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On the Integral Sliding-Mode Control for Sample-data Systems with State Time-Delay ⋆

Lijun Mu * **Cunchen Gao** * **Juan Li** **

* *Department of Mathematics, Ocean University of China, 266071, Qingdao,China (e-mail: cherrymu81@yahoo.com.cn;ccgao@ ouc.edu.cn).*
** *Institute of Information and Engineering, Ocean University of China, 266071, Qingdao,China (e-mail: lijuan@ouc.edu.cn).*

Abstract: A discrete-time integral sliding mode control (DISMC) scheme is proposed for sample-data systems with state time-delay and disturbance. A steady-state error about the magnitude of $O(T^2)$ is achieved for some delay systems. A variable replacement is applied to transform the original system to a new delay-free system; then a DISMC scheme is designed for the new system. Comparing with the existing methods dealing with time-delay systems, the new scheme is different and simple for some systems. The illustrative example demonstrates the validity of the proposed scheme.

1. INTRODUCTION

With the development of computer, research in discrete-time control has been intensified in recent years. This also necessitated a rework in the sliding-mode control (SMC) strategy for discrete-time delay-free systems(see Gao et al. [1995],Hui et al. [1999],Koshkouei et al. [2000],Goloa et al. [2000],Gao et al. [1995]). However, the reaching law method is not very good wether in theory or practice; and the equivalent control method based on the equivalent control and disturbance estimation (see Cheng et al. [2000]) just drives the sliding-mode into a region of $O(T^2)$. The integral sliding-mode control method (see Abidi et al. [2007]) eliminates the reaching phase and drives both sliding-mode and state to the region magnitude of $O(T^2)$.

In fact, discrete-time systems with state-delay have strong background in engineering applications. Though a great number of research results concerning time-delay systems have existed(see Xu et al. [2001],Richard [2003],Gao et al. [2007]), little progress has been reported for the sliding-mode control strategy in discrete-time systems with time-delay. So it motivates the present study.

In this work, aiming at transforming state time-delay system to delay-free one and improving control performance, a new variable is introduced and an integral sliding manifold is applied. First, a feedback control is selected to get proper poles for the nominal model; then a variable replacement is used to transform the time-delay system to a delay-free one; then, for the new system, the closed-loop system can achieve the desired control performance and achieve the boundary layer of the region magnitude of $O(T^2)$ for state regulator. And the two systems are connected by assumption1.

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Note: $\|z\| = \max_i\{|z_i|\}, \|A\| = \max_i\{\sum_{j=1}^{n} |a_{ij}|\}$.

2. PROBLEM FORMULATION

2.1 Sample-data System

Consider the continuous-time system with state-delay and matched disturbance

$$\begin{array}{l} x(t) = Ax(t) + A_1x(t-\tau) + B[u(t) + f(t)] \\ x(\theta) = 0 \qquad \text{for} \quad \theta \in [-\tau, 0) \\ x(0) = x_0 \end{array} \tag{1}$$

where the state $x \in \Re^n$, the control $u \in \Re^m$, and the disturbance $f \in \Re^m$ is assumed smooth and bounded. τ is the constant time-delay in state. If T is the sampling period and $\tau = hT + \tau_1, 0 \le \tau_1 < T$, then the discretized counterpart of (1) can be given by

$$\begin{array}{l} x_{k+1} = \Phi x_k + \Phi_0 x_{k-h} + \Phi_1 x_{k-h-1} + \Gamma u_k + d_k \\ x_i = 0 \qquad \text{for} \quad i = -1, -2, \cdots \\ x_0 = x_0 \end{array} \tag{2}$$

where

$$\Phi = e^{AT}, \Phi_0 = \int_{\tau_1}^{T} e^{As} ds \cdot A_1,$$

$$\Phi_1 = \int_{0}^{\tau_1} e^{As} ds \cdot A_1, \Gamma = \int_{0}^{T} e^{As} ds \cdot B,$$

the disturbance $d_k = \int_0^T e^{As} Bf((k+1)T - s)ds$ represents the influence accumulated from kT to $(k+1)T$. From the definition of Γ, it can be shown that

$$\Gamma = BT + \frac{1}{2!}ABT^2 + \cdots = BT + MT^2 + O(T^3)$$

where $M = \dfrac{1}{2!}AB$ is a constant matrix, and it can be concluded that the magnitude of Γ is $O(T)$.

Lemma 1. If the disturbance $f(t)$ in (1) is bounded and smooth, then

$$d_k = \Gamma f_k + \frac{1}{2}\Gamma g_k T + O(T^3) \tag{3}$$

$$d_k - d_{k-1} = O(T^2) \tag{4}$$

$$d_k - 2d_{k-1} + d_{k-2} = O(T^3) \tag{5}$$

where $f_k = f(kT),\ g_k = g(kT),\ g(t) = (d/dt)f(t)$.

See Abidi et al. [2007].

Note that the magnitude of the mismatched part d_k in the disturbance is of the order $O(T^3)$.

2.2 Transformation

Consider the linear nominal model

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{6}$$

Lemma 2. With a feedback control law

$$u_k^n = -Kx_k \tag{7}$$

the closed-loop system of (6) is asymptotically stable. K is chosen such that $\Phi - \Gamma K = G$ has all different poles inside the unit disk in the complex $z-$plane.

For G has different poles in unit disk, it can be written as diagonal form

$$G = PJP^{-1}$$

where P is a nonsingular transformation matrix and J is the diagonal matrix of $\{\lambda_i, i = 1, \cdots, n\}$ and $\{\lambda_i\}$ are the poles of G, where J is that

$$J = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Obviously, $\|J\| = \lambda, \lambda = \max_{1\le i\le n} |\lambda_i|$.

Consider a controller with the following structure

$$u_k = u_k^n + v_k$$

then, the discretized system (2) is changed to

$$x_{k+1} = Gx_k + \Phi_0 x_{k-h} + \Phi_1 x_{k-h-1} + \Gamma v_k + d_k \tag{8}$$

Lemma 3. The transformation

$$z_k = x_k - \sum_{j=h}^{k-1} G^{k-j-1}\Phi_0 x_{j-h} - \sum_{j=h+1}^{k-1} G^{k-j-1}\Phi_1 x_{j-h-1} \tag{9}$$

can transfer the state time-delay system (8) to delay-free system

$$z_k = Gz_k + \Gamma v_k + d_k \tag{10}$$

Proof. Using (9) and (2), we obtain

$$\begin{aligned} z_{k+1} &= x_{k+1} - \sum_{j=h}^{k} G^{k-j}\Phi_0 x_{j-h} - \sum_{j=h+1}^{k} G^{k-j}\Phi_1 x_{j-h-1} \\ &= Gx_k + \Phi_0 x_{k-h} + \Phi_1 x_{k-h-1} + \Gamma v_k + d_k \\ &- \sum_{j=h}^{k} G^{k-j}\Phi_0 x_{j-h} - \sum_{j=h+1}^{k} G^{k-j}\Phi_1 x_{j-h-1} \\ &= Gz_k + \Gamma v_k + d_k \end{aligned}$$

Assumption 1. If G is an asymptotically stable matrix and x_k is bounded, then x_k has the same magnitude with z_k.

Remark: This assumption is necessary for the control design of the system (8)and system (10); fortunately, there are some systems satisfy it. In fact

$$\begin{aligned} &\frac{\| z_k \|}{\| x_k \|} \ge 1 - \sum_{j=h}^{k-1} \| G^{k-j-1}\Phi_0 \frac{x_{j-h}}{x_k} \| \\ &- \sum_{i=h}^{k-2} \| G^{k-i-2}\Phi_1 \frac{x_{i-h}}{x_k} \| \\ &\ge 1 + \| P \| \| J^{-1} \| \| P^{-1} \| \| \Phi_1 \| \| \frac{x_{k-1-h}}{x_k} \| \\ &- \| P \| (\sum_{j=h}^{k-1} \| J^{k-j-1}(P^{-1}\Phi_0 + J^{-1}P^{-1}\Phi_1) \| \| \frac{x_{j-h}}{x_k} \|) \end{aligned}$$

So, by selecting proper G, $\frac{\|z_k\|}{\|x_k\|} \ge O(T)$ can be obtained.

3. STATE REGULATION WITH ISMC

From assumption 1, it is reasonable to consider the delay-free system (10). Define the integral sliding manifold as follows

$$\begin{aligned} \sigma_k &= Dz_k + \varepsilon_k - Dz_0 \\ \varepsilon_k &= \varepsilon_{k-1} + Ez_{k-1} \end{aligned} \tag{11}$$

where $\sigma_k \in \Re^m, \varepsilon_k \in \Re^m,\quad k = 0, 1, \cdots, D, E \in \Re^{m\times n}$ are constant matrices of rank m and $D\Gamma$ is invertible. It can be concluded that $\sigma_0 = 0, \varepsilon_0 = 0$; this means that the reaching phase is eliminated. In order to force the state trajectory to stay on the sliding manifold, the equivalent control is calculated by $\sigma_{k+1} = 0$. This leads to

$$(v_k)_{eq} = (D\Gamma)^{-1}z_0 - (D\Gamma)^{-1}[(DG + E)z_k + Dd_k + \varepsilon_k]$$

Lemma 4. With the integral sliding manifold and disturbance estimation

$$\widehat{d}_k = d_{k-1} = z_k - Gz_{k-1} - \Gamma v_{k-1}$$

the practical control law

$$v_k = (D\Gamma)^{-1}Dz_0 - (D\Gamma)^{-1}[(DG+E)z_k + D\widehat{d}_k + \varepsilon_k] \tag{12}$$

can make the sliding manifold σ_k go into a neighborhood of $O(T^2)$.

Proof. Using (11), (12) and (4), we obtain

$$\begin{aligned} \sigma_{k+1} &= Dz_{k+1} + \varepsilon_{k+1} - Dz_0 \\ &= DGz_k + D\Gamma v_k + Dd_k + \varepsilon_k + Ez_k - Dz_0 \\ &= DGz_k + Dd_k + \varepsilon_k + Ez_k - Dz_0 + D\Gamma\{ \\ &(D\Gamma)^{-1}Dz_0 - (D\Gamma)^{-1}[(DG + E)z_k + D\widehat{d}_k + \varepsilon_k]\} \\ &= D(d_k - \widehat{d}_k) = D(d_k - d_{k-1}) = O(T^2) \end{aligned}$$

Assumption 2. $\|I - \Gamma(D\Gamma)^{-1}D\| = O(1)$ and $\|P\|\|P^{-1}\| = O(T^{-1})$.

Remark: This assumption is easy to be satisfied by selecting proper matrix D, and enough small T.

Theorem 1. If assumption 1 and 2 are satisfied, then with the control law (12), the closed-loop dynamic system

$$z_{k+1} = Gz_k + \xi_k \tag{13}$$

is set by the magnitude $O(T^3)$,the ultimate bound of z_k is set by the magnitude $O(T^2)$with $E = D - DG$ and $\xi_k \in \Re^n$. So, the original state vector x_k goes into the boundary layer with the magnitude $O(T^2)$.

Proof. Substituting (12) into (10), we obtain

$$\begin{aligned} z_{k+1} &= [G - \Gamma(D\Gamma)^{-1}(DG + E)]z_k - \Gamma(D\Gamma)^{-1}\varepsilon_k \\ &+\Gamma(D\Gamma)^{-1}Dz_0 + d_k - \Gamma(D\Gamma)^{-1}D\widehat{d}_k \end{aligned} \quad (14)$$

using the integral sliding manifold (11), we obtain

$$\varepsilon_k = \sigma_k - Dz_k + Dz_0 \quad (15)$$

Substituting (15),$E = D - DG, \sigma_{k+1} = D(d_k - d_{k-1})$ into (14), we obtain

$$\begin{aligned} z_{k+1} &= Gz_k + d_k - \Gamma(D\Gamma)^{-1}Dd_{k-1} \\ &-\Gamma(D\Gamma)^{-1}D(d_{k-1} - d_{k-2}) \end{aligned} \quad (16)$$

Let

$$\xi_k = d_k - \Gamma(D\Gamma)^{-1}Dd_{k-1} - \Gamma(D\Gamma)^{-1}D(d_{k-1} - d_{k-2}) \quad (17)$$

then using (3), (5),$\|I - \Gamma(D\Gamma)^{-1}D\| = O(1)$, we obtain

$$\begin{aligned} \xi_k &= d_k - 2d_{k-1} + d_{k-2} \\ &+(I - \Gamma(D\Gamma)^{-1}D)(2d_{k-1} - d_{k-2}) \\ &= O(T^3) + (I - \Gamma(D\Gamma)^{-1}D)[\Gamma(2f_{k-1} - f_{k-2}) \\ &+\Gamma(g_{k-1} - \frac{1}{2}g_{k-2})T + O(T^3)] \\ &= O(T^3) + (I - \Gamma(D\Gamma)^{-1}D)O(T^3) = O(T^3) \end{aligned}$$

The solution of (13) is

$$\begin{aligned} z_k &= G^k z_0 + \sum_{i=0}^{k-1} G^{k-i-1}\xi_i \\ &= PJ^kP^{-1}z_0 + P\sum_{i=0}^{k-1} J^{k-i-1}P^{-1}\xi_i \end{aligned}$$

So

$$\begin{aligned} \|z_k\| &\le \|P\|\|J^k\|\|P^{-1}\|\|z_0\| + \|P\|\|\sum_{i=0}^{k-1} J^{k-i-1}P^{-1}\xi_i\| \\ &\le \|P\|\|P^{-1}\|[\lambda^k\|z_0\| + \sum_{i=0}^{k-1}\lambda^{k-i-1}O(T^3)] \\ &\le \|P\|\|P^{-1}\|\lambda^k\|z_0\| + \|P\|\|P^{-1}\|\frac{1}{1-\lambda}O(T^3) \end{aligned}$$

For λ is predict given, so it is easy to select $\lambda \le 0.9$ to make $\frac{1}{1-\lambda} = O(1)$, so $\|P\|\|P^{-1}\|\frac{1}{1-\lambda} = O(T^{-1})$. So, when $k \to \infty, \|z_k\| = O(T^2)$. And from assumption 1, the original states vector x_k go into the boundary layer of $O(T^2)$.

4. NUMERICAL EXAMPLES

Consider (1) with the following parameters:

$$A = \begin{bmatrix} 8 & 12 \\ 10 & 6 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 8 \\ 4 & 7 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$f(t) = 0.1(\cos(t) - \sin(t)), \tau = 0.265$$

The initial states are $x_0 = [1 \ \ -0.5]^T$. Take the sample time $T = 0.01s$, then $h = 26$, and $\tau_1 = 0.005$. The discretized counterpart is given by

$$\Phi = \begin{bmatrix} 1.0898 & 0.12896 \\ 0.10747 & 1.0683 \end{bmatrix}, \Phi_0 = \begin{bmatrix} 0.0108 & 0.4191 \\ 0.0206 & 0.0366 \end{bmatrix}$$

$$\Phi_1 = \begin{bmatrix} 0.0108 & 0.4191 \\ 0.0206 & 0.0366 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.11376 \\ 0.16015 \end{bmatrix}, \|d(k)\| < 0.03$$

The poles are selected as $\lambda_1 = 0.7, \ \lambda_2 = 0.65$, then the gain matrices can be obtained

$$K = [\, 8.5728 \ \ -1.0443 \,]$$

Then

$$G = \begin{bmatrix} 0.1145 & 0.2478 \\ -1.2655 & 1.2355 \end{bmatrix}, P = \begin{bmatrix} -0.4199 & -0.3897 \\ -1.9076 & -0.9209 \end{bmatrix}$$

So, $\|P\|\|P^{-1}\| = 73.5 < O(T^{-1})$. Select any matrix D which make $D\Gamma$ is invertible, here $D = [1 \ \ 1.4]$, then $\|I - \Gamma(D\Gamma)^{-1}D\| = 1.1346 = O(1)$. According to $E = D - DG$

$$E = [\, 2.6572 \ \ -0.5775 \,]$$

The delayed disturbance is used. Fig.1 shows that the system states are ultimately bounded and are set by the magnitude $O(T^2)$ from Fig.2. The ISMC goes into a boundary after the second step and is set by the magnitude $O(T^2)$ from Fig.4; this is because that for the first step it is equal to Dd_0. The control input is bounded and avoids chattering from Fig.5 and Fig.6.

5. CONCLUSION

This work proposes a method to deal with some sample-data systems with time-delay. It is effective for some systems which satisfy assumption1 and 2. So, it is meaningful to be after the necessary and sufficient conditions to select proper systems which satisfy the assumptions.

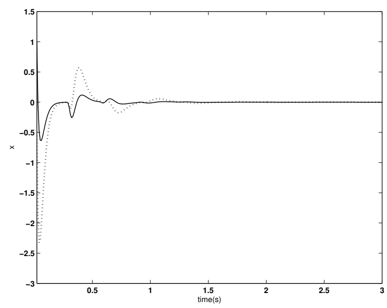

Fig. 1. state x

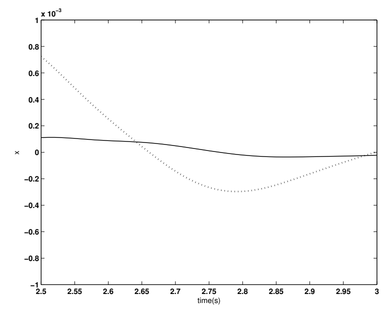

Fig. 2. magnified state x

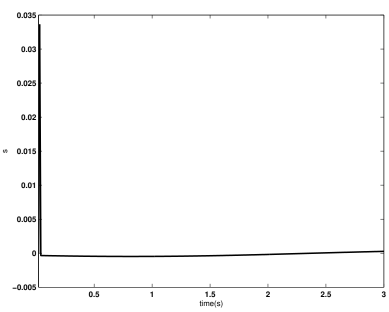

Fig. 3. ISM s

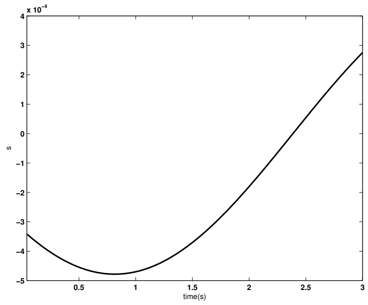

Fig. 4. magnified ISM s

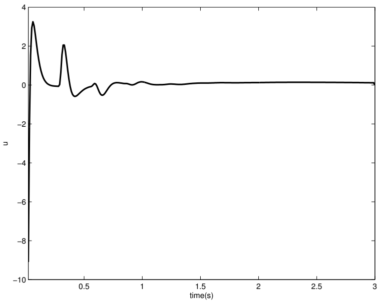

Fig. 5. control output u

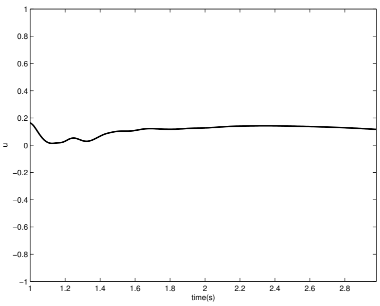

Fig. 6. magnified control output u

REFERENCES

Y. W. Discrete control systems. *Science Press*, 1989.

H. J. Gao and T. W. Chen. New Results on Stability of Discrete-Time Systems With Time-Varying State Delay. *IEEE Transanctions on Automatic Control*, pages 328-334, 52(2), 2007.

S. Y. Xu, J. Lamb, C. G. Yang. Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay. *Systems &Control Letters*, pages 77-84, 43, 2001.

J. P. Richard. Time-delay systems:an overview of some recent advances and open problems. *Automatica*, pages 1667-1694, 39, 2003.

Y. H. Roh, J. H. Oh. Sliding Mode Control with Delay Compensation for Uncertain Input-delay Systems. *Proceedings of the American Control Conference, San Diego, California*, pages 309-313, 1999.

K. Abidi, J. X. Xu, and X. H. Yu. On the Discrete-Time Integral Sliding-Mode Control. *IEEE Transanctions on Automatic Control*, pages 709-715, 52(4), 2007.

W. C. Su, S. V. Drakunov and Ü. Özgüner. An $O(T^2)$ Boundary Layer in Sliding Mode for Sampled-Data Systems. *IEEE Transanctions on Automatic Control*, pages 482-485, 45(3), 2000.

C. C. Cheng, M. H. Lin and J. M. Hsiao. Sliding mode controllers design for linear discrete-time systems with matching perturbations. *Automatica*, pages 1205-1211, 36, 2000.

G. Goloa, Ĉ. Milosavljević. Robust discrete-time chattering free sliding mode control. *Systems &Control Letters*, pages 19-28, 41, 2000.

S. Hui, S. H. ZaK. On discrete-time variable structure sliding mode control. *Systems &Control Letters*, pages 283-288, 38, 1999.

A. J. Koshkouei, A. S. I. Zinober. Sliding Mode Control of Discrete-Time Systems. *Journal of Dynamic Systems, Measurement, and Control*, pages 793-802, 122, 2000.

W. B. Gao, Y. F. Wang and A. Homaifa. Discrete-time variable structure control systems. *IEEE Transanction on Industrial Electronics*, pages 117-122, 42(2), 1995.