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Minimal Representations for Delay Systems

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Abstract: There are many, nonequivalent notions of minimality in state space representations for delay systems. In this class, one can express the transfer function as a ratio of two exponential polynomials. Then one can introduce various notions of coprimeness in such a representation. For example, if there is no common zeros between the numerator and denominator, it corresponds to a spectrally minimal realization, i.e., all eigenspaces are reachable. Another fact is that if the numerator and denominator are approximately coprime in some sense, then it corresponds to approximate reachability. All these are nicely embraced in the class of pseudorational transfer functions introduced by the author. The central question here is to characterize the Bézout identity in this class. This is shown to correspond to a non-cancellation property in the extended complex plane, including infinity. This leads to a unified understanding of coprimeness conditions for commensurate and non-commensurable delay cases. Various examples are examined in the light of the general theorem obtained here.

1. INTRODUCTION

Consider the following simple delay-differential system

$$\begin{cases} \dot{x}(t) = x(t-1) + u(t) \\ y(t) = x(t). \end{cases} \quad (1)$$

Taking the Laplace transform, one can compute, at least formally, its transfer function:

$$G(s) = \frac{1}{s - e^{-s}} = \frac{e^s}{se^s - 1}. \quad (2)$$

which is a ratio of exponential polynomials. In general, a delay-differential system with point delays, commensurate or non-commensurate, its transfer function is expressible as the ratio of exponential polynomials. On the other hand, for systems with distributed delays this is not true, but their transfer functions are still expressible as the ratio of entire (i.e., analytic on the whole plane) functions of exponential type. This is due to the fact that delay systems involve only finite-time memory, and this yields a representation that is a ratio of entire functions. This property is a consequence of the Paley-Wiener theorem.

One may then raise the following questions regarding such expressions:

(1) How can one associate a state space realization to such a representation (2)?
(2) When is a representation like (2) minimal?
(3) Given such a notion of minimality, how is it related to minimality of a realization?

For infinite-dimensional systems, there can be various different notions of minimality of fractional representations of a transfer function. For example, when there are common zeros between the numerator and denominator, the corresponding eigenspace is not controllable or observable. Likewise, there exist several distinct notions of controllability which correspond to coprimeness notions of such fractional representations.

This paper gives a comprehensive account on these questions. We first place delay systems in a more general framework of systems with pseudorational transfer functions, and associate a standard observable realization to them. We then introduce various notions of coprimeness for fractional representations in this class, and their relationship to various notions of controllability. A recently obtained result on the Bézout condition is discussed in relation to an extended notion of exact reachability. Finally, a relationship with behavioral controllability is also discussed.

The notation and nomenclature used are explained in the Appendix.

2. REVIEW: PSEUDORATIONAL IMPULSE RESPONSES

2.1 Pseudorationality

To motivate, we first consider system (1). An algebraic approach toward such a system is that of systems over rings (see, e.g., Kamen (1982); Khargonekar and Sontag (1982); Fliess and Mounier (1998)). It is also studied in the context of $2D$ (or nD) systems also. This amounts to writing (1) as

$$\begin{cases} \dot{x}(t) = \sigma x(t) + u(t) \\ y(t) = x(t). \end{cases} \quad (3)$$

by formally introducing the delay operator $(\sigma x)(t) := x(t-1)$ (often without specifying the domain of the operator), and viewing the systems as given over the ring $\mathbb{R}[s,\sigma]$ of two indeterminates instead of $\mathbb{R}[s]$. Various control synthesis problems are solved in this setting, but it possesses an inherent difficulty that it ignores some crucial analytical properties. For example, consider the pair $(z -$

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$2, sz)$, where z denotes e^s. Over the ring $\mathbb{R}[s,z]$ this pair is not coprime because they both vanish at $s=0$, $z=2$. On the other hand, this is impossible under the constraint $z=e^s$.

We thus conduct a study that reflects analytical properties more faithfully. Note that the Dirac delta distribution δ_1 acts on functions as a delay operator as $(\delta * x)(t) = x(t-1)$ via convolution. Then (1) can be expressed as $x = (\delta' - \delta_1)^{-1} * u$. Shifting the time axis by 1, we obtain $x = (\delta'_{-1} - \delta)^{-1} * \delta_{-1} * u$. The crux here is that both $\delta'_{-1} - \delta$ and δ have compact support, and this is a consequence of the finite-time delay. This property clearly holds for general delay-differential systems, and hence it is natural to view them as a subclass of impulse responses which have this property.

One may be tempted to develop a theory in the algebra consisting of δ' and δ_{-1}. In the Laplace transform domain, this corresponds to considering $\mathbb{R}[s, e^s]$. However, this ring is not adequate for studying minimality even in the delay systems. For example, it is not closed under pole-zero cancellation. Take as an example $(se^s - 1)/s$. The numerator and denominator both vanish at $s = 0$. On the other hand, cancelling s does not yield an element in $\mathbb{R}[s, e^s]$. We are thus led to consider a more general class of algebra. To this end, we consider space of distributions generated by δ_a, $a \le 0$. This yields the space $\mathcal{E}'(\mathbb{R}_-)$. For the notation and nomenclature, see the Appendix.

Theorem 2.1. An impulse response function $p \times m$ matrix G ($\operatorname{supp} G \subset [0,\infty)$) is said to be *pseudorational* (Yamamoto (1988)) if there exist matrices Q and P having entries in $\mathcal{E}'(\mathbb{R}_-)^{p\times p}$ and $\mathcal{E}'(\mathbb{R}_-)^{p\times m}$, respectively, such that

(1) $G = Q^{-1} * P$ where the inverse is taken with respect to convolution;
(2) $\operatorname{ord} \det Q^{-1} = -\operatorname{ord} \det Q$.

From here on, we deal only with the SISO systems for simplicity, i.e., assume that $m = p = 1$, although this is not at all necessary. We will write $G = q^{-1} * p$ in place of $Q^{-1} * P$. This class covers all practical cases of delay differential equations and some others Yamamoto (1988). For some basic notions and results, the reader is referred to Hale (1977); Kolmanovskii and Nosov (1986). See also Fliess et al. (2002) for some practical examples that also falls into the present framework.

Hence for a pseudorational impulse response G, its Laplace transform, i.e., *transfer function*, $\hat{G}(s)$ is $\hat{p}(s)/\hat{q}(s)$, and hence it is the ratio of entire functions satisfying the estimate (24) in the Appendix.

2.2 Input/output Operators

In the subsequent three subsections, we give how a standard delay-differential system model can arise from an abstract realization setting. Consider the following simple retarded system:

$$\frac{dx(t)}{dt} = Ax(t-1) + Bu(t)$$
$$y(t) = Cx(t)$$

This system is often modeled by the following so-called M^2 space model:

$$\frac{d}{dt}\begin{bmatrix} z_t(\theta) \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial\theta} z_t(\theta) \\ Az_t(1) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t)$$
$$y(t) = Cz_t(1) \tag{4}$$

where $[z_t(\theta), x_t]^T \in L^2[0,1] \times \mathbb{R}^{\bullet}$. The domain of the right-hand side differential operator is given by $\{[z(\theta), x]^T | (d/d\theta) z(\theta) \in L^2[0,1] \text{ and } z(0) = x\}$. This is called an M^2 space model Delfour and Mitter (1972), and we will give a realization formalism that naturally leads to such a model.

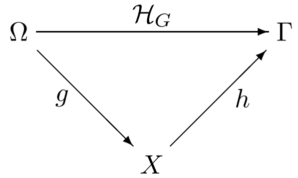
Let G be an impulse response function. The *input/output* or a *Hankel operator* associated with G is the continuous linear mapping $\mathcal{H}_G : \Omega \to \Gamma$ defined by

$$\mathcal{H}_G(\omega)(t) := \int_{-\infty}^{0} G(t-\tau)\omega(\tau)d\tau.$$

We now introduce the notion of a linear, time-invariant system.

Definition 2.2. A *(linear, time-invariant) system* Σ is a quadruple (X, Φ, g, h) such that

- X is a Banach space, and $\Phi(t)$ is a strongly continuous semigroup defined on it;
- $g : \Omega \to X$ is a continuous linear mapping such that $g\sigma_t = \Phi(t)g$ for all $t \ge 0$;
- $h : X \to \Gamma$ is also a continuous linear map satisfying $h\Phi(t) = \sigma_t h$ for all $t \ge 0$.



$\Phi(t)$ is called the *state transition semigroup.* The mappings g and h are called *reachability map* and *observability map*, respectively. Σ is said to be *approximately reachable* if g has dense image, and *observable* if h is one to one. It is *topologically observable* if h gives a topological homomorphism (i.e., continuously invertible when its codomain is restricted to $\operatorname{im} h$). Σ is *weakly canonical* if it is approximately reachable and observable; it is *canonical* if it is further topologically observable. Σ is said to be a *realization* of G if $\mathcal{H}_G = hg$.

Remark 2.3. Various other notions of controllability have been studied in the literature; see, for example, Fliess and Mounier (1998) more in the sense of systems over rings.

2.3 Realization

The definition above is motivated by the following functional differential equation model. We assume sufficient smoothness when necessary. (see Yamamoto (1988) for details):

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

where A is the infinitesimal generator of the state transition semigroup $\Phi(t)$. Now define g and h as follows:

$$g(\omega)(t) = \int_{-\infty}^{0} \exp(-At)B\omega(t)dt$$
$$h(x)(t) = C\exp(At)x.$$

The impulse response G is given by $Ce^{At}B$. Under certain regularity assumptions, one can check that they satisfy the requirements of Definition 2.2.

We are interested in how this applies to the context of pseudorational impulse responses. Let $G = q^{-1} * p$ be pseudorational. One can always associate with it a topologically observable realization $\Sigma^{q,p}$ as follows (Yamamoto (1988)):

Define X^q as follows:

$$X^q := \{x \in \Gamma \mid \pi(q * x) = 0\}$$

where π is the truncation to $(0, \infty)$. It is easy to check X^q is a σ_t-invariant closed subspace of Γ. To define $\Sigma^{q,p}$, take this X^q as the state space with σ_t (restricted to X^q) as its semigroup. Then define $g : \Omega \to X^q$ and $h : X^q \to \Gamma$ as follows.

$$g(\omega) := \pi(q^{-1} * p * \omega) \tag{5}$$

$$h(x) = x \text{ (injection).} \tag{6}$$

Since h is clearly a topological homomorphism, $\Sigma^{q,p}$ is topologically observable. It is approximately reachable if the pair (q, p) is further approximately coprime (Yamamoto (1988)).

2.4 Computation of a canonical realization

Let us show how these abstract formulae lead to a concrete realization (Yamamoto (1988)). Take the transfer function $\hat{G}(s) = 1/(se^s - 1)$. The inverse Laplace transform gives $G = (\delta'_{-1} - \delta)^{-1}$, hence this is pseudorational, with $q = \delta'_{-1} - \delta$ and $p = \delta$ (except the order condition which becomes clear from formula (7) below. Expanding this inverse, we obtain

$$G(t) = H(t-1) * \left(\sum_{n=0}^{\infty} H(t-1)^{*n}\right) \tag{7}$$

where $H(t)$ denotes the Heaviside unit step function.

We claim

$$X^q \equiv L^2[0,1] \times \mathbb{R}. \tag{8}$$

Indeed, suppose $x \in X^q$. Then $(\delta'_{-1} - \delta) * x \in \mathcal{E}'(\mathbb{R}_-)$, i.e., $\mathrm{supp}(\delta'_{-1} - \delta) * x \subset (\infty, 0]$. This means that $\dot{x}(t+1) = x(t)$, $t > 0$, that is,

$$\dot{x}(t) = x(t-1), \quad t > 1. \tag{9}$$

This gives no constraint on $x(t)$, $0 \le t < 1$. Hence at least $x|_{[0,1)}$ ($x \in X^q$) should be freely chosen from $L^2[0,1]$. Since $x|_{[0,1)} \in L^2[0,1]$, x should be absolutely continuous fro $t \ge 1$ in order that (9) be satisfied. Then (9) can be successively integrated as

$$x(1+t) = x(1) + \int_1^{1+t} \dot{x}(\tau)d\tau = x(1) + \int_0^t x(\tau)d\tau. \tag{10}$$

This means that for solutions of (9), $[x|_{[0,1)}, x(1)] \in L^2[0,1] \times \mathbb{R}$ completely determines $x(t)$ for all t. This clearly yields (8).

Let us now compute the representation of A under (8). Take $[z(\cdot), x]^T \in L^2[0,1] \times \mathbb{R}$. In order that it be the restriction of an element of $\gamma \in D(A)$, it is necessary that

(1) $z(\theta) = \gamma(\theta)$, $0 \le \theta < 1$,
(2) $z(\cdot) \in H^1[0,1]$,
(3) $z(1) = x$,

where $H^1[0,1]$ is the Sobolev space of the first order, and $D(A)$ denotes the domain of the closed operator A. The first condition should be obvious, and the second condition is required since γ should belong to H^1 on any compact interval in order that it be differentiable. The third condition guarantees that the extension of the pair $[z(\cdot), x]^T$ is continuous. This, along with (9), indeed guarantees that the extension of $[x(\cdot), x]^T$ to $[0, \infty)$ is indeed absolutely continuous, and hence belongs to $D(A)$. From (10) we have

$$\gamma(1+\epsilon) = \gamma(1) + \int_1^{1+\epsilon} \dot{\gamma}(t)dt = x + \int_0^{\epsilon} z(t)dt.$$

This implies that

$$\frac{1}{\epsilon}\left(\gamma(1+\epsilon) - \gamma(1)\right) = \frac{1}{\epsilon}\int_0^{\epsilon} z(t)dt \to z(0) \tag{11}$$

as $\epsilon \to 0$. (Note that γ is absolutely continuous.) This yields

$$A \begin{bmatrix} z(\cdot) \\ x \end{bmatrix} = \begin{bmatrix} \dfrac{d}{dt} z(\cdot) \\ z(0) \end{bmatrix}$$

with

$$D(A) = \{[z(\cdot), x]^T | z(\cdot) \in H^1[0,1], z(1) = x\}.$$

According to Yamamoto (1988), B and C are also computed as follows:

$$(Bu)(t) = [G(\cdot), G(1)]^T u$$
$$C[z(\cdot), x]^T = z(0).$$

Since $G|_{[0,1)} \equiv 0$ and $G(1) = 1$ by (7), we obtain the functional differential equation

$$\frac{d}{dt}\begin{bmatrix} z_t(\theta) \\ x_t \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial\theta} z_t(\theta) \\ z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{12}$$

$$y(t) = z_t(0), \tag{13}$$

which is what we encountered in (4).

It is not difficult to see that this gives a realization of

$$\dot{x}(t) = x(t-1) + u(t)$$
$$y(t) = x(t).$$

3. MINIMALITY AND COPRIMENESS

Let $G = q^{-1} * p$ be pseudorational. Then $\hat{G}(s) = \hat{p}(s)/\hat{q}(s)$, and this is a ratio of entire functions.

In the previous section we have given a construction of a topologically observable realization $\Sigma^{q,p}$. Reachability of this realization is related to coprimeness of (p, q), and there are various different notions of coprimeness here.

Definition 3.1. Let $G = q^{-1} * p$ be pseudorational. The pair (p, q) is said to be *spectrally coprime* if $\hat{p}(s)$ and $\hat{q}(s)$ have no common zeros. It is *approximately coprime* if there exist sequences ϕ_n and ψ_n in $\mathcal{E}'(\mathbb{R}_-)$ such that $p * \phi_n + q * \psi_n \to \delta$ in $\mathcal{E}'(\mathbb{R}_-)$ where δ denotes the Dirac delta distribution. It is a *Bézout pair* if there exist $\phi, \psi \in \mathcal{E}'(\mathbb{R}_-)$ such that

$$p * \phi + q * \psi = \delta. \tag{14}$$

If (p, q) is a Bézout pair, then $\hat{p}(s)$ and $\hat{q}(s)$ satisfies

$$\hat{p}(s)\hat{\phi}(s) + \hat{q}(s)\hat{\psi}(s) = 1. \tag{15}$$

Identity (14) or (15) is called the *Bézout identity.*

The following theorem is crucial.

Theorem 3.2. Let (p, q) be pseudorational, and $\Sigma^{q,p}$, X^q as above. $\Sigma^{q,p}$ is approximately reachable if and only if (p, q) is approximately coprime. The pair (p, q) is spectrally coprime if and only if every eigenspace of $\Sigma^{q,p}$ is reachable. It is Bézout if and only if every state $x \in X^q$ is reachable if there exists input $u \in \mathcal{E}'(\mathbb{R}_-)$ such that $x = g(u)$, by extending the input space to $\mathcal{E}'(\mathbb{R}_-)$.

Proof We first prove the last statement. For the other two, see Yamamoto (1988, 1989).

Suppose (14) holds. This yields

$$q^{-1} = q^{-1} * p * \phi + \psi. \tag{16}$$

Take any $x \in X^q$. Then $\omega := q * x \in \mathcal{E}'(\mathbb{R}_-)$ by definition. It follows that $x = q^{-1} * \omega = (q^{-1} * p * \phi + \psi) * \omega$. Applying π on both sides, we obtain $x = \pi(q^{-1} * p * \phi * \omega) = g(\phi * \omega)$ by (5) because $\pi(\psi * \omega) = 0$. Hence every x is reachable with $\mathcal{E}'(\mathbb{R}_-)$ input $\phi * \omega$.

Conversely, suppose $\Sigma^{q,p}$ is reachable with $\mathcal{E}'(\mathbb{R}_-)$ inputs. In particular, $q^{-1} \in X^q$ is reachable. This implies $q^{-1} = \pi(q^{-1} * p * \phi)$ for some $\phi \in \mathcal{E}'(\mathbb{R}_-)$. Hence $q^{-1} = q^{-1} * p * \phi + \psi$, for some $\psi \in \mathcal{E}'(\mathbb{R}_-)$, which readily yields (14). □

The relationship between approximate coprimeness and spectral coprimeness is rather striking. We recall the following result obtained in Yamamoto (1989):

Theorem 3.3. Let $G = q^{-1} * p$ be pseudorational. The pair (p, q) is approximately coprime if and only if the following two condition holds:

(1) (p, q) is spectrally coprime, and
(2) $\min\{r(p), r(q)\} = 0$.

The proof is given in Yamamoto (1989), but let us illustrate what this means.

Let q and X^q be as above. As described earlier, $\hat{q}(s)$ is an entire function of exponential type, and it has at most countably many zeros $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$. Suppose, for simplicity, that their multiplicity is all one. Then $e^{\lambda_n t}$, $n = 1, 2, \ldots$ are eigenfunctions. Let M be the closure of the linear span of all such eigenfunctions in X^q. We know from Yamamoto (1989) that $M = X^q$ if and only if $r(q) = 0$. That is, X^q is eigenfunction complete if and only if q "touches" the origin. While it requires quite technical arguments, let us see why. Suppose q is written as $q = q_1 * \delta_{-a}$, $a > 0$. It is easy to see that $X^q \cong X^{q_1} \oplus X^{\delta_{-a}}$, and $X^{\delta_{-a}} \cong L_{[0,a]}$. The latter space actually refers to the space of functions that becomes identically zero for $t > a$. It is now clear that such functions cannot be spanned by exponential functions $e^{\lambda_n t}$.

If the second condition in Theorem 3.3 is satisfied, then one can modify q to $q + kp$ without altering the reachability property. Since by the first condition $(p, q + kp)$ do not possess a common zero, its eigenspaces are reachable, and hence the totality is dense in X^q. This is the structure of the theorem.

We also note that there is indeed a gap between approximate coprimeness and the Bézout condition. The following counterexample is given in Yamamoto (1998):

Example 3.4. Let $q := \delta_{-1}$ and p be any C^∞ function with support contained in $[-1/2, 0]$ such that $r(p) = 0$. Then by Theorem 3.3, this pair is approximately coprime. But it is not Bézout. For if we had $p * \phi + q * \psi = \delta$, $\text{supp}(q * \psi) \subset (-\infty, -1]$, and hence in a neighborhood of the origin, $p * \phi = \delta$. Since p is a C^∞ function, this is clearly impossible.

4. COPRIMENESS CONDITIONS

Let us further examine coprimeness conditions for delay-differential systems. So far, we have seen

- Spectral coprimeness < approximate coprimeness < Bézout condition.
- The gap between spectral coprimeness and approximate coprimeness is not very large: if $(\hat{p}(s), \hat{q}(s)$ has a common factor of type e^{as}, $a > 0$, then the pair is not approximately coprime; otherwise, spectral coprimeness implies approximate coprimeness.
- there is a gap between approximate coprimeness and Bézout condition (Example 3.4).
- On the other hand, in the behavioral systems context, Glüsing-Lürssen (1997); Rocha and Willems (1997) proved that spectral coprimeness implies Bézout condition (behavioral controllability). This is mysterious, and now is the theme of this section.

Let us start by quoting the following (Glüsing-Lürssen (1997)):

Theorem 4.1. Let $\hat{p}, \hat{q}$ be elements of $\mathbb{R}[s, z]$, where z denotes e^s. Suppose that $\hat{p}(s, e^s)$ and $\hat{q}(s, e^s)$ possess no common zeros. Then there exists $\phi, \psi \in \mathcal{E}'(\mathbb{R})$ such that $\hat{p}\hat{\phi} + \hat{q}\hat{\psi} = 1$.

In other words, if (p, q) is a spectrally coprime pair, then it is Bézout over the ring $\mathcal{E}(\mathbb{R})$ where ϕ and ψ have compact support but not necessarily in the negative half line.

Actually, what she proved is stronger than this. She has shown that ϕ and ψ can be taken in a much smaller ring which is obtained as the finite Laplace transform arising from delay-differential operators. She also showed that this ring is a Bézout domain. But for our present purposes, the above form is enough.

Let us first note that it is *not* mysterious that spectral coprimeness implies approximate coprimeness in the behavioral context. As we have seen in Theorem 3.3, a spectrally coprime pair (p, q) is approximately coprime if and only if they have no common factor of type δ_{-a}, $a > 0$. While δ_{-a} is not invertible over $\mathcal{E}'(\mathbb{R}_-)$, it is over $\mathcal{E}(\mathbb{R})$ over the whole line, because $\delta_a \in \mathcal{E}(\mathbb{R})$.

Hence the gap between spectral coprimeness and approximate coprimeness is not so large. What is mysterious is why the gap between approximate coprimeness and Bézout condition can be circumvented somehow. Example 3.4 is not a delay differential system, hence outside the scope of Theorem 4.1. The clue is the "cancellation at infinity." Before discussing the detail, let us give some intuitive ideas.

Let us consider the simplest case of measures, i.e., distributions of order zero. That is, we consider

$$p * \phi + q * \psi = \delta \tag{17}$$

where not only are p and q measures, but also ϕ, ψ are. Then

$$\hat{p}(s)\hat{\phi}(s) + \hat{q}(s)\hat{\psi}(s) = 1. \tag{18}$$

must hold. Since they have compact support, they are also bounded, and hence $\hat{p}, \hat{q}, \hat{\phi}, \hat{\psi}$ are all bounded on the whole plane. Suppose that there exists a sequence $\lambda_n \in \mathbb{C}$ such that $\hat{p}(\lambda_n), \hat{p}(\lambda_n) \to 0$. Then, since $\hat{\phi}(S)$ and $\hat{\psi}(s)$ are bounded, $\hat{p}(\lambda_n)\hat{\phi}(\lambda_n) + \hat{q}(\lambda_n)\hat{\psi}(\lambda_n) \to 0$. This clearly contradicts (18).

In fact, the following theorem was obtained in Yamamoto (2007):

Theorem 4.2. Let $G = q^{-1} * p$ be pseudorational, and suppose further that $q, p \in \mathfrak{M}_{(-\infty,0]}$, i.e., measures. Suppose that there exists $\sigma \in \mathbb{R}$ such that $\hat{p}(s), \hat{q}(s) \neq 0$ for $\operatorname{Re} s \geq \sigma$. A necessary and sufficient condition for (p, q) to be Bézout in the ring of measures is that there exists $c > 0$ such that

$$|p(s)| + |q(s)| \geq c > 0 \tag{19}$$

for every $s \in \mathbb{C}$.

One may notice that condition (19) is the same as the celebrated Corona condition. In fact, the situation here is quite similar in that the proof amounts to characterization of maximal ideals. This is in the context of the theory of Banach algebras, especially Gel'fand algebras. However, there is an important difference here in that $\hat{q}(s)$ is an entire function so that it can have only countably many discrete zeros. This makes the analysis of the limiting behavior of $\hat{p}(s)$ along the zeros of $\hat{q}(s)$. The detail is given in Yamamoto (2007), and omitted here.

However this is not the whole story. What we need a characterization of the Bézout identity in $\mathcal{E}'(\mathbb{R}_-)$, not in the space of measures. To see the difference, consider the following example:

Example 4.3. Consider the pair $(se^s - 1, e^s)$. The first component $se^s - 1$ is not a measure. It has infinitely many zeros $\lambda_n \to \infty$. At these points, $e^{\lambda_n} = 1/\lambda_n \to 0$. Hence this pair approximately cancels at infinity and does not satisfy (19).

On the other hand, it satisfies the following Bézout identity:

$$(se^s - 1) \cdot (-1) + s \cdot e^s = 1. \tag{20}$$

This is outside the scope of Theorem 4.2.

We have the following theorem Yamamoto (2007):

Theorem 4.4. Let $q^{-1} * p$ be pseudorational, and suppose that there exists $\sigma \in \mathbb{R}$ such that $\hat{p}(s), \hat{q}(s) \neq 0$ for $\operatorname{Re} s \geq \sigma$. Suppose also that the algebraic multiplicity of each zero of $\hat{q}(s)$ is globally bounded. If there exists a nonnegative integer m such that

$$|\lambda_n^m \hat{p}(\lambda_n)| \geq c, n = 1, 2, \ldots \tag{21}$$

Then the pair (p, q) is Bézout.

The following corollary is an easy consequence of Theorem 4.4, and gives a symmetric condition on p and q:

Corollary 4.5. Let (p, q) be as above. If there exist a nonnegative integer m and $c > 0$ such that

$$|s^m \hat{p}(s)| + |s^m \hat{q}(s)| \geq c > 0, \tag{22}$$

Then the pair (p, q) is Bézout.

5. EXAMPLES IN DELAY SYSTEMS

Let us now re-examine Example 4.3. The pair $(se^s - 1, e^s)$ clearly satisfies

$$|s(se^s - 1)| + |se^s| \geq c > 0, \tag{23}$$

and hence by Corollary 4.5, this pair is Bézout. Multiplication by s actually cancels the decay rate of e^s along the zeros of $se^s - 1$. This is precisely why the result of Glüsing-Lürssen (1997) guarantees the Bézout condition under spectral coprimeness; see also Rocha and Willems (1997).

Generally speaking, this is a characteristic of delay systems with commensurate delays.

On the other hand, when delays are not commensurable, it is also known that her result does not carry over. Let us see this with a typical example.

5.1 Commensurable Delay Case

In this case p and q belong to the polynomial ring $\mathbb{R}[s, z]$ where we allow the interpretation $z = e^s$ (normalizing the delay length to be 1). Now consider $p(s, z)$ as a polynomial of two variables. Then $p(s, z)$ as $s \to \infty$ can go to zero only at most with polynomial order in s, z. Hence if there is an asymptotic cancellation as $s \to \infty$, this can be removed by multiplying a suitable factor s^m, because such a cancellation must be of polynomial order.

Theorem 4.1 claims that (p, q) is Bézout if it is spectrally coprime; this is proven in the behavioral context, and hence time shift does not count. Hence one can always assume $r(q) = 0$ without loss of generality, so that spectral coprimeness indeed implies approximate coprimeness Yamamoto (1989). The argument above shows that there is no cancellation at infinity in this commensurable delay case, and hence the Bézout condition results.

Example 5.1. Consider the pair $(z, sz - 1)$, $z = e^s$. This pair has an asymptotic cancellation for $z = 1/s$, as $s \to \infty$. But this cancellation can be removed by multiplying s to the first component z. This is why the pair $(e^s, se^s - 1)$ is Bézout over $\mathcal{E}'(\mathbb{R}_-)$ while it is not over $\mathfrak{M}_{(-\infty,0]}$.

5.2 Noncommensurable Delay Case

The case for noncommensurable delay case is completely different. In such a case, spectral coprimeness does not imply Bézout. We will see why by showing an example.

In this case, the pair p and q are elements of the ring $\mathbb{R}[s, z_1, z_2]$, where $z_1 = e^{h_1 s}$, and $z_2 = e^{h_2 s}$ and h_2 is not a rational multiple of h_1 (and vice versa).

Let $h_1 = 1$, and h_2 an irrational, say π for example. Consider the pair $(e^s - 1, e^{\pi s} - 1)$. By Kronecker's approximation theorem (Apostol (1997)), for any integer n and $\epsilon > 0$, there exists m such that $|n - m\pi| < \epsilon$. It then follows that $e^{\pi s} - 1|_{s=2m\pi i} = e^{2m\pi^2 i} - 1$ is very close to $e^{2n\pi i} - 1$ which is zero. Of course, $e^s - 1|_{s=2m\pi i} = e^{2m\pi i} - 1 = 0$, so that $(e^s - 1, e^{\pi s} - 1)$ asymptotically cancels each other at infinity. Moreover, since ϵ is arbitrary, this asymptotic cancellation cannot be removed if we consider $s^m(e^s - 1)$ in place of $e^s - 1$, because for zeros of $s^m(e^s - 1)$, there always exists λ that is close to a zero of $e^{\pi s} - 1$. In a sense, one can say that $e^s - 1$ and $e^{\pi s} - 1$ are asymptotically close with almost arbitrary order, and this is why the result of the single-delay case does not carry over to the noncommensurable delay case.

6. CONCLUSION

We have given some fundamental facts on minimality of state space and transfer function representations for delay systems. The whole discussions are given in the scope of distributions with compact support. It is worth noting that the general condition for Bézout identity involves cancellation at infinity in the Laplace transform domain, and also the order of such cancellation for distributions.

APPENDIX: NOTATION AND NOMENCLATURE

Let $\mathcal{E}'(\mathbb{R}_-)$ denote the space of distributions having compact support contained in the negative half line $(-\infty, 0]$. Distributions such as Dirac's delta δ_a placed at $a \le 0$, its derivative δ_a' are examples of elements in $\mathcal{E}'(\mathbb{R}_-)$. In contrast, $\mathcal{E}'(\mathbb{R})$ denotes the space of distributions with compact support in $(-\infty, \infty)$. $\mathfrak{M}_{(-\infty,0]}$ denotes the subspace of $\mathcal{E}'(\mathbb{R}_-)$ consisting of measures, i.e., distributions of order 0. The *order* of a distribution ψ is denoted by $\operatorname{ord}\psi$ (Schwartz (1966).)

The following Paley-Wiener theorem gives a representation of pseudorationality in the Laplace domain, i.e., transfer functions.

Theorem 6.1. (Schwartz (1966)). A necessary and sufficient condition for a complex function $\chi(s)$ to be the Laplace transform of a distribution $f \in \mathcal{E}'(\mathbb{R}_-)$ is that

(1) $\chi(s)$ is an entire function; and
(2) $\chi(s)$ satisfies the growth estimate

$$|\chi(s)| \le C(1+|s|)^m e^{a\operatorname{Re}s}, \operatorname{Re}s \ge 0, \\ \le C(1+|s|)^m, \operatorname{Re}s \le 0. \tag{24}$$

for some $C > 0, a > 0$ and integer $m \ge 0$.

We will refer to (24) as the Paley-Wiener estimate.

Note that the zeros of $\chi(s)$ are discrete, and each zero has a finite multiplicity, because $\chi(s)$ is entire.

Since $\chi(s)$ is an entire function of exponential type, the following Hadamard factorization holds (Boas (1954)):

$$\chi(s) = s^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{25}$$

Since there are no finite accumulation point for $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$.

Hence for a pseudorational impulse response G, its Laplace transform, i.e., *transfer function*, $\hat{G}(s)$ is $\hat{p}(s)/\hat{q}(s)$, and hence it is the ratio of entire functions satisfying the estimate (24) above.

Let $\Omega := \lim\limits_{\to} L^2[-n, 0]$ denote the *inductive limit* of the spaces $\{L^2[-n,0]\}_{n>0}$; it is the union $\cup_{n=1}^{\infty} L^2[-n, 0]$, endowed with the finest topology that makes all injections $j_n : L^2[-n, 0] \to \Omega$ continuous; see, e.g., Treves (1967). Dually, $\Gamma := L^2_{loc}[0, \infty)$ is the space of all *locally* Lebesgue square integrable functions with obvious family of seminorms:

$$\|\phi\|_n := \left\{ \int_0^n |\phi(t)|^2 dt \right\}^{1/2}.$$

This is the *projective limit* of spaces $\{L^2[0,n]\}_{n>0}$. Ω is the space of past inputs, and Γ is the space of future outputs, with the understanding that the present time is 0. These spaces are equipped with the following natural *left shift* semigroups:

$$(\sigma_t \omega)(s) := \begin{cases} \omega(s+t), & s \le -t, \\ 0, & -t < s \le 0, \end{cases} \tag{26}$$
$$\omega \in \Omega, t \ge 0, s \le 0.$$

$$(\sigma_t \gamma)(s) := \gamma(s+t), \quad \gamma \in \Gamma, t \ge 0, s \ge 0. \tag{27}$$

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