdot\hat{\Xi}(p)^{-1}\hat{B}(p) = \hat{C}(t)\Phi_{\hat{A}}(t,p)\hat{B}(p), \end{aligned}$$

$\forall t, \forall p \in \mathbb{R}$, which means that the triplet (15) gives an (n_0+1)-dimensional realization of $W(t,p)$. The first equation in (14), the nonsingularity of $\hat{Q}$ and the definition of $\hat{A}(t)$ in (15) imply that $\hat{A}(t)$ has the period T. Since $\hat{Q}$ is block-diagonal, we see that

$$\hat{L}(t+T) = \hat{L}(t)\hat{Q}, \quad \hat{R}(t+T) = \hat{Q}^{-1}\hat{R}(t), \quad \forall t \in \mathbb{R}.$$

These with (14) and the definitions of $\hat{B}(t)$ and $\hat{C}(t)$ in (15) imply that $\hat{B}(t)$ and $\hat{C}(t)$ have also the period T. Hence, we conclude that $(\hat{A}(t),\ \hat{B}(t),\ \hat{C}(t))$ defined in (15) is an (n_0+1)-dimensional T-periodic realization of $W(t,p)$.

It remains only to show that for the case $q > 0$, there is no T-periodic realization with a dimension less than n_0 and that for the case $q < 0$, no T-periodic realization with a dimension less than n_0+1. We know by Proposition 4 that we have no realization with a dimension less than n_0 even though we include non-periodic realization candidates. Therefore, we need to only prove that there is no T-periodic realization with a dimension equal to n_0 in the case $q < 0$. The proof is by contradiction. Assume that we have an n_0-dimensional T-periodic realization of $W(t,p)$, say $(\tilde{A}(t),\ \tilde{B}(t),\ \tilde{C}(t))$ with $\tilde{A}(t) \in \mathbb{R}^{n_0\times n_0}$, in the case $q < 0$. By definition, $W(t,p) = \tilde{C}(t)\Phi_A(t,p)\tilde{B}(p) = \tilde{C}(t)\tilde{\Phi}(t)\tilde{\Phi}(p)^{-1}\tilde{B}(p)$ with an arbitrary real fundamental matrix $\tilde{\Phi}(t)$ of $\dot{x} = \tilde{A}(t)x$. Consequently, we obtain a separation $W(t,p) = \tilde{L}(t)\tilde{R}(p)$ with $\tilde{L}(t) = \tilde{C}(t)\tilde{\Phi}(t)$ and $\tilde{R}(t) = \tilde{\Phi}(t)^{-1}\tilde{B}(t)$. Note here that this separation is globally reduced, since the number of columns of $L(t)$ is equal to the order n_0 of $W(t,p)$. If we let Q_0 be the monodromy matrix of $\tilde{\Phi}(t)$, we have $\det Q_0 > 0$ because $\tilde{A}(t)$ is real. This leads to, $\forall t \in \mathbb{R}$,

$$\begin{aligned} \tilde{L}(t+T) &= \tilde{C}(t+T)\tilde{\Phi}(t+T) = \tilde{C}(t)\tilde{\Phi}(t)Q_0 = \tilde{L}(t)Q_0, \\ \tilde{R}(t+T) &= \tilde{\Phi}(t+T)^{-1}\tilde{B}(t+T) = Q_0^{-1}\tilde{\Phi}(t)^{-1}\tilde{B}(t) \\ &= Q_0^{-1}\tilde{R}(t). \end{aligned}$$

Thus, we conclude that the sign corresponding to T of $W(t,p)$ must be positive, which contradicts the assumption $q < 0$.

Remark 1. The proof of Theorem 11 is closely related to Floquet factorization. For a continuous and real T-periodic $A(t) \in \mathbb{R}^{n\times n}$, let $W(t,p) := \Phi_A(t,p)$. Note that the sign q corresponding to T of $\Phi_A(t,p)$ is positive, or equivalently, a monodromy matrix Q defined by $Q = \Phi(t)^{-1}\Phi(t+T)$ belonging to any real fundamental matrix $\Phi(t)$ of $\dot{x} = A(t)x$ has a positive determinant, because $A(t)$ is real matrix-valued. Then, the procedure for the case $q > 0$ generates a *Floquet-like* factorization for any real fundamental matrix $\Phi(t)$ of $\dot{x} = A(t)x$, $\Phi(t) = C(t)\Xi(t)$, $\forall t \in \mathbb{R}$, where $C(t)$ is nonsingular real and periodic with the period T, and $\Xi(t)$ is a nonsingular real matrix-valued function generated from the monodromy matrix Q of $\Phi(t)$ by Lemma 12. Note that the factorization holds true with the common period T and on the real number. In contrast, in the standard Floquet theory, factorizations can be performed with the common period T, but the factors may be non-real. Floquet-type factorizations on the real number with a non-common period are discussed in [12, 13].

Let us illustrate the period-specific minimal realization procedure given above by the following two examples: one for $q > 0$ and the other for $q < 0$.

Example 2. To see the former type of example, we revisit $W(t,p)$ in (9). We already know that the triplet (11) is a realization which is minimal as well as 1-periodic, since the dimension of (11) is equal to 2, the order of $W(t,p)$. This is consistent with Theorem 11, because $\det Q = e > 0$ which means that the sign corresponding to $T = 1$ of $W(t,p)$ is positive. We then try to "realize" the triplet (11) along the proof of Theorem 11 and Lemma 12. Although Q in (10) has no real logarithm, $-Q$ has a real logarithm

$$A_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

With this A_- and

$$H = \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix},$$

define

$$\Xi(t) = e^{Ht}e^{A_-t} = \begin{bmatrix} \cos\pi t & -e^t\sin\pi t \\ \sin\pi t & e^t\cos\pi t \end{bmatrix}.$$

The realization triplet (11) is then immediately obtained by substitution into (13).

Example 3. To see the latter type of example, consider a real valued function

$$W(t,p) = \cos \pi t \sin \pi p - 2\cosh(p-t) \sin \pi t \cos \pi p$$

which has the order $n_0 = 3$, a period $T = 1$, and a globally reduced form $W(t,p) = L_0(t)R_0(p)$ given by

$$L_0(t) = \left[-e^{-t} \sin \pi t \ \ -e^{t} \sin \pi t \ \ \cos \pi t \right],$$

$$R_0(t) = \left[e^{t} \cos \pi t \ \ e^{-t} \cos \pi t \ \ \sin \pi t \right]^{\mathrm{T}}.$$

We have a nonsingular real matrix $Q = \mathrm{diag}[-e^{-1}, -e, -1]$ as a solution to the equation (7) with $T = 1$. We thus identify the sign q corresponding to $T = 1$ of $W(t,p)$ as $q = -1$. Since the three eigenvalues of Q are negative real and distinct, the procedure in the proof of Theorem 9 generates an 1-periodic realization with a 6-dimensional constant A-matrix. Here we shall observe how an 1-periodic realization with the dimension $n_0 + 1 = 4$ is generated by following the proof of Theorem 11. Define an augmented matrix

$$\hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Observing that $-\hat{Q}$ has a real logarithm

$$A_- = \mathrm{diag}\,[\,-1,\ 1,\ 0,\ 0\,]$$

and setting

$$H = \mathrm{diag}\left[\begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix} \right],$$

we obtain

$$\begin{aligned} \hat{\Xi}(t) &= e^{Ht} e^{A_- t} \\ &= \mathrm{diag}\left[\begin{bmatrix} e^{-t}\cos \pi t & -e^{t} \sin \pi t \\ e^{-t} \sin \pi t & e^{t} \cos \pi t \end{bmatrix}, \begin{bmatrix} \cos \pi t & -\sin \pi t \\ \sin \pi t & \cos \pi t \end{bmatrix} \right]. \end{aligned}$$

With these functions, we arrive at a realization (15) calculated as

$$\hat{A}(t) = \mathrm{diag}\left[\begin{bmatrix} -\cos 2\pi t & -\pi - \sin 2\pi t \\ \pi - \sin 2\pi t & \cos 2\pi t \end{bmatrix}, \begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix} \right],$$

$$\hat{B}(t) = \frac{1}{2} \begin{bmatrix} 1 + \cos 2\pi t - \sin 2\pi t \\ 1 + \cos 2\pi t + \sin 2\pi t \\ \sin 2\pi t \\ 1 - \cos 2\pi t \end{bmatrix},$$

$$\hat{C}(t) = \frac{1}{2} \begin{bmatrix} 1 - \cos 2\pi t - \sin 2\pi t \\ -1 + \cos 2\pi t - \sin 2\pi t \\ 1 + \cos 2\pi t \\ \sin 2\pi t \end{bmatrix}^{\mathrm{T}},$$

which is 4-dimensional and 1-periodic.

We have examined in Section 2.2 that Silverman's procedure provides, for a T-periodic weighting pattern matrix with the order n_0, $2T$-periodic, in particular n_0-dimensional realizations. In order to interpret this situation through Theorem 11, let q denote the sign corresponding to T of $W(t,p)$. Then, we have the sign $q^2 = 1$ corresponding to $2T$ of $W(t,p)$, for both the cases $q = 1$ and $q = -1$. Theorem 11 states for this situation that $W(t,p)$ has a $2T$-periodic realization with the dimension equal to the order n_0 of $W(t,p)$. Thus, the dimension as well as the period of this realization coincides with those by Silverman's procedure.

4. CONCLUSIONS

In this paper, we have proposed and answered the period-specific realization problem, the question when a given weighting pattern matrix has a periodic realization with a given period. One of the crucial steps toward the proofs of the first and second results in this paper has been to utilize the freedom of dimension of realization candidates. It would seem to be unnecessary and redundant that we should take account of realization candidates with dimensions larger than the order n_0 of a given weighting pattern matrix as in the results of this paper, in view of the fact that the realization problem for general linear systems has always a solution system with the dimension equal to the order n_0. Nevertheless, we have revealed that it is necessary and concise in the period-specific realization problem.

This situation can be interpreted as the relation between minimality of realizations and the notion of controllability and observability; it turns out by the second result that period-specific minimal realizations are not necessarily controllable and observable. In fact, the realization for a negative sign of a weighting pattern matrix shown in the proof of the second result has rank-deficient controllability and observability Grammians. More precisely, the ranks of these Garmmians are both equal to n_0, while the sizes of the Grammians are $n_0 + 1$. This situation is noteworthy, even paradoxical, in view of the well-known fact that minimal realizations are inevitably controllable and observable (and the converse is also true), in the realization problems for linear time-varying/time-invariant systems, or the linear periodic realization problems with an unspecified period.

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