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Production Control and Steady-State Performance Analysis for A Two-stage Manufacturing System with Finite Buffer Sizes

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Abstract: A two-station tandem manufacturing system with limited buffer sizes and production capacity is considered. The problem is to examine the stability of the system and to control the service rates to meet an exogenous Poisson demand. A sufficient and necessary condition for the system stability is provided. It is shown that the optimal control policy for this finite buffer capacity problem has the similar structural properties to that of the infinite buffer capacity problem. Three threshold-type control policies are presented and their stability conditions are shown to be the same. The stationary distributions under two threshold policies are obtained, which can then be used to compute steady-state performance measures and find the optimal threshold values. Numerical examples are given to demonstrate the results.

1. INTRODUCTION

Production control in discrete manufacturing systems with uncertain customer demands and processing times has attracted much attention. Research has been carried out in the following two aspects: (i) finding the optimal control policy and its structural properties; (ii) evaluating the performance of specific control mechanisms. In the first aspect, a base-stock policy was shown to be optimal for a single stage system (Gavish and Graves 1980, Li 1992). Switching structure of the optimal policy in a two-station tandem system was addressed in Veatch and Wein (1994). Monotonic and asymptotic characteristics of the optimal feedback control policy in a stochastic serial production line with failure-prone machines were established in Song and Sun (1998, 1999). The structural properties are useful to construct simple sub-optimal policies, particularly in situations where the optimal policies are too complicated.

In the second aspect, extensive studies have been performed. Well-known specific control mechanisms include kanban control (Mitra and Mitrani 1991), base-stock control (Lee and Zipkin 1992), CONWIP control (Spearman et al 1990), buffer control (Conway et al 1988), extended kanban control (Dallery and Liberopoulos 2000), control point policy (Gershwin 2000, Véricourt & Gershwin 2004). Comparative research for some of the above control mechanisms was reported in (Veatch and Wein 1994, Karaesmen and Dallery 2000, Bollon et al 2004). These specific control mechanisms are attractive because they are relatively easy to implement and perform very well in certain settings.

This paper analyses a two-workstation tandem manufacturing system with Poisson demand arrivals and exponential service times. However, unlike the most research in the above literature, we consider the finite buffer spaces for storing work-in-progress (WIP) and finished goods (FG). Once the WIP/FG buffer is full, the first/second workstation will be blocked. Exogenous demand that cannot be met from the FG inventory is backordered and met by the next available finished item. Because the system has limited production capacity and limited buffer space, a more fundamental question than optimality of a policy is the stability, e.g. does a given policy allow the system to meet demands or become increasingly backordered? Another interesting issue is to find the stationary distribution of the inventories and backlogs under a stable control policy. Our contributions include: (i) a sufficient and necessary condition for the stability of the system is provided; (ii) it is shown that the optimal policy has the similar structural properties to that of infinite buffer capacity problem, i.e. characterized by two monotonic switching curves. However, in our situations the asymptotic convergence of the switching curves is guaranteed; (iii) three threshold control policies are presented and their stability conditions are established; (iv) the stationary distribution under two threshold policies are derived. An analytical optimisation procedure is presented to determine the optimal threshold values.

2. PROBLEM FORMULATION

Consider a manufacturing system consisting of two tandem workstations (Fig. 1). Production time at workstation (WS) i follows an exponential distribution with rate λ_i, which is controllable in [0, r_i]. The system has finite buffer sizes with the WIP buffer size M and the FG buffer size N. Here the WIP buffer size includes one unit at WS two, e.g. $M \geq 1$. The objective is to control the production rates in order to meet an exogenous Poisson demand process with rate d as close as possible. The unmet demands are backordered.

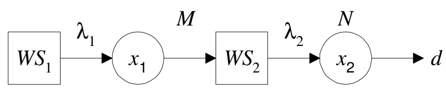

Fig. 1. A two-station tandem manufacturing system

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Let $x_1(t)$ be the number of WIPs and $x_2(t)$ be the inventory-on-hand of FGs. When $x_2(t)$ is negative, it represents the backordered demands. We have $0 \le x_1(t) \le M$ and $x_2(t) \le N$. Define $\mathbf{x}(t) := (x_1(t), x_2(t))$. The system state space $X = \{(x_1, x_2) \mid 0 \le x_1 \le M \text{ and } x_2 \le N\}$.

Let $\Omega = \{u(t) = (\lambda_1(\mathbf{x}(t)), \lambda_2(\mathbf{x}(t)) \mid 0 \le \lambda_1(\mathbf{x}(t)) \le r_1;\ 0 \le \lambda_2(\mathbf{x}(t)) \le r_2;\ \lambda_1(\mathbf{x}(t))=0 \text{ if } x_1(t) = M;\ \lambda_2(\mathbf{x}(t)) = 0 \text{ if } x_1(t) = 0 \text{ or } x_2(t) = N)$ be the set of state-feedback admissible controls. The state transition map of the induced Markov chain under a control policy $u \in \Omega$ is shown in Fig. 2.

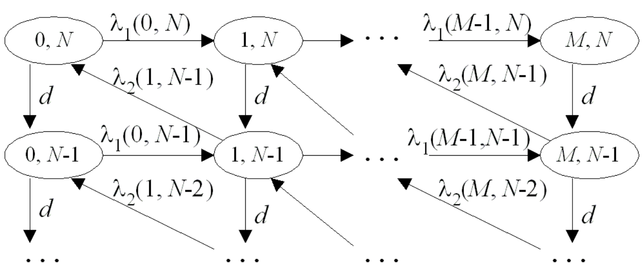

Fig. 2. State transition map

The problem is to find an optimal control policy $u \in \Omega$ to minimise the expected long-run average cost, i.e.

$$J(u) = \lim_{T\to\infty} \inf \frac{1}{T} E_u \int_0^T [g(\mathbf{x}(t))]dt \tag{1}$$

where $g(\mathbf{x}(t))$ is a cost function consisting of WIP holding costs, FG holding costs, and demand backlog costs.

3. SYSTEM STABILITY CONDITION

It is fundamental to determine whether a given policy allow the system to meet demands without increasingly backordered. This is about the system stability issue.

Definition 1. A system is stable if there exists a stable admissible control policy $u \in \Omega$, under which the induced Markov chain has steady-state probability distribution.

Clearly, the maximum production capacity can be achieved by allowing two workstations to produce whenever possible, namely, let $\lambda_1(\mathbf{x}) = r_1$ if $x_1 < M$ and $\lambda_2(\mathbf{x}) = r_2$ if $x_1 > 0$ and $x_2 < N$. Such policy is determined by two buffer sizes M and N, denoted by $u_{M,N}$. We will provide a sufficient and necessary condition to ensure that the induced Markov chain under $u_{M,N}$ is positive recurrent, and therefore has a unique stationary distribution.

Sequence the system state as follows: $(0, N)$, $(1, N)$, …, (M, N), $(0, N\text{-}1)$, $(1, N\text{-}1)$, …, $(M, N\text{-}1)$, …, in which the WIP state (i.e. x_1) is treated as different phases from 0 to M and x_2 is treated as different stages from N to $-\infty$. This is a quasi-birth-death (QBD) Markov chain. Using the matrix analytic method (Latouche and Ramaswami 1999), the following result can be derived (Song 2006).

Proposition 1. The sufficient and necessary condition for the stability of the system is: $\rho(M, r_1, r_2) < 1$, where $\rho(M, r_1, r_2) = d\cdot(1+\rho+\ldots+\rho^M) / (r_1\cdot(1+\rho+\ldots+\rho^{M-1}))$ and $\rho=r_1/r_2$.

The following insights can be obtained: (i) The stability condition is related to the WIP buffer size M, but does not depend on the FG buffer size N. (ii) If the system with the WIP buffer size M is stable, it is also stable with any WIP buffer size that is greater than M. (iii) As M tends to infinity, the system stability condition becomes $d/\min\{r_1, r_2\} < 1$, which is intuitively true by queuing theory.

4. AVERAGE COST OPTIMAL CONTROL POLICY

This section investigates the optimal control policy to minimize the average cost. Define mappings R_1, R_2 and R_3 from X to X as follows:

$R_1\mathbf{x} := (x_1+1, x_2)$ if $x_1 < M$; $R_1\mathbf{x} := \mathbf{x}$, otherwise.

$R_2\mathbf{x} := (x_1\text{-}1, x_2+1)$ if $x_1 > 0$ and $x_2 < N$; $R_2\mathbf{x} := \mathbf{x}$, otherwise.

$R_3\mathbf{x} := (x_1, x_2\text{-}1)$, $\mathbf{x} \in X$.

We uniformise the process by defining the potential event rate $v = r_1 + r_2 + d$. Since it has been shown that the induced Markov chain under the policy $u_{M,N}$ is positive recurrent and covers the whole state space X. The average optimality equation holds and can be expressed by (Sennott 1999):

$$w(\mathbf{x}) + J^* / v = [g(\mathbf{x}) + d\cdot w(R_3\mathbf{x}) + r_1\min\{w(R_1\mathbf{x}), w(\mathbf{x})\} + r_2\min\{w(R_2\mathbf{x}), w(\mathbf{x})\}] / v \tag{2}$$

Where J^* is the optimal average cost and $w(\mathbf{x})$ is a finite function.

Lemma 1. If $g(R_l\mathbf{x}) - g(\mathbf{x}) \ge g(R_jR_l\mathbf{x}) - g(R_j\mathbf{x})$ holds for any $l, j \in \{1, 2, 3\}$ and $l \ne j$, then $w(R_l\mathbf{x}) - w(\mathbf{x}) \ge w(R_jR_l\mathbf{x}) - w(R_j\mathbf{x})$ holds for $\forall\ l \ne j \in \{1, 2, 3\}$.

This can be proved by replacing the average cost problem with a discounted cost one for the same Markov chain, then following the procedures in Song and Sun (1998). However, here we have to consider the additional boundary constraints caused by the finite buffer capacities for WIP and FG. These constraints incur extensive additional checking work in the proof.

Proposition 2. If $g(R_l\mathbf{x}) - g(\mathbf{x}) \ge g(R_jR_l\mathbf{x}) - g(R_j\mathbf{x})$ holds for any $l \ne j \in \{1, 2, 3\}$, then the optimal average cost control policy can be determined by two switching curves $S_1(x_2)$ and $S_2(x_1)$:

(i) $S_1(x_2)$ is non-increasing and $S_2(x_1)$ is non-decreasing.

(ii) $\lim_{x_2\to-\infty} S_1(x_2) = x_{11}^*$, where $x_{11}^* \in (0, M)$.

(iii) The optimal control policy is: $\lambda_1^*(\mathbf{x}) = r_1$ if $\mathbf{x} \in B_1$; $\lambda_1^*(\mathbf{x}) = 0$ otherwise; $\lambda_2^*(\mathbf{x}) = r_2$ if $\mathbf{x} \in B_2$; $\lambda_2^*(\mathbf{x}) = 0$ otherwise; where $B_1 = \{\mathbf{x} \mid 0 \le x_1 \le S_1(x_2) < x_{11}^*, x_2 \le x_2^*\}$ and $B_2 = \{\mathbf{x} \mid x_2 \le S_2(x_1) < N, 0 < x_1 \le x_{12}^*\}$ with $x_2^* < N$, $x_{11}^* < M$, and $x_{12}^* < M$, as shown in Fig. 3.

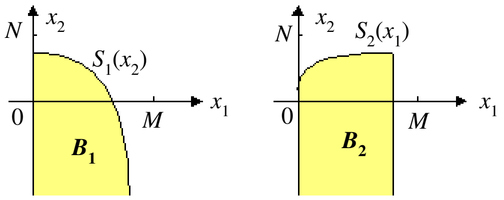

Fig. 3. Optimal control structure

The assertions can be proved using Lemma 1 and following the similar arguments in Song and Sun (1998, 1999). A commonly used cost structure is a linear form, e.g., $g(x_1, x_2) = c_1x_1 + c_2^+\max(0, x_2) + c_2^-\max(0, -x_2)$, where c_1, c_2^+ and c_2^- are non-negative constants. In this case, it is easy to show that the condition in Lemma 1 and Proposition 2 holds.

Proposition 2 indicates that the optimal control policy for the finite buffer capacity situation has the similar structural properties to that for the infinite buffer capacity (Veatch and Wein 1994, Song and Sun, 1998). The key difference is that both B_1 and B_2 in our situations (see Fig. 3) are limited into a finite band in terms of x_1 (i.e. $0\le x_1 < x_{11}^*$ for B_1 and $0< x_1 \le x_{12}^*$ for B_2). Therefore, the asymptotic convergence of the switching curves is guaranteed.

It is difficult to find the closed-form solution to this control problem. However, Proposition 2 provides good insights into the structure of the optimal policy. This can be used to derive the sub-optimal threshold-type control policies analogous to those in the literature (Veatch and Wein 1994, Gershwin 2000, Véricourt and Gershwin 2004). As pointed out by Gershwin (2000), although there are no assurances of optimality of such decentralized policies, experience suggests that these policies have desirable characteristics.

5. THRESHOLD TYPE CONTROL POLICIES AND STABILITY CONDITION

By approximating the control regions B_1 and B_2, simple threshold-type policies can be constructed.

5.1 Buffer Threshold Control

A buffer threshold control is characterised by two parameters m and n, which are used to impose the inventory levels for WIP and FG respectively. More specifically, $BTC_{m,n} := (\lambda_1(\mathbf{x}), \lambda_2(\mathbf{x}))$ for $m\le M$ and $n\le N$ is defined as:

$$\lambda_1(\mathbf{x}) = \begin{cases} r_1 & x_1 < m, \\ 0 & otherwise \end{cases}; \quad \lambda_2(\mathbf{x}) = \begin{cases} r_2 & x_1 > 0,\ x_2 < n \\ 0 & otherwise \end{cases}$$

Corollary 1. The system under a buffer threshold control $BTC_{m,n}$ is stable if and only if $\rho(m, r_1, r_2) < 1$.

5.2 Buffer Basestock Control

The sum of x_1 and x_2 can be defined as the basestock of WS1, which is the difference between the cumulative production of WS1 and the cumulative demands. Combining the buffer threshold parameter with the basestock level yields a buffer basestock threshold policy, $BBC_{m,n}$ for $m\le M$ and $n\le N$:

$$\lambda_1(\mathbf{x}) = \begin{cases} r_1 & x_1 < m,\ x_1 + x_2 < n \\ 0 & otherwise \end{cases}; \quad \lambda_2(\mathbf{x}) = \begin{cases} r_2 & x_1 > 0,\ x_1 + x_2 < n \\ 0 & otherwise \end{cases}$$

Comparing the induced Markov chain under $BBC_{M,N}$ with the induced Markov chain under $BTC_{M,N}$, a one-to-one mapping can be established, which leads to (Song 2006):

Corollary 2. The system under a buffer basestock control $BBC_{m,n}$ is stable if and only if $\rho(m, r_1, r_2) < 1$.

From Corollary 1 and 2, it can be seen that the sufficient and necessary conditions for the stability of the system under $BTC_{m,n}$ and $BBC_{m,n}$ are the same. This may be intuitively explained by the fact that when x_2 is sufficient negative, $x_1+x_2\le n$ is always satisfied and the production of WS1 is actually controlled by the single parameter m, which is the same as $BTC_{m,n}$.

5.3 Multiple Threshold Control

By introducing control parameters to WIP inventory, FG inventory, and basestock WIP+FG, a multiple threshold control $MTC_{m,n,h}$ for $m\le M$ and $n\le N$ can be defined:

$$\lambda_1(\mathbf{x}) = \begin{cases} r_1 & x_1 < m,\ x_1 + x_2 < h \\ 0 & otherwise \end{cases}; \quad \lambda_2(\mathbf{x}) = \begin{cases} r_2 & x_1 > 0,\ x_2 < n,\ x_1 + x_2 < h \\ 0 & otherwise \end{cases}$$

Without loss of the generality, we assume $n\le h\le n+m$. The system state-space under $MTC_{m,n,h}$ is given by $\{(x_1, x_2) \mid 0\le x_1\le m,\ x_2\le n,\ x_1+x_2\le h\}$. It is clear that when $h\ge n$, we have $\{(x_1, x_2) \mid 0\le x_1\le m,\ x_2\le n,\ x_1+x_2\le h\} \supseteq \{(x_1, x_2) \mid 0\le x_1\le m,\ x_1+x_2\le n\}$. In other words, the state-space under $MTC_{m,n,h}$ includes the state-space under $BBC_{m,n}$. This implies that $MTC_{m,n,h}$ allows both workstations to operate with more opportunities than $BBC_{m,n}$. Therefore, the effective production capacity of the system under $MTC_{m,n,h}$ is not less than the effective production capacity under $BBC_{m,n}$. With the similar argument, when $h\le n+m$, the effective production capacity under $MTC_{m,n,h}$ is not greater than the effective production capacity under $BTC_{m,n}$. From Corollaries 1 and 2, both $BTC_{m,n}$ and $BBC_{m,n}$ have the same stability condition, which yields that $MTC_{m,n,h}$ must have the same stability condition.

Corollary 3. The system under a multiple threshold control $MTC_{m,n,h}$ for $n\le h\le n+m$ is stable if and only if $\rho(m, r_1, r_2) < 1$.

To implement these threshold-type policies, an important issue arising is how to determine the optimal threshold parameters. One way to solve this problem is to use the value iteration algorithm to numerically find the optimal average costs and the optimal control parameters. The disadvantage of this approach is that it cannot provide information on other interesting steady-state performance measures such as service level and stock-out probability. We present an alternative approach consisting of three steps: i) deriving the steady-state distribution of the induced Markov chain under a threshold policy; ii) obtaining the explicit forms of interesting performance measures; iii) optimising the threshold parameters. This analysis will only be performed on $BTC_{m,n}$ and $BBC_{m,n}$ because it appears difficult to derive the stationary distribution for the induced Markov chain under $MTC_{m,n,h}$ due to its complex structure.

6. STEADY-STATE PERFORMANCE MEASURES UNDER $BTC_{M,N}$ AND $BBC_{M,N}$

If the system stability condition is satisfied, the induced Markov chains under $BTC_{M,N}$ or $BBC_{M,N}$ are positive recurrent, irreducible and therefore ergodic.

6.1 Stationary distribution under $BTC_{M,N}$

Let $\{p_i^{M,N}(j) \mid i=0, 1, \ldots, M$ and $j=N, N\text{-}1, \ldots\}$ be the stationary distribution under control policy $BTC_{M,N}$. From the state transition map (Fig. 1), it is clear that $p_i^{M,N}(j) \equiv p_i^{M,0}(j\text{-}N)$ for any i and j, where $p_i^{M,0}(j)$ is the stationary distribution under $BTC_{M,0}$. To simplify the narrative, we drop the superscript from $p_i^{M,0}(j)$, i.e. $p_i(j) := p_i^{M,0}(j)$. Their balance equations can be easily established (Song 2006).

To find the stationary distribution of the recurrent QBD process, we use spectral expansion approach (Mitrani and Chakka 1995). Let $Q(x)$ be the associated characteristic matrix polynomial. The eigenvalues of the characteristic equation can be found by solve the scalar polynomial $\det(Q(x)) = 0$, where $\det(Q(x))$ is the determinant of matrix $Q(x)$. The following result can be derived based on Mitrani and Chakka (1995) and Grassmann (2002).

Proposition 3. $\det(Q(x)) = 0$ is a polynomial of order M+2, i.e. $\text{degree}(\det(Q(x))) = M+2$. It has M+2 distinct real roots: $x_0, x_1, \ldots, x_{M+1}$, such that $0 < x_0 < x_1 < \ldots < x_M < x_{M+1} = 1$.

From Proposition 3, a general solution to the balance equations can be expressed as

$p_i(j) = a_{0i}x_0^{-j} + a_{1i}x_1^{-j} + \ldots + a_{Mi}x_M^{-j}$ for $i=0, 1, \ldots, M$; $j\leq 0$. (3)

Where a_{ik} for $i,k = 0, 1, \ldots, M$ are undetermined coefficients. Using the balance equations and the normalisation condition, the M+1 unknown variables $\{a_{00}, a_{10}, \ldots, a_{M0}\}$ can be determined uniquely (Song 2006). This gives the explicit form of the stationary distribution of the induced Markov chain under the buffer threshold control $BTC_{M,0}$.

Proposition 4. The stationary distribution under the buffer threshold policy $BTC_{M,N}$ is given by $p_i^{M,N}(j) \equiv p_i^{M,0}(j\text{-}N) = p_i(j\text{-}N)$ for $i=0, \ldots, M$ and $j \leq N$, where $p_i(.)$ is given in (3).

6.2 Steady-state performance measures under $BTC_{m,n}$

In terms of the steady-state performance, the system under control by $BTC_{m,n}$ is equivalent to a system with WIP buffer size m and FG buffer size n. The stationary distribution $\{p_i^{m,n}(j) \mid i=0, 1, \ldots, m$ and $j\leq n\}$ can be obtained by replacing M and N with m and n respectively in Section 6.1. Due to the ergodicity of the induced Markov chain, the long-run average cost function under the buffer threshold control $BTC_{m,n}$ can be rewritten as

$$J(BTC_{m,n}) = \sum_{j=n}^{-\infty}\sum_{i=0}^{m} p_i^{m,n}(j)g(i,j) = \sum_{j=0}^{-\infty}\sum_{i=0}^{m} p_i^{m,0}(j)g(i,j+n)$$

Apart from the long-run average cost, other steady-state performance measures may also be interesting, e.g. the stock-out probability P_{so} (that may be defined as the fraction of time that there exist unmet demands), the service level (that may be defined as the fraction of time that customer demands can be met immediately from inventory), the average WIP buffer utilisation U_{WIP}, the average FG buffer utilisation U_{FG}, the expected backordering or FG inventory level L_{FG} These performance measures can be easily calculated from the stationary distribution. For example,

$$P_{so} = \sum_{j=-n-1}^{-\infty}\sum_{i=0}^{m} p_i^{m,0}(j); \qquad U_{WIP} = \sum_{j=0}^{-\infty}\sum_{i=0}^{m}(p_i^{m,0}(j)\cdot i)/M;$$

$$U_{FG} = \sum_{j=0}^{-n}\sum_{i=0}^{m}(p_i^{m,0}(j)\cdot(j+n))/N; \quad L_{FG} = \sum_{j=0}^{-\infty}\sum_{i=0}^{m} p_i^{m,0}(j)\cdot(j+n).$$

6.3 Optimal threshold values for $BTC_{m,n}$

This section addresses the optimisation of the threshold parameters and discusses some structural properties of the optimal *BTC*.

Proposition 5. The optimal buffer threshold control u_{m^*,n^*} can be determined by $(m^*, n^*) = \arg\min J(BTC_{m,n})$ subject to $\rho(m, r_1, r_2) < 1$.

However, the two-variable optimisation problem in Proposition 5 may not be trivial. We aim at providing more insights into the structural properties of the optimal *BTC*.

Proposition 6. For a fixed WIP threshold value m, if $g(i, j)$ is convex in j and $g(i, j) - g(i, j\text{-}1) \leq 0$ for any $j\leq 0$, then

(i) The cost function $J(BTC_{m,n})$ is convex in n;

(ii) The optimal threshold value n^* is non-negative;

(iii) The optimal threshold value n^* is the maximum non-negative integer such that

$$\sum_{j=0}^{-\infty}\sum_{i=0}^{m} p_i^{m,0}(j)(g(i,j+n) - g(i,j+n-1)) \leq 0$$

The assumption $g(i, j) - g(i, j\text{-}1) \leq 0$ for any $j\leq 0$ is reasonable. This represents the fact that more backlogs incur higher cost. E.g. a commonly used cost structure, $g(i, j) = c_1 i + c_2^+\max(0, j) + c_2^-\max(0, -j)$, satisfies the conditions in Proposition 6.

Proposition 7. Assuming $g(i, j)$ is convex in j and $g(i, j) - g(i, j\text{-}1) \leq 0$ for any $j\leq 0$. The optimal buffer threshold policy can be determined in two steps:

(i) Step 1: for any fixed m, the stationary distribution $p_i^{m,0}(j)$ is calculated by Proposition 4 and the optimal $n^*(m)$ is obtained from Proposition 6;

(ii) Step 2: the optimal WIP inventory level m^* is determined by:

$$m^* = \arg\min_m\{J(BTC_{m,n^*(m)}) \mid 0 \leq m \leq M,\ \rho(m, r_1, r_2) < 1\}.$$

In Proposition 7, the determination of $n^*(m)$ is relatively simple since the stationary distribution $\{p_i^{m,0}(j)\}$ is independent of n. However, the determination of m^* is much more involving since the stationary distribution must be recalculated for different m. A straightforward approach is to start with the minimum m that satisfies the stability condition $\rho(m, r_1, r_2)<1$, then calculate the cost until it is no longer decreasing in m.

6.4 Steady-state performance measures and optimal threshold values for $BBC_{m,n}$

Now consider the buffer basestock control policy. Let $\{q_i^{m,n}(j) \mid i=0, 1, \ldots, m \text{ and } i+j\leq n\}$ be the stationary distribution of the induced Markov chain under $BBC_{m,n}$. Note that this Markov chain can be transformed into the exact same Markov chain under $BTC_{m,n}$ by the state-mapping $\varphi(x_1, x_2) := (M-x_1, x_2+x_1)$ and swapping the transition rates r_1 and r_2. Therefore, we can establish a relationship between two stationary distributions for $BTC_{m,n}$ and $BBC_{m,n}$.

To reflect the dependence of the stationary distribution on transition rates r_1 and r_2, we rename the symbols $p_i^{m,n}(j)$ to be $p_i^{m,n}(j, r_1, r_2)$ and $q_i^{m,n}(j)$ to be $q_i^{m,n}(j, r_1, r_2)$.

Proposition 8. The stationary distribution under $BBC_{m,n}$ is given by $q_i^{m,n}(j, r_1, r_2) = p_{m-i}^{m,n}(i+j, r_2, r_1)$ for i=0, 1, …, m and $i+j\leq n$, where $p_{m-i}^{m,n}(i+j, r_2, r_1)$ is the stationary distribution under $BTC_{m,n}$, in which the production rate of the first WS is r_2 and the production rate of the second WS is r_1.

Since we have obtained the stationary distribution, it is straightforward to compute the stead-state performance measures under $BBC_{m,n}$. Similar results to Propositions 6 and 7 can also be established.

7. NUMERICAL EXAMPLES

We first briefly examine the sensitivity of the system stability to the buffer size, then compare the performance of the threshold policies and the optimal policy.

Assume that both WSs have the same production rate 0.8. Let demand take three levels: 0.4, 0.6 and 0.7. For different buffer size M, the stability index $\rho(M, r_1, r_2)$ can be calculated easily. It is found that the system is stable when d=0.4 with $M\geq 2$, d=0.6 with $M\geq 4$, and d=0.7 with $M\geq 8$. The stability index is much more sensitive to the buffer size when the actual buffer size is smaller. Moreover, as the demand rate is increasingly approaching to the production rate, the system tends to be less stable and therefore bigger buffer sizes are required.

Now, take the cost function $g(x_1, x_2) = c_1x_1 + c_2^+\max\{0, x_2\} + c_2^-\max\{0, -x_2\}$, where c_1=1, c_2^+=2 and c_2^-=10. Consider a case with balanced WSs: r_1=0.8, r_2=0.8, d=0.4. Different combinations of the buffer sizes M and N will be tested. Both the dynamic programming value iteration algorithm and the analytical optimisation method (in Section 6) are performed. In the value iteration algorithm we limit the state space into a finite region with $0\leq x_1\leq M$ and $-100\leq x_2\leq N$ and take the maximum iteration number to be 8000. The value iteration algorithm can numerically compute the optimal long-run average costs under the optimal policy, and the optimal threshold parameters and costs under threshold policies such as $BTC_{m,n}$, $BBC_{m,n}$ and $MTC_{m,n,h}$. The analytical optimisation method can yield the various steady-state performance measures, and the optimal threshold parameters and costs for $BTC_{m,n}$ and $BBC_{m,n}$.

For each scenario, the system stability condition is checked first. If it is stable, the above two methods are applied. It was found that two methods produce the same optimal threshold values $m^*$and n^* for $BTC_{m,n}$ and $BBC_{m,n}$ respectively in all scenarios. They also produce the same costs for $BTC_{m,n}$ and $BBC_{m,n}$ respectively in all scenarios. This verifies the results given in Section 6. Table 1 gives the results.

Table 1. Performance measures and optimal threshold values

(M, N)	(1,2)	(2,2)	(2,10)	(5,10)	(10,10)
Stability	No	Yes	Yes	Yes	Yes
$J(BTC)$	-	13.05	10.85	8.90	8.90
(m^*, n^*)	-	(2, 2)	(2, 4)	(3, 3)	(3, 3)
P_{so}	-	0.3057	0.1463	0.1154	0.1154
U_{WIP}	-	0.6500	0.6500	0.4349	0.2175
U_{FG}	-	0.4481	0.2379	0.1910	0.1910
L_{FG}	-	-0.1000	1.9000	1.6202	1.6202
$J(BBC)$	-	16.00	10.84	8.05	8.00
(m^*, n^*)	-	(2, 2)	(2, 5)	(5, 4)	(7, 4)
P_{so}	-	0.4233	0.1455	0.1126	0.1068
U_{WIP}	-	0.3500	0.3500	0.1891	0.0962
U_{FG}	-	0.3042	0.2678	0.2238	0.2289
L_{FG}	-	-0.8000	2.2000	1.9761	2.0438
$J(MTC)$	-	13.05	10.80	7.86	7.81
(m^*,n^*,h^*)	-	(2,2,4)	(2,4,5)	(5,3,4)	(7,3,4)
J^*	-	13.05	10.80	7.86	7.80

In Table 1, the first block includes the stability, optimal cost under $BTC_{m,n}$, optimal threshold values m^* and n^*, various steady-state performance measures (e.g. stock-out probability P_{so}, the WIP buffer utilisation U_{WIP}, the FG buffer utilisation U_{FG}, and the average backlog or FG inventory level L_{FG}) under the optimal buffer threshold control. The second block includes the results for $BBC_{m,n}$. The third block includes the optimal threshold parameters and costs for $MTC_{m,n,h}$, and the optimal average cost J^* under the optimal policy.

When M=1, we have ρ=1, which indicates the system is unstable. For all other scenarios, the system is stable. The following points can be observed and interpreted:

When buffer sizes are very small (e.g. ≤2), the optimal threshold values tend to be the buffer sizes M and N. The performance of the BTC (i.e. $J(BTC)$) is close to the optimal cost J^*.

As M and N increase, the costs $J(BTC)$, $J(BBC)$, $J(MTC)$ and J^* are decreasing and converging to finite numbers, which are essentially approaching to the case with infinite buffer capacity. For *BTC* policy, as both buffer sizes increase, the optimal threshold values $m^*$and n^* tend to be constants (e.g. $m^*\to 3$ and $n^* \to 3$). For *BBC* policy, the optimal value $n^* \to 4$; whereas m^* is increasing as M increases. Similar properties can be found for the *MTC* policy. These observations reveal that sufficiently large WIP buffer sizes do not affect the optimal *BTC*, but may affect the optimal *BBC* and *MTC*.

In terms of the long-run average cost, *MTC* achieves almost the same performance as the optimal policy in those scenarios. Comparing *BTC* with *BBC*, it appears that *BTC* is better than *BBC* when *M* is very small, while *BBC* is better than *BTC* when *M* becomes large.

The stock-out probability (P_{so}) is much more sensitive to the buffer sizes when they are smaller, particularly to the FG buffer size *N*. This is in agreement with intuition that very small FG buffer size has low ability to meet demands immediately. On the whole, the buffer utilisation (i.e. U_{WIP} and U_{FG}) is decreasing as buffer sizes increase.

8. CONCLUSIONS

This paper considers the optimal production control in a two-station manufacturing system with random service times and demand arrivals. Limited production capacity and limited buffer spaces give rise to not only the production control problem but also the system stability problem. A sufficient and necessary condition for the system stability is provided. It is shown that the optimal production policy has the similar structural properties to those with infinite buffer capacities. The key difference is that the asymptotic convergence of the switching curves is guaranteed for the systems with finite buffer capacity.

A few threshold-type control policies, e.g. *BTC*, *BBC*, *MTC*, are presented. It is shown that the stability condition of the system under these threshold policies is the same. The stationary distributions under *BTC* and *BBC* are obtained, which are used to derive the explicit forms of stead-state performance measures. The optimal threshold values can then be found by an analytical optimisation procedure based on the analysis of the explicit performance measures.

Numerical examples using the value iteration algorithm and the analytical optimisation method confirm the results. It is found that the system stability does not depend on the FG buffer size and is more sensitive to the WIP buffer size when it is small. It also reveals that the optimal MTC achieves almost the same performance as the optimal policy. Comparing *BTC* with *BBC*, it appears that *BTC* is better than *BBC* when *M* is very small, while *BBC* is better than *BTC* when *M* becomes large. It is also observed that the stock-out probability is much more sensitive to the buffer sizes when they are smaller, particularly to the FG buffer size *N*.

Further work includes extending the model to more complicated systems with multistage and multiple part types.

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