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Tracking Control for Switched Linear Systems with Time-Delay ⋆

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Abstract: Tracking control for switched linear systems with time-delay is investigated in this paper. Sufficient conditions for the solvability of the tracking control problem are given respectively for the cases that the state of system is measurable and unmeasurable.When the state is measurable, we design a switching control law to achieve the H_∞ model reference tracking performance. When the state is not available, the design of a switching control law based on measured output instead of the state information is considered. Lyapunov function methods are utilized to the stability analysis and controller design for the switched linear systems with time-delay. By using linear matrix inequalities and convex optimization techniques, the controller design problem can be solved efficiently. The simulation examples show the validity of the switching control laws.

1. INTRODUCTION

Switched systems, due to their significance both in theory development and practical applications, have been attracting considerable attention in recent years (see, e.g., Liberzon [1999], Hespanha and Morse [1999], Zhao and Dimirovski [2004]). As an important class of hybrid systems, switched systems have hybrid features comprising of a family of subsystems described by continuous or discrete time dynamics, and a rule specifying the switching among them. As useful tools, Lyapunov functions can deal with the stability problems for switched systems, although certain switching laws incorporated with compatible information sometimes should be designed (see, e.g., Branicky [1998], Hespanha and Morse [1999]).
On the other hand, it is well known that time-delays, which are the inherent features of many engineering process, are great sources of instability and poor performance. So, many researchers have devoted to the study of systems with time-delay (see, Hale [1977], Dugard and Verrist [1998], Kharitonov [1999]). Since switched systems with time-delay have strong engineering background, special attention has been attracted, and several useful results have been reported in the literature such as the issues on stability analysis (Zhai *et al.* [2000], Sun *et al.* [2006]), optimal control (Wu *et al.* [2006]), and so on. The importance of the study of tracking control for switched systems with time-delay arises from the extensive applications in robot tracking control (Zhou *et al.* [1996]), guided missile tracking control, etc. However, to the authors' best knowledge, up to now, the issue of tracking control, which has been well addressed for non-switched systems without delay (Schmitendorf and Barmish [1986]), has been rarely investigated for switched linear systems with time-delay.
In this paper, we investigate the problem of tracking control for switched linear systems with time-delay. Sufficient conditions for the solvability of the tracking control problem are given for the cases that the state of a system is measurable and unmeasurable, respectively. When the state is measurable, we use single Lyapunov function technique to design tracking controllers and a switching law such that the H_∞ model reference tracking performance is satisfied; and when the state is not available, we design observer-based tracking control laws. The method in (Sun and Ge [2005]) is extended to the design of the switching controllers. Meanwhile, multiple Lyapunov functions are used to the design of the tracking control problem. The feasibility of the problem can be realized by convex optimization techniques and linear matrix inequalities (LMIs). Finally, the simulation examples show the validity of the proposed methods.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this paper, we use $P > 0\,(\geq, <, \leq\ 0)$ to denote a positive definite (semi-definite, negative definite, semi-negative definite) matrices P. The superscript "T" stands for matrix transpose; and the symmetric terms in a matrix are denoted by $*$, $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $L_2[0,\infty)$ is the space of square integrable functions on $[0,\infty)$ and $\|\cdot\|$ stands for the usual 2-norm. Let x_t be defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$ and $\|x_t\|_{cl} =$

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$\sup_{-\tau \le t \le 0} \|x(t+\theta)\|$.

Consider the switched linear time-delay system

$$\begin{cases} \dot{x}(t) = A_\sigma x(t) + D_\sigma x(t-\tau) + B_\sigma u(t) + \omega(t), \\ \phi(\theta) = x(t+\theta),\, \theta \in [-\tau, 0],\, x(0) = \phi(0) = 0, \\ y(t) = C_\sigma x(t), \quad t \in [0, \infty), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $\omega(t) \in \mathbb{R}^n$ is bounded exogenous disturbance; $y(t) \in \mathbb{R}^q$ is the output, $\phi(t)$ is the continuous vector valued function specifying the initial state of the system, $\tau > 0$ is the constant, the right continuous function $\sigma(t) : [0, \infty) \to \underline{N} \triangleq \{1, 2, \cdots, N\}$ is the switching signal which can be characterized by the switching sequence $\Sigma = \{x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_j, t_j), \cdots | i_j \in \underline{N}, j = 0, 1, \cdots\}$. Moreover, $\sigma(t) = i$ implies that the ith subsystem (A_i, D_i, B_i, C_i) is active, where A_i, D_i, B_i and C_i are constant matrices of appropriate dimensions, $i \in \underline{N}$. For simplicity, we denote $\sigma := \sigma(t)$.

Given a reference model

$$\dot{x}_r(t) = A_r x_r(t) + r(t), \quad x_r(0) = 0, \tag{2}$$

and performance index

$$\int_0^{t_f} e_r^T(t) e_r(t) dt < \gamma^2 \int_0^{t_f} \varpi^T(t) \varpi(t) dt, \tag{3}$$

where $x_r(t) \in \mathbb{R}^n$ is reference state, A_r is a Hurwitz matrix, $r(t)$ is bounded reference input; $e_r(t) = x(t) - x_r(t)$ denotes the error between the real state of the switched system (1) and the reference state; t_f is the control terminated time; $\varpi(t) = (\omega^T(t), r^T(t))^T$, $\gamma > 0$ is disturbance attenuation level.

Combining (1) with (2), we get the augmented system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_r(t) \end{bmatrix} = \begin{bmatrix} A_\sigma x(t) + D_\sigma x(t-\tau) + B_\sigma u(t) \\ A_r x_r(t) \end{bmatrix} + \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}. \tag{4}$$

Definition 1. For system (4), if there exist control input $u = u(t)$ and switching signal $\sigma = \sigma(t)$ such that (4) is asymptotically stable when $\varpi \equiv 0$ and (3) is satisfied when $\varpi \neq 0$ under the initial conditions stated in (1) and (2), then the switched system (1) is said to have H_∞ model reference tracking performance.

Our purpose is to design a tracking controller $u(t) = K_{\sigma(t)} e_r(t)$ and a switching law such that system (1) has the H_∞ model reference tracking performance.

To conclude this section, we recall the following lemma.

Lemma 1 (Cao [1998]). Let M, N be real matrices of appropriate dimensions. Then, for any matrix $Q > 0$ of appropriate dimension and any scalar $\gamma > 0$, the following inequality holds

$$MN + N^T M^T \le \gamma^{-1} M Q^{-1} M^T + \gamma N^T Q N. \tag{5}$$

3. PERFORMANCE ANALYSIS AND CONTROLLER DESIGN

3.1 The measurable state case

We first consider the case that the state of system (1) is measurable. We will show how to design state feedback gain K_i and a switching law $\sigma(t)$ such that the H_∞ model reference tracking performance is satisfied.

For a fixed switching signal $\sigma(t) = i$, consider the ith subsystem with the state feedback controller $u(t) = K_i e_r(t)$. The augmented system (4) can be rewritten as

$$\dot{\bar{x}}(t) = \bar{A}_i \bar{x}(t) + \bar{D}_i \bar{x}(t-\tau) + \varpi(t), \tag{6}$$

where

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x_r(t) \end{bmatrix},\ \bar{A}_i = \begin{bmatrix} A_i + B_i K_i & -B_i K_i \\ 0 & A_r \end{bmatrix},\ \bar{D}_i = \begin{bmatrix} D_i & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the following closed-loop switched linear system with time-delay,

$$\dot{\bar{x}}(t) = \bar{A}_\sigma \bar{x}(t) + \bar{D}_\sigma \bar{x}(t-\tau) + \varpi(t). \tag{7}$$

We have the following result.

Theorem 1. For the augmented system (7), if there exist positive definite matrices P, S, matrices K_i, and scalars $\alpha_i > 0, i \in \underline{N}$, $\sum_{i=1}^N \alpha_i = 1$, such that

$$\begin{bmatrix} \sum_{i=1}^N \alpha_i \Xi_i + \bar{Q} & \sum_{i=1}^N \alpha_i P \bar{D}_i & P \\ * & -S & 0 \\ * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{8}$$

holds, then the feedback controller $u(t) = K_\sigma e_r(t)$ for system (4), such that the H_∞ model reference tracking performance in (1) is guaranteed, the corresponding switching law is given by

$$\sigma(t) = i, \text{ if } \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-\tau) \end{bmatrix} \triangleq \xi(t) \in \Omega_i,\ \forall t \ge 0, \tag{9}$$

where

$$\Xi_i = \bar{A}_i^T P + P \bar{A}_i + S,\ \bar{Q} = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

$$\Omega_i = \left\{ y \in R^{4n} | y^T \begin{bmatrix} \Pi_i & P\bar{D}_i \\ * & -S \end{bmatrix} y < 0 \right\}, \tag{10}$$

in which $\Pi_i = \Xi_i + \gamma^{-2} PP + \bar{Q}$.

Proof. By Schur complement lemma, the condition (8) is equivalent to the following inequality

$$\begin{bmatrix} \sum_{i=1}^N \alpha_i \Pi_i & \sum_{i=1}^N \alpha_i P \bar{D}_i \\ * & -S \end{bmatrix} < 0. \tag{11}$$

Obviously, we have $\bigcup \Omega_i = R^{4n} \backslash \{0\}$. Define a Lyapunov-Krasovskii functional candidate

$$V(\bar{x}(t)) = \bar{x}^T(t) P \bar{x}(t) + \int_{t-\tau}^t \bar{x}^T(\sigma) S \bar{x}(\sigma) d\sigma, \tag{12}$$

which is positive definite since P and S are positive definite matrices.

First, we will prove that the system (7) is asymptotically stable while $\varpi(t) \equiv 0$.

For any $t \ge 0$, there exists $i \in \underline{N}$ such that $\xi(t) \in \Omega_i$, which means $\sigma(t) = i$, that is , the ith subsystem is active. From (10) we can get

$$\begin{bmatrix} \Xi_i & P\bar{D}_i \\ * & -S \end{bmatrix} < \begin{bmatrix} -\bar{Q} - \gamma^{-2} PP & 0 \\ 0 & 0 \end{bmatrix} \le 0. \tag{13}$$

Therefore, we have

$$\frac{dV(\bar{x}(t))}{dt} = \xi^T(t) \begin{bmatrix} \Xi_i & P\bar{D}_i \\ * & -S \end{bmatrix} \xi(t) < 0, \tag{14}$$

which implies asymptotic stability of the switched systems (7) with $\varpi(t) \equiv 0$.

Next, we prove $\int_0^{t_f} e_r^T(t) e_r(t) dt < \gamma^2 \int_0^{t_f} \varpi^T(t) \varpi(t) dt$ under the zero initial condition and with $\varpi(t) \neq 0$.

Differentiating the Lyapunov-Krasovskii functional candidate along the trajectories $\bar{x}(t)$ of the system (7) gives

$$\frac{dV(\bar{x}(t))}{dt} = \xi^T(t) \begin{bmatrix} \Xi_i & P\bar{D}_i \\ * & -S \end{bmatrix} \xi(t) + 2\bar{x}^T(t) P \varpi(t). \tag{15}$$

Applying Lemma 1, we get

$$2\bar{x}^T(t)P\varpi(t) \leq \gamma^{-2}\bar{x}^T(t)PP\bar{x}(t) + \gamma^2\varpi^T(t)\varpi(t). \tag{16}$$

Then

$$\frac{dV(\bar{x}(t))}{dt} \leq \xi^T(t)\begin{bmatrix}\Xi_i + \gamma^{-2}PP & P\bar{D}_i \\ * & -S\end{bmatrix}\xi(t) + \gamma^2\varpi^T(t)\varpi(t). \tag{17}$$

By the condition (10), it holds that

$$\begin{bmatrix}\Xi_i + \gamma^{-2}PP & P\bar{D}_i \\ * & -S\end{bmatrix} < \begin{bmatrix}-\bar{Q} & 0 \\ 0 & 0\end{bmatrix}. \tag{18}$$

Substituting (18) into (17), we obtain

$$\frac{dV(\bar{x}(t))}{dt} < -\bar{x}^T(t)\bar{Q}\bar{x}(t) + \gamma^2\varpi^T(t)\varpi(t), \tag{19}$$

where

$$\bar{x}^T(t)\bar{Q}\bar{x}(t) = \begin{bmatrix}x(t) \\ x_r(t)\end{bmatrix}^T\begin{bmatrix}I & -I \\ -I & I\end{bmatrix}\begin{bmatrix}x(t) \\ x_r(t)\end{bmatrix} = e_r^T(t)e_r(t). \tag{20}$$

Substituting (20) into (19) results in

$$\frac{dV(\bar{x}(t))}{dt} < -e_r^T(t)e_r(t) + \gamma^2\varpi^T(t)\varpi(t), \tag{21}$$

Integrating both sides of (21) from zero to t_f yields

$$\begin{aligned}&\int_0^{t_f}\sum_{i_j\in\underline{N}}\dot{V}(\bar{x}(t))dt = \sum_{j=0}^{t_f}\sum_{i_j\in\underline{N}}\int_{t_{i_j}}^{t_{i_{j+1}}}\dot{V}(\bar{x}(t))dt \\ &= V(\bar{x}(t_f)) - V(\bar{x}(0)) \\ &< -\int_0^{t_f}e_r^T(t)e_r(t)dt + \gamma^2\int_0^{t_f}\varpi^T(t)\varpi(t)dt.\end{aligned}$$

According to the zero initial condition and $V(\bar{x}(t))$ being positive definite, it is easy to derive

$$\int_0^{t_f}e_r^T(t)e_r(t)dt < \gamma^2\int_0^{t_f}\varpi^T(t)\varpi(t)dt$$

which completes the proof. □

Remark 1. Theorem 1 presents a sufficient condition for the solvability of the problem of H_∞ model reference tracking control. Although we might seek N controllers $u(t) = K_ie_r(t)$ for N subsystems according to (8). It is noticed that the ith subsystem of (1) usually cannot achieve the H_∞ model reference tracking performance, this is because the Lyapunov function does not decrease along the solution of the subsystem whenever $\begin{bmatrix}x^T(t) & x^T(t-\tau)\end{bmatrix}^T \notin \Omega_i$. Therefore, in order to get the whole H_∞ model reference tracking performance for switched system, switching should be designed among subsystems.

Remark 2. Theorem 1 does not give a method of getting the positive definite matrices P, S, and K_i. We now convert (8) into LMIs, then apply convex optimization techniques. Denote $\hat{P} = P^{-1}, \hat{S} = \hat{P}S\hat{P}$, and let $\hat{P} = \begin{bmatrix}\tilde{P} & 0 \\ 0 & \tilde{P}\end{bmatrix}, \hat{S} = \begin{bmatrix}\tilde{S}_1 & 0 \\ 0 & \tilde{S}_2\end{bmatrix}$. Multiplying both sides of (8) by the matrix $diag\{P^{-1}, P^{-1}, I\}$, we have

$$\sum_{i=1}^N\alpha_i\begin{bmatrix}\wp_i & \bar{D}_i\hat{P} & \hat{P} \\ * & \hat{S} & 0 \\ * & * & -\gamma^2I\end{bmatrix} < 0, \tag{22}$$

where

$$\wp_i = \begin{bmatrix}\tilde{P}A_i^T + A_i\tilde{P} + X_i^TB_i^T + B_iX_i + \tilde{S}_1 & -B_iX_i \\ -X_i^TB_i^T & \tilde{P}A_r^T + A_r\tilde{P} + \tilde{S}_2\end{bmatrix}$$

with $X_i = K_i\tilde{P}$.

Once we have $\tilde{P}, \tilde{S}_1, \tilde{S}_2$ from (22), the tracking controller $u(t) = K_ie_r(t)$, with $K_i = X_i\tilde{P}^{-1}, i \in \underline{N}$ can be constructed.

3.2 The unmeasurable state case

In this subsection, we will investigate the possibility of designing observer-based tracking control laws when the state is not available. For the convenience of designing, we will use multiple Lyapunov function method rather than single Lyapunov function method which used for the measurable state case.

Consider the state estimator given by

$$\dot{\hat{x}}(t) = A_\sigma\hat{x}(t) + D_\sigma\hat{x}(t-\tau) + B_\sigma u(t) + L_\sigma(y(t) - \hat{y}(t)) \tag{23a}$$

$$\hat{y}(t) = C_\sigma\hat{x}(t) \tag{23b}$$

in which $y(t)$ and $\sigma(t)$ are the measurable output and switching signal of system (1), respectively. The matrices $L_1, L_2, \cdots, L_N \in R^{n\times q}$ are to be determined later.

Define the difference between the real state and the estimator state as

$$e(t) = x(t) - \hat{x}(t). \tag{24}$$

From (1) and (23a), we have

$$\dot{e}(t) = (A_\sigma - L_\sigma C_\sigma)e(t) + D_\sigma e(t-\tau) + \omega(t). \tag{25}$$

Assumption 1. There exist positive definite matrices X, G and matrices Y_i, such that

$$\begin{aligned}\Phi_i :=& A_i^TX + XA_i - C_i^TY_i^T - Y_iC_i \\ &+ G + XD_iG^{-1}D_i^TX < 0.\end{aligned} \tag{26}$$

Remark 3. The above assumption asserts the existence of a common Lyapunov-Krasovskii functional candidate $V(e(t))$ for the switched linear time-delay system

$$\dot{e}(t) = (A_\sigma - L_\sigma C_\sigma)e(t) + D_\sigma e(t-\tau). \tag{27}$$

In fact, let $L_i = X^{-1}Y_i$, and choose

$$V(e(t)) = e^T(t)Xe(t) + \int_{t-\tau}^t e^T(s)Ge(s)ds$$

as a Lyapunov-Krasovskii functional candidate. It is easy to show that there exist scalars $\alpha_1 > 0$, $\alpha_2 > 0$, such that

$$\alpha_1\|e(t)\|^2 \leq V(e(t)) \leq \alpha_2\|e(t)\|_{cl}^2.$$

Moreover,

$$\|e(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\frac{\lambda}{2\alpha_1}(t-t_0)}\|e(t_0)\|_{cl}$$

holds with λ being the smallest eigenvalue of the matrices Φ_i. This implies that the system (27) is exponentially stable under arbitrary switching.

Now, define the estimation error between the observer state and the reference state as

$$\hat{e}_r(t) = \hat{x}(t) - x_r(t), \tag{28}$$

and the difference between the real state and the reference state as

$$e_r(t) = x(t) - x_r(t),$$

Design the estimation error feedback control law

$$u(t) = K_\sigma\hat{e}_r(t). \tag{29}$$

Combining (23a), (2) with (24), (25), (28) and (29), we have the augmented switching linear time-delay system as follows:

$$\dot{e}(t) = (A_\sigma - L_\sigma C_\sigma)e(t) + D_\sigma e(t-\tau) + \omega(t), \tag{30a}$$

$$\begin{cases} \dot{\hat{x}}(t) = A_\sigma \hat{x}(t) + D_\sigma \hat{x}(t-\tau) + B_\sigma K_\sigma \hat{e}_r(t) \\ \qquad +L_\sigma C_\sigma e(t), \\ \dot{x}_r(t) = A_r x_r(t) + r(t). \end{cases} \tag{30b}$$

Let

$$\tilde{x}(t) = \begin{bmatrix} \hat{x}(t) \\ x_r(t) \end{bmatrix}, \ \tilde{D} = \begin{bmatrix} D_\sigma 0 \\ 0 \ 0 \end{bmatrix}, \ \tilde{Q} = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$$

$$\tilde{A}_\sigma = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma & -B_\sigma K_\sigma \\ 0 & A_r \end{bmatrix}, \ f_\sigma(t) = \begin{bmatrix} L_\sigma C_\sigma e(t) \\ r(t) \end{bmatrix}.$$

Then, (30b) can be rewritten as

$$\dot{\tilde{x}}(t) = \tilde{A}_\sigma \tilde{x}(t) + \tilde{D}_\sigma \tilde{x}(t-\tau) + f_\sigma(t). \tag{30b'}$$

Take $\dot{\tilde{x}}(t) = \tilde{A}_\sigma \tilde{x}(t) + \tilde{D}_\sigma \tilde{x}(t-\tau)$ as the nominal system of (30b') and $e(t)$ as the exoteric disturbance of (25), and further, $f_\sigma(t)$ can also be viewed as the exoteric disturbance of (30b').

Assumption 2. There exist $T \geq t_0$, and $\beta > 0$, such that when $t > T$, it holds that

$$f_i^T(t) f_i(t) < \beta \tilde{x}^T(t)\tilde{x}(t).$$

Theorem 2. For system (30), suppose that Assumption 1 and Assumption 2 hold. If there exist positive definite matrices P_i, S, matrices K_i, and scalars $\alpha_{ij} > 0, (i, j \in \underline{N})$, such that the following matrices inequalities hold,

$$\begin{bmatrix} \Delta_i & P_i\tilde{D}_i \\ * & -S \end{bmatrix} < 0, \tag{31}$$

where

$$\Delta_i = \tilde{A}_i^T P_i + P_i \tilde{A}_i + S + \beta I + P_i P_i + \tilde{Q} + \sum_{j \neq i,\, j \in \underline{N}} \alpha_{i,j}(P_j - P_i),$$

then the feedback controller $u(t) = K_\sigma \hat{e}_r(t)$ for the augmented system (23), such that the H_∞ model reference tracking performance in (1) is guaranteed, the corresponding switching law is given as

$$\sigma(t) = arg \min_{i \in \underline{N}} \{\tilde{x}^T(t) P_i \tilde{x}(t)\}. \tag{32}$$

Proof. Design Lyapunov-Krasovskii functional candidate

$$V(\tilde{x}(t)) = \tilde{x}^T(t) P_{\sigma(t)} \tilde{x}(t) + \int_{t-\tau}^{t} \tilde{x}^T(s) S \tilde{x}(s) ds. \tag{33}$$

Obviously, the Lyapunov-Krasovskii functional candidate is positive definite.

First, we prove asymptotic stability of system (30) with $\varpi(t) \equiv 0$. Let $\zeta(t) = \left[\tilde{x}^T(t) \ \tilde{x}^T(t-\tau)\right]^T$. For any $t > 0$, the jth switching instant is denoted by $t_{j-1} (j \geq 1)$. During any time interval $[t_{j-1}, t_j)$, suppose that the ith subsystem is active. The time derivative of $V(\tilde{x}(t))$ along the trajectory of (30b') with $\varpi(t) = 0$ is

$$\frac{dV(\tilde{x}(t))}{dt} = \zeta^T(t) \begin{bmatrix} \tilde{A}_i^T P_i + P_i \tilde{A}_i + S & P_i \tilde{D}_i \\ * & -S \end{bmatrix} \zeta(t) + 2\tilde{x}^T(t) P_i f_i(t). \tag{34}$$

Note that $f_i(t) = \begin{bmatrix} L_i C_i e(t) \\ 0 \end{bmatrix}$, by Lemma 1, we have

$$2\tilde{x}^T(t) P_i f_i(t) \leq f_i^T(t) f_i(t) + \tilde{x}^T(t) P_i P_i \tilde{x}(t). \tag{35}$$

Assumption 1 guarantees $e(t) \to 0 \, (t \to \infty)$, which in turn gives $f_i(t) \to 0 \, (t \to \infty)$. Assumption 2 guarantees that there exist $T > t_0$, and scalar $\beta > 0$, such that when $t > T$, it holds

$$f_i^T(t) f_i(t) < \beta \tilde{x}^T(t) \tilde{x}(t) \tag{36}$$

It follows from (34), (35) and (36), that

$$\frac{dV(\tilde{x}(t))}{dt} < \zeta^T(t) \begin{bmatrix} \Sigma_i & P_i\tilde{D}_i \\ * & -S \end{bmatrix} \zeta(t). \tag{37}$$

where $\Sigma_i = \tilde{A}_i^T P_i + P_i \tilde{A}_i + S + \beta I + P_i P_i$.

By virtue of the designed switching law (32), there holds

$$\tilde{x}^T(t) \Big(\sum_{j \neq i,\, j \in \underline{N}} \alpha_{ij}(P_j - P_i) \Big) \tilde{x}(t) \geq 0, \forall t \in R^{2n}.$$

Also we note that $\tilde{Q} = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$, we obtain

$$\frac{dV(\tilde{x}(t))}{dt} < \begin{bmatrix} \tilde{x}(t) \\ \tilde{x}(t-\tau) \end{bmatrix}^T \begin{bmatrix} \Delta_i & P_i\tilde{D}_i \\ * & -S \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{x}(t-\tau) \end{bmatrix}.$$

in which $\Delta_i = \Sigma_i + \tilde{Q} + \sum_{j \neq i,\, j \in \underline{N}} \alpha_{ij}(P_j - P_i)$.

With the condition (31), during $[t_{j-1}, t_j)$, we easily get $\frac{dV(\tilde{x}(t))}{dt} < 0$ when $\zeta(t) = \left[\tilde{x}^T(t) \ \tilde{x}^T(t-\tau)\right]^T \neq 0$.

In addition, by the switching law (32), at the switching instant t_j, we have

$$\tilde{x}^T(t_j) P_{\sigma(t_j)} \tilde{x}(t_j) \leq \lim_{t \to t_j^-} \tilde{x}^T(t) P_{\sigma(t)} \tilde{x}(t),$$

which implies $V(\tilde{x}^T(t_j)) \leq \lim_{t \to t_j^-} V(\tilde{x}^T(t))$. So, with the multiple Lyapunov functions technique (Branicky [1998]), system (30) with $\varpi(t) = 0$ is asymptotically stable under the switching law (32).

Secondly, we prove under zero initial condition with $\varpi(t) \neq 0$ that $\int_0^{t_f} e_r^T(t) e_r(t) dt < \gamma^2 \int_0^{t_f} \varpi^T(t) \varpi(t) dt$.

Again, assume $\sigma(t) = i$, $t \in [t_{j-1}, t_j)$. Therefore

$$\tilde{x}^T(t) \Big(\sum_{j \neq i,\, j \in \underline{N}} \alpha_{ij}(P_j - P_i) \Big) \tilde{x}(t) \geq 0.$$

Differentiating the Lyapunov-Krasovskii functional candidate $V(\tilde{x}(t))$ along the trajectory of the system (30b') with $\varpi(t) \neq 0$, and taking (31) into account, we have

$$\begin{aligned} \frac{dV(\tilde{x}(t))}{dt} &< \zeta^T(t) \begin{bmatrix} \Sigma_i & P_i\tilde{D}_i \\ * & -S \end{bmatrix} \zeta(t) \\ &\leq \zeta^T(t) \begin{bmatrix} -\tilde{Q} 0 \\ 0 \ 0 \end{bmatrix} \zeta(t) = -\hat{e}_r^T(t) \hat{e}_r(t). \end{aligned} \tag{38}$$

Note that $\hat{e}_r(t) = e_r(t) - e(t)$, using Lemma 1 with $Q = diag\{\frac{1}{2}, \cdots, \frac{1}{2}\} \in \mathbb{R}^{n \times n}$ gives

$$\begin{aligned} -\hat{e}_r^T(t)\hat{e}_r(t) &= -e_r^T(t) e_r(t) - e^T(t) e(t) + 2 e_r^T(t) e(t) \\ &\leq -e_r^T(t) e_r(t) - e^T(t) e(t) \\ &\quad + e_r^T(t) Q e_r(t) + e^T(t) Q^{-1} e(t) \\ &= -\frac{1}{2} e_r^T(t) e_r(t) + \|e(t)\|^2. \end{aligned} \tag{39}$$

Assumption 1 gives that the nominal system of (30a) is exponentially stable, according to Variation-of-constants (Hale [1977]), when $e_t(0) = e(0) = 0$, there exist constants $\alpha > 0$, $0 < k \leq 1$ such that

$$\|e_t(t, \omega)\| \leq \int_0^t k e^{-\alpha(t-s)} \|\omega(s)\| ds \tag{40}$$

holds for (30a), and according to Cauchy-Schwartz Inequality, there has (Li *et al.* [2008])

$$\|e(t)\|^2 \le \frac{k^2}{\beta}\int_0^t e^{-\beta(t-s)}\|\omega(s)\|^2 ds. \tag{41}$$

Substituting (39), (41) into (38) gives rise to

$$\frac{dV(\tilde{x}(t))}{dt} < -\frac{1}{2}e_r^T(t)e_r(t) + \frac{k^2}{\beta}\int_0^t e^{-\beta(t-s)}\|\omega(s)\|^2 ds. \tag{42}$$

Integrating (42) from zero to t_f , we get

$$\begin{aligned}
&\int_0^{t_f}\sum_{i_j\in \underline{N}}\dot{V}(\tilde{x}(t))dt = \sum_{j=0}^{t_f}\sum_{i_j\in \underline{N}}\int_{t_{i_j}}^{t_{i_{j+1}}}\dot{V}(\tilde{x}(t))dt\\
&< -\frac{1}{2}\int_0^{t_f} e_r^T(t)e_r(t)dt + \frac{k^2}{\beta}\int_0^{t_f}\int_0^t e^{-\beta(t-s)}\|\omega(s)\|^2 dsdt\\
&< -\frac{1}{2}\int_0^{t_f} e_r^T(t)e_r(t)dt + \frac{k^2}{\beta^2}\int_0^{t_f}\|\omega(s)\|^2 dt\\
&< -\frac{1}{2}\int_0^{t_f} e_r^T(t)e_r(t)dt + \frac{1}{2}\gamma^2\int_0^{t_f}\varpi^T(t)\varpi(t)dt, \qquad (43)
\end{aligned}$$

where $\frac{2k^2}{\beta^2} \le \gamma^2$.
Again, taking the switching law (32) into account, on the switching instant t_j, it holds

$$V(\tilde{x}(t_j)) \le V(\tilde{x}(t_j^-)). \tag{44}$$

Substituting (44) into the expansion of the left side of (43), yields

$$\begin{aligned}
&V(\tilde{x}(t_f)) - V(\tilde{x}(t_0))\\
&\le V(\tilde{x}(t_f)) - V(\tilde{x}(t_{f-1})) + V(\tilde{x}(t_{f-1}^-))\\
&\quad - V(\tilde{x}(t_{f-2})) + \cdots + V(\tilde{x}(t_1^-)) - V(\tilde{x}(t_0))\\
&= \int_0^{t_f}\sum_{i_j\in \underline{N}}\dot{V}(\tilde{x}(t))dt = \sum_{j=0}^{t_f}\sum_{i_j\in \underline{N}}\int_{t_{i_j}}^{t_{i_{j+1}}}\dot{V}(\tilde{x}(t))dt\\
&< -\int_0^{t_f} e_r^T(t)e_r(t)dt + \gamma^2\int_0^{t_f}\varpi^T(t)\varpi(t)dt.
\end{aligned}$$

By the zero initial condition and the positive definiteness of $V(\tilde{x}(t))$, $\int_0^{t_f} e_r^T(t)e_r(t)dt < \gamma^2\int_0^{t_f}\varpi^T(t)\varpi(t)dt$ holds.

4. NUMERICAL EXAMPLES

We illustrate the main results by examples in this section.
Example 1. Consider the systems (1) and the reference system (2) with

$$A_1 = \begin{bmatrix}1.2 & 0\\ 3.6 & -2.2\end{bmatrix}, D_1 = \begin{bmatrix}0.5 & 0.8\\ -0.1 & -0.4\end{bmatrix}, A_r = \begin{bmatrix}-1.5 & -1.2\\ 2 & 1.2\end{bmatrix};$$
$$A_2 = \begin{bmatrix}1.5 & 1.7\\ 0 & -3.3\end{bmatrix}, D_2 = \begin{bmatrix}0.3 & 0.2\\ 0.1 & 0.3\end{bmatrix}, B_1 = \begin{bmatrix}0\\ -0.3\end{bmatrix}, B_2 = \begin{bmatrix}1.3\\ 0\end{bmatrix}.$$

Consider the closed-loop switched linear time-delay systems (7) with the measurable state case.
To solve the inequality (22), we take the following parameters: $\gamma = 0.7$, $\tau = 3$, so we have

$$P = \begin{bmatrix}\tilde{P}^{-1} & 0\\ 0 & \tilde{P}^{-1}\end{bmatrix}, \text{ where } \tilde{P} = \begin{bmatrix}0.2227 & -0.1388\\ -0.1388 & 0.2677\end{bmatrix};$$

$S = P^{-1}\begin{bmatrix}\tilde{S}_1 & 0\\ 0 & \tilde{S}_2\end{bmatrix}P^{-1}$, where

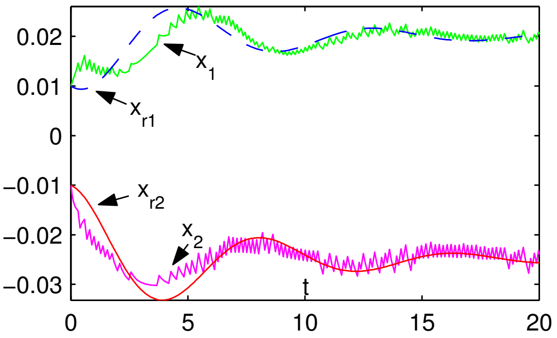

Fig. 1. State tracking with switching control.

$$\tilde{S}_1 = \begin{bmatrix}0.4370 & -0.1028\\ -0.1028 & 6.5786\end{bmatrix}, \tilde{S}_2 = \begin{bmatrix}6.9846 & 0.3230\\ 0.3230 & 0.7246\end{bmatrix};$$
$$X_1 = [-0.1019\ 0.3278], X_2 = [-0.7544\ 0.1639];$$
$$K_1 = [4.0344\ 14.3356], K_2 = [-4.4402\ -1.6902].$$

According to Theorem 1, the switching region are

$$\Omega_i = \{y \in R^{4n} | y^T W_i y < 0\},\ i = 1, 2,$$

where

$$W_1 = \begin{bmatrix}X_{11} & X_{12}\\ * & X_{22}\end{bmatrix}, W_2 = \begin{bmatrix}Y_{11} & Y_{12}\\ * & Y_{22}\end{bmatrix},$$

in which

$$X_{11} = \begin{bmatrix}165.8484 & 169.1334 & -1 & 0\\ * & 189.6761 & 22.8229 & 26.8988\\ * & * & 359.4153 & 236.2133\\ * & * & * & 187.2741\end{bmatrix},$$

$$Y_{11} = \begin{bmatrix}89.3348 & 142.9769 & 44.8530 & 31.9824\\ * & 208.5541 & 0 & -1\\ * & * & 359.4153 & 236.2133\\ * & * & * & 187.2741\end{bmatrix},$$

$$X_{22} = Y_{22} = \begin{bmatrix}-2.2968 & -0.0359 & 0 & 0\\ * & -0.1526 & 0 & 0\\ * & * & -0.1462 & 0.0652\\ * & * & * & -1.4091\end{bmatrix}.$$

$$X_{12} = \begin{bmatrix}2.9731 & 3.9315 & 00\\ 1.1680 & 0.5442 & 00\\ 0 & 0 & 00\\ 0 & 0 & 00\end{bmatrix}, Y_{12} = \begin{bmatrix}2.3342 & 2.3588 & 00\\ 1.5838 & 2.3437 & 00\\ 0 & 0 & 00\\ 0 & 0 & 00\end{bmatrix},$$

Thus, the switching law is designed as follows,

$$\sigma(t) = \begin{cases}1, \text{ if } y \in \Omega_1\\ 2, \text{ if } y \in \Omega_2\backslash\Omega_1\end{cases} \text{when } y = \begin{bmatrix}\tilde{x}(t)\\ \tilde{x}(t-\tau)\end{bmatrix}.$$

With $t_f = 20$, $r(t)$ and $\omega(t)$ are generated by *square* wave form, the simulation result is depicted in Fig.1, it is obvious that neither subsystem 1 nor subsystem 2 tracks the reference system, while the switching control achieves tracking control.
Example 2. Consider the systems (1) and the reference system (2) with

$$A_1 = \begin{bmatrix}-1.5 & -1.2\\ -1.2 & 1\end{bmatrix}, D_1 = \begin{bmatrix}-0.5 & 0.8\\ -0.1 & -0.4\end{bmatrix}, B_1 = \begin{bmatrix}-0.1\\ -0.3\end{bmatrix};$$
$$A_2 = \begin{bmatrix}1.5 & -1\\ -1 & -2.3\end{bmatrix}, D_2 = \begin{bmatrix}-0.3 & -0.2\\ 0.1 & -0.3\end{bmatrix}, B_2 = \begin{bmatrix}-1.3\\ -0.1\end{bmatrix};$$
$$A_r = \begin{bmatrix}-1.5 & -1.2\\ 2 & -0.2\end{bmatrix}, C_1 = [-0.1\ 0.5], C_2 = [1.3\ -0.7].$$

We now consider the unmeasurable state case. First, by Assumption 1, we have the candidate observer gains via arbitrary switching as

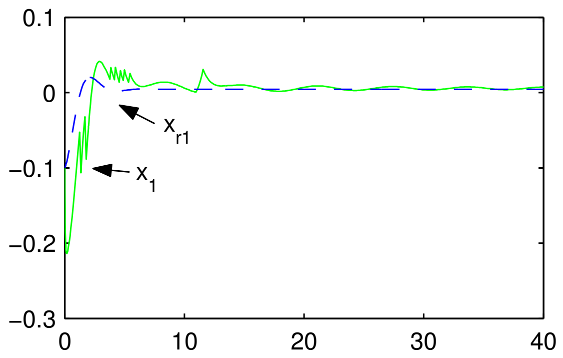

Fig. 2. State x_1 tracking the reference state x_{r1}.

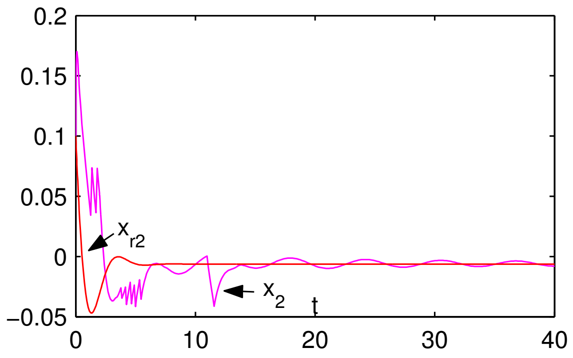

Fig. 3. State x_2 tracking the reference state x_{r2}.

$$L_1 = \begin{bmatrix} -39.3909 \\ 41.7974 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 12.9363 \\ -11.9392 \end{bmatrix}.$$

Consider the closed-loop system (32). We adopt the parameters below: $\gamma = 1$, $\tau = 3$. Solving the inequality (31) by using LMIs, we get

$P_1 = \begin{bmatrix} \tilde{P}_1^{-1} & 0 \\ 0 & \tilde{P}_1^{-1} \end{bmatrix}, P_2 = \begin{bmatrix} \tilde{P}_2^{-1} & 0 \\ 0 & \tilde{P}_2^{-1} \end{bmatrix}$, in which

$\tilde{P}_1 = \begin{bmatrix} 0.5426 & -0.2007 \\ * & 0.4784 \end{bmatrix}, \tilde{P}_2 = \begin{bmatrix} 0.5026 & -0.0527 \\ * & 0.7023 \end{bmatrix};$

$S = \begin{bmatrix} S_1 0 \\ 0 \ \ S_1 \end{bmatrix}$, where $S_1 = \begin{bmatrix} 6.5442 & -0.0466 \\ * & 6.4185 \end{bmatrix}$

$K_1 = [4.0344 \ 14.3356], K_2 = [-4.4402 \ -1.6902]$.

According to theorem 2, the switching control law are given by

$$\sigma(t) = arg \min_{i \in \underline{N}} \{\tilde{x}^T(t) P_i \tilde{x}(t)\}, \ u(t) = K_{\sigma(t)} \hat{e}_r(t).$$

With $t_f = 40$, $r(t)$ and $\omega(t)$ are generated by *sine* wave form, the simulation results are given in Fig.2-Fig.3. Due to the complicated design of the switching control, for example, the imposed assumptions restrict the simulation conditions, the result can not compare beauty with the measurable state case, see in Fig.2 and Fig.3.

5. CONCLUSION

In this paper, tracking control for switched linear systems with time-delay is investigated. When the state is measurable, we use single Lyapunov function technique to design a tracking control law such that the H_∞ model reference tracking performance is satisfied, and when the state is not available, the observer-based tracking control laws and multiple Lyapunov functions techniques are utilized for the stability analysis and control synthesis. The controller design problem can be solved efficiently by using LMIs and convex optimization techniques.

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