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Swing-up Control Based on Virtually Composite Links for an n-Link Under-actuated Robot with Passive First Joint ⋆

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Abstract: This paper concerns a swing-up control problem for an n-link revolute robot in a vertical plane with its first joint being passive and the rest being active. The objectives of this paper are: 1) to design a controller under which the robot can be brought into any arbitrarily small neighborhood of the upright equilibrium point, where all links of the robot remain in their upright positions; 2) to attain a global analysis of the motion of the robot under the controller. To achieve the above challenging objectives, first, this paper addresses how to devise iteratively a series of virtually composite links for designing a coordinate transformation on the angles of all active joints. Second, this paper constructs a novel Lyapunov function based on the transformation, and proposes an energy based swing-up controller. Third, this paper carries out a global analysis of the motion of the robot under the controller, and establishes some conditions on control parameters for achieving the swing-up control objective. This paper provides insight into the energy or passivity based control to underactuated multi-degree-of-freedom systems.

1. INTRODUCTION

Recent years, many researchers have studied underactuated mechanical systems, which possess fewer actuators than degrees of freedom. Such mechanisms arise in several ways, e.g., the intentional designs as the pendulum type robots in Hauser and Murray [1990], Spong and Block [1995]; rigid robots with elastic joints/flexible links in De Luca et al. [2001]; robots with actuator failure in Arai et al. [1998]. Since these systems usually have nonholonomic second-order constraints, their control problems are challenging, see Grizzle et al. [2005], Ortega et al. [2002].

There are many researches on 2-DOF (degree of freedom) pendulum type robots, e.g., Åström and Furuta [2000], Hauser and Murray [1990], Spong [1995]. The energy based control approach, which has been developed in the seminal works of Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002], Spong [1996], has been shown theoretically effective for solving the swing-up control problem for the Pendubot in Fantoni et al. [2000], Kolesnichenko and Shiriaev [2002] and the Acrobot in Xin and Kaneda [2007a].

Note that the complete analysis of the swing-up control for a 3-link planar robot in a vertical plane with its first joint being passive and the rest being active, has not been reported in Spong [1996]. A swing-up control problem was studied in Xin and Kaneda [2007b] for this 3-link planar robot. Different from the 2-DOF case, it is shown in the above paper that it is difficult to analyze the motion of the 3-link robot under the swing-up controller designed by using the Lyapunov function which contains the energy of the robot, the angles and angular velocities of two active joints. To overcome this difficulty, Xin and Kaneda [2007b] treated the links 2 and 3 as a virtually composite link and proposed a coordinate transformation of the angles of two active joints. Moreover, based on the transformation, a new Lyapunov function was constructed for designing an energy based swing-up controller and for achieving a global analysis of the motion of the 3-link robot under the controller.

In this paper, we investigate how to extend the design and analysis for the 3-link robot in Xin and Kaneda [2007b] to a general n-link planar robot with passive first joint. The objectives of this paper are: 1) to design a controller under which the robot can be brought to any arbitrarily small neighborhood of the upright equilibrium point, where all links remain in their upright positions; 2) to attain a global analysis of the motion of the robot under the controller.

For the n-link robot, first, we address how to devise a series of virtually composite links in an iterative way for designing a coordinate transformation on the angles of all active joints, which is a main contribution of this paper. Second, we construct a Lyapunov function based on the transformation, and propose a swing-up controller. Third, we carry out a global analysis of the motion of the robot under the controller. Indeed, by using the devised virtually composite links, we succeed in revealing the relationship between the closed-loop equilibrium points and a control parameter, and we establish some conditions on control

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parameters for achieving the swing-up control objective. This is another main contribution of this paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Model of an n-Link Underactuated Robot

Consider an n-link revolute robot with passive first joint shown in Fig. 1, where for the ith $(i = 1, \ldots, n)$ link, m_i is its mass, l_i is its length, l_{ci} is the distance from joint i to its center of mass (COM), and I_i is the inertia moment around its COM.

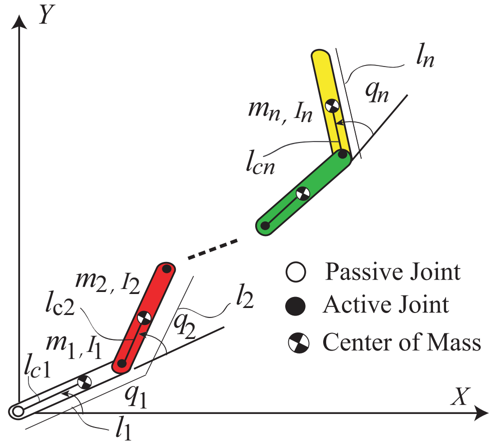

Fig. 1. An n-link underactuated planar robot.

Partition the generalized coordinate vector q as $q = \left[\, q_1 \; q_a^{\mathrm{T}} \,\right]^{\mathrm{T}}$, with $q_a = \left[\, q_2 \; \ldots \; q_n \,\right]^{\mathrm{T}}$. The motion equations of the robot are:

$$M_{11}\ddot{q}_1 + M_{1a}\ddot{q}_a + H_1 + G_1 = 0, \tag{1}$$

$$M_{a1}\ddot{q}_1 + M_{aa}\ddot{q}_a + H_a + G_a = \tau, \tag{2}$$

where

$$M(q_a) = \left[\begin{array}{c:c} M_{11} & M_{1a} \\ \hdashline M_{a1} & M_{aa} \end{array}\right] \tag{3}$$

is a symmetric positive definite inertia matrix containing only q_a; $H_1 \in \mathbb{R}$ and $H_a \in \mathbb{R}^{n-1}$ contain Coriolis and centrifugal terms, $G_1 \in \mathbb{R}$ and $G_a \in \mathbb{R}^{n-1}$ contain gravitational terms, and $\tau = \left[\, \tau_2 \; \ldots \; \tau_n \,\right]^{\mathrm{T}} \in \mathbb{R}^{n-1}$ is the input torque vector produced by $n-1$ actuators at active joints 2, …, n.

The energy of the robot is expressed as

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^{\mathrm{T}} M(q_a)\dot{q} + P(q), \tag{4}$$

where $P(q)$ is the potential energy and is defined as

$$P(q) = \sum_{i=1}^{n} m_i g Y_{Gi} = \sum_{i=1}^{n} \beta_i \sin \sum_{j=1}^{i} q_j, \tag{5}$$

where g is the acceleration of the gravity, Y_{Gi} is the Y-axis coordinate of the COM of link i, and

$$\beta_i := m_i l_{ci} g + \Big(\sum_{j=i+1}^{n} m_j\Big) l_i g, \quad \text{for } i = 1, \ldots, n. \tag{6}$$

Note that $P(q)$, $G_1(q)$, and $G_a(q)$ satisfy

$$G_1(q) = \frac{\partial P}{\partial q_1}, \quad G_a(q) = \frac{\partial P}{\partial q_a}. \tag{7}$$

2.2 Problem Formulation

Consider the following upright equilibrium point:

$$q_1 = \frac{\pi}{2} \ (\mathrm{mod}\ 2\pi), \ q_a = 0, \ \dot{q} = 0, \tag{8}$$

where the equality of passive joint angle q_1 holds modulo 2π, that is, q_1 is treated in S^1, which denotes a unit circle; and active joint angle vector q_a is treated in $\mathbb{R}^{n-1}$.

For $E(q, \dot{q})$, $\dot{q}_a$, and q_a, if we can design τ such that

$$\lim_{t\to\infty} E(q, \dot{q}) = E_r, \quad \lim_{t\to\infty} \dot{q}_a = 0, \quad \lim_{t\to\infty} q_a = 0, \tag{9}$$

where $E_r := \sum_{i=1}^{n} \beta_i$ is the energy of the robot at the upright equilibrium point, then from the analysis of Case 1 in Section 5, we know that the robot can be swung up to any arbitrarily small neighborhood of the upright equilibrium point.

A Lyapunov function candidate for designing such τ is

$$V_A = \frac{1}{2}(E - E_r)^2 + \frac{1}{2}k_D \dot{q}_a^{\mathrm{T}}\dot{q}_a + \frac{1}{2}k_P q_a^{\mathrm{T}} q_a, \tag{10}$$

where scalars $k_D > 0$ and $k_P > 0$ are control parameters. However, similar to the discussion in Xin and Kaneda [2007b], we find that it is difficult to complete the motion analysis of the robot under the controller designed via such V_A. Thus, to devise a Lyapunov function for designing τ and fulfilling the motion analysis for the robot, we shall propose a coordinate transformation on q_a.

3. VIRTUALLY COMPOSITE LINKS AND COORDINATE TRANSFORMATION

For links 2 to n of the robot shown in Fig. 1, we devise $n-1$ virtually composite links (VCLs) described as follows: for $i = 2, \cdots, n$, we consider links i to n of the robot as VCL i, which starts from joint i, and whose COM is the same as the joint COM of links i to n, see Fig. 2. Although the definition of VCL n as link n of the robot is a little abuse of words of virtually composite link, under such a definition, we can construct VCLs iteratively. Indeed, for $i = 2, \cdots, n-1$, we can see that **VCL i is a composite link of link i and VCL $i+1$**, see Fig. 3. This facilitates expressing the results in this paper in a concise and unified way.

For VCL i shown in Fig. 3, we define:

$\overline{l}_{ci}$: the distance between joint i and the COM of VCL i;
$\overline{q}_i$: the angle of VCL i with respect to link $i-1$;
θ_{i+1}: the angle of VCL i with respect to link i.

If $\overline{l}_{ci} = 0$, that is, VCL i shrinks to a point at joint i, then neither $\overline{q}_i$ nor θ_{i+1} can be well defined. To avoid occurrence of such case, we make the following assumption, which is discussed further at the end of this section.

Assumption 1: $\overline{l}_{ci} > 0$ holds for all $\left[\, q_{i+1} \; \ldots \; q_n \,\right] \in \mathbb{R}^{n-i}$.

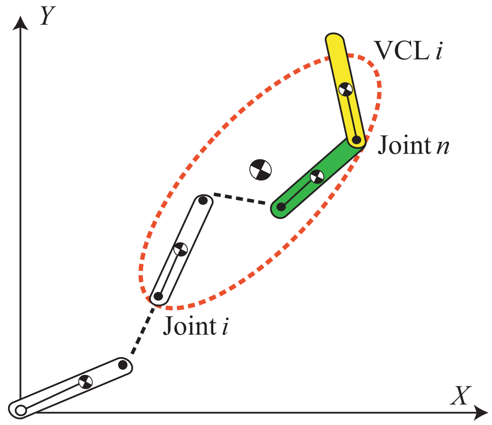

Fig. 2. Links i to n and a VCL i.

Consider the following coordinate transformation on q_a:

$$\overline{q}_a := [\,\overline{q}_2\ \dots\ \overline{q}_n\,]^{\mathrm{T}}. \tag{11}$$

We shall derive $\overline{q}_a$ in terms of q_a. For $2 \le i \le n$, since q_i is the angle of link i with respect to link $i-1$, from the definitions of $\overline{q}_i$ and θ_{i+1}, we obtain

$$\overline{q}_i = q_i + \theta_{i+1}, \quad \text{for } 2 \le i \le n, \text{ with } \theta_{n+1} = 0. \tag{12}$$

We express θ_{i+1} in what follows by using the fact that VCL i is a composite link of link i and VCL $i+1$. First, we use a Cartesian coordinate system (x_i, y_i) with its origin at joint i and its x-axis lying on link i, see Fig. 3. Since in these coordinates the coordinates of COMs of link i and VCL $i+1$ are $(l_{ci}, 0)$ and $(l_i + \overline{l}_{c(i+1)} \cos \overline{q}_{i+1}, \overline{l}_{c(i+1)} \sin \overline{q}_{i+1})$, respectively, letting (x_{ci}, y_{ci}) be the coordinates of the COM of VCL i, we obtain

$$(x_{ci}, y_{ci}) = \frac{(\beta_i + \overline{\beta}_{i+1} \cos \overline{q}_{i+1}, \overline{\beta}_{i+1} \sin \overline{q}_{i+1})}{\overline{m}_i g}, \tag{13}$$

where

$$\overline{\beta}_i := \overline{m}_i \overline{l}_{ci} g, \tag{14}$$

where $\overline{m}_i := \sum_{j=i}^{n} m_j$ is the mass of VCL i.

Next, from (13), we obtain

$$\begin{cases} \sin \theta_{i+1} = \dfrac{y_{ci}}{\overline{l}_{ci}} = \dfrac{\overline{\beta}_{i+1} \sin \overline{q}_{i+1}}{\overline{\beta}_i}, \\[2ex] \cos \theta_{i+1} = \dfrac{x_{ci}}{\overline{l}_{ci}} = \dfrac{\beta_i + \overline{\beta}_{i+1} \cos \overline{q}_{i+1}}{\overline{\beta}_i}, \end{cases} \tag{15}$$

which yields

$$\dot{\theta}_{i+1} = w_{i+1} \dot{\overline{q}}_{i+1} + v_{i+1} \dot{\overline{\beta}}_{i+1}, \quad \text{for } 2 \le i \le n-1, \tag{16}$$

where

$$w_{i+1} := \frac{\overline{\beta}_{i+1}(\overline{\beta}_{i+1} + \beta_i \cos \overline{q}_{i+1})}{\overline{\beta}_i^2}, \quad v_{i+1} := \frac{\beta_i \sin \overline{q}_{i+1}}{\overline{\beta}_i^2}. \tag{17}$$

Next, to treat $\dot{\overline{\beta}}_{i+1}$, we derive the following iterative relation between $\overline{\beta}_i$ and $\overline{\beta}_{i+1}$ from (14) and $\overline{l}_{ci} = \sqrt{x_{ci}^2 + y_{ci}^2}$:

$$\overline{\beta}_i = h(\beta_i, \overline{\beta}_{i+1}, \overline{q}_{i+1}), \text{ for } 2 \le i \le n-1, \text{ with } \overline{\beta}_n = \beta_n, \tag{18}$$

where

$$h(a, b, z) := \sqrt{a^2 + b^2 + 2ab \cos z}. \tag{19}$$

Thus, we obtain

$$\dot{\overline{\beta}}_i = p_{i+1} \dot{\overline{q}}_{i+1} + f_{i+1} \dot{\overline{\beta}}_{i+1}, \tag{20}$$

where

$$p_{i+1} := -\frac{\beta_i \overline{\beta}_{i+1} \sin \overline{q}_{i+1}}{\overline{\beta}_i}, \quad f_{i+1} := \frac{\overline{\beta}_{i+1} + \beta_i \cos \overline{q}_{i+1}}{\overline{\beta}_i}. \tag{21}$$

Using (20) and $\dot{\overline{\beta}}_n = 0$, we can express $\dot{\theta}_{i+1}$ in (16) as

$$\dot{\theta}_{i+1} = -\sum_{j=i}^{n-1} \phi_{i(j+1)} \dot{\overline{q}}_{j+1}, \quad \text{for } 2 \le i \le n-1, \tag{22}$$

where $\phi_{i(j+1)}$ is a function of $\overline{q}_a$ and its expression is omitted due to page limitations.

When $\overline{q}_{i+1} = 0$, that is, link i and VCL $i+1$ are stretched straight out, it is reasonable to define

$$\theta_{i+1} = 0, \quad \text{when } \overline{q}_{i+1} = 0, \quad \text{for } 2 \le i \le n-1. \tag{23}$$

Thus, θ_{i+1} can be determined by (22) and (23).

To summarize, as to $\overline{q}_a$ and q_a, owing to (12) and (23),

$$\overline{q}_a = 0 \Longleftrightarrow q_a = 0 \tag{24}$$

holds. As to $\dot{\overline{q}}_a$ and $\dot{q}_a$, from (12) and (22), we have

$$\dot{q}_a = \Phi(\overline{q}_a) \dot{\overline{q}}_a, \tag{25}$$

where $\Phi(\overline{q}_a)$ is the following *upper triangular* matrix:

$$\Phi(\overline{q}_a) := \begin{bmatrix} 1 & \phi_{23} & \cdots & \phi_{2i} & \cdots & \phi_{2n} \\ 0 & 1 & \cdots & \phi_{3i} & \cdots & \phi_{3n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & \phi_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}. \tag{26}$$

Finally, as to Assumption 1, from (14) and (18), we know that $\overline{l}_{ci} = 0$ holds if and only if $\beta_i = \overline{\beta}_{i+1}$ and $\cos \overline{q}_{i+1} = -1$. Thus, $\overline{l}_{ci} \ne 0$ holds for all $q_a \in \mathbb{R}^{n-1}$ if

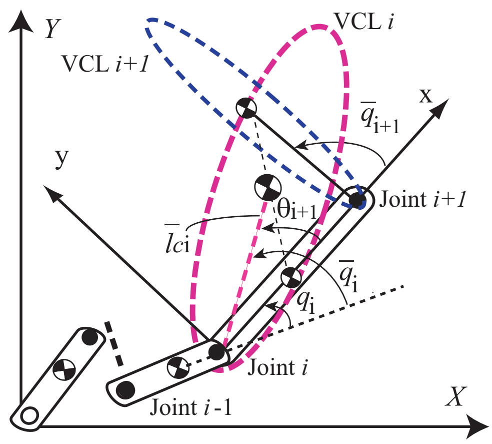

Fig. 3. Links $i-1$ and i; VCLs i and $i+1$.

$$\beta_{n-1} \neq \beta_n, \;\; \beta_k > \sum_{j=k+1}^{n} \beta_j, \quad \text{for } 2 \leq k \leq n-2. \tag{27}$$

Note that a 4-link robot with $m_j = m$, $l_{cj} = l_j/2 = l/2$ for $1 \leq j \leq 4$ is an example of the robots satisfying (27).

4. SWING-UP CONTROLLER FOR THE ROBOT

We take the following Lyapunov function candidate:

$$V = \frac{1}{2}(E - E_r)^2 + \frac{1}{2}k_D \dot{q}_a^{\mathrm{T}} \dot{q}_a + \frac{1}{2}k_P \overline{q}_a^{\mathrm{T}} \overline{q}_a. \tag{28}$$

Instead of q_a in V_A, we use $\overline{q}_a$ in V. Obviously, from (24), we know that $\lim_{t\to\infty} V = 0$ is equivalent to (9).

Taking the time-derivative of V along the trajectories of (1) and (2), and using $\dot{E} = \dot{q}_a^{\mathrm{T}}\tau$ and using $\dot{\overline{q}}_a = \Phi^{-1}\dot{q}_a$ owing to (25), we obtain

$$\dot{V} = \dot{q}_a^{\mathrm{T}}\Big((E - E_r)\tau + k_D \ddot{q}_a + k_P \Phi^{-\mathrm{T}}\overline{q}_a\Big).$$

where $\Phi^{-\mathrm{T}}$ denotes $(\Phi^{-1})^{\mathrm{T}}$. If we can choose τ such that

$$(E - E_r)\tau + k_D \ddot{q}_a + k_P \Phi^{-\mathrm{T}}\overline{q}_a = -k_V \dot{q}_a \tag{29}$$

holds for some constant $k_V > 0$, then we have

$$\dot{V} = -k_V \dot{q}_a^{\mathrm{T}} \dot{q}_a \leq 0. \tag{30}$$

We discuss under what condition (29) *is solvable with respect to τ for any $(q, \dot{q})$*. From (1) and (2), we obtain

$$\widehat{M}_{aa}\ddot{q}_a = \tau - \widehat{H}_a - \widehat{G}_a, \tag{31}$$

where $\widehat{H}_a := H_a - M_{a1}M_{11}^{-1}H_1$, $\widehat{G}_a := G_a - M_{a1}M_{11}^{-1}G_1$, and

$$\widehat{M}_{aa} := M_{aa} - M_{a1}M_{11}^{-1}M_{1a} > 0.$$

Substituting (31) into (29) yields

$$\Lambda(q,\dot{q})\tau = k_D(\widehat{H}_a + \widehat{G}_a) - \widehat{M}_{aa}(k_V \dot{q}_a + k_P \Phi^{-\mathrm{T}}\overline{q}_a), \tag{32}$$

where

$$\Lambda(q,\dot{q}) := k_D I + (E(q,\dot{q}) - E_r)\widehat{M}_{aa}(q_a), \tag{33}$$

where I denotes *an identity matrix* of size $(n-1)\times(n-1)$. Therefore, when

$$\det \Lambda(q,\dot{q}) \neq 0, \quad \text{for } \;\forall q, \;\forall \dot{q} \tag{34}$$

holds, the following controller obtained from (32) has no singular points for any $(q, \dot{q})$:

$$\tau = \Lambda^{-1}\Big(k_D(\widehat{H}_a + \widehat{G}_a) - \widehat{M}_{aa}(k_V \dot{q}_a + k_P \Phi^{-\mathrm{T}}\overline{q}_a)\Big). \tag{35}$$

We are ready to present the following theorem.

Theorem 1. Consider the robot in (1) and (2). Suppose that $k_D > 0$, $k_P > 0$, and $k_V > 0$ hold. Then the controller (35) has no singular points for any $(q, \dot{q})$ if and only if

$$k_D > \max_{q_a}\Big\{(E_r + \psi(q_a))\lambda_{\max}\big(\widehat{M}_{aa}(q_a)\big)\Big\}, \tag{36}$$

where $\lambda_{\max}(\widehat{M}_{aa})$ denotes the maximal eigenvalue of $\widehat{M}_{aa}$,

$$\psi(q_a) = \Big(\sum_{i=1}^{n}\beta_i^2 + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n}\beta_i\beta_j \cos\sum_{k=i+1}^{j} q_k\Big)^{1/2}. \tag{37}$$

In this case,

$$\lim_{t\to\infty} V = V^*, \quad \lim_{t\to\infty} E = E^*, \tag{38}$$

$$\lim_{t\to\infty} q_a = q_a^*, \quad , \lim_{t\to\infty} \overline{q}_a = \overline{q}_a^*, \tag{39}$$

where V^*, E^*, q_a^* and $\overline{q}_a^*$ are some constants. Moreover, as $t \to \infty$, every closed-loop solution $(q(t), \dot{q}(t))$ approaches the following invariant set:

$$W = \left\{(q,\dot{q}) \mid \dot{q}_1^2 = \frac{2(E^* - P(q_1, q_a^*))}{M_{11}(q_a^*)};\; q_a \equiv q_a^*\right\}. \tag{40}$$

Proof. See Appendix A. ∎

5. MOTION ANALYSIS OF THE ROBOT

We shall characterize the invariant set W in (40) by analyzing the convergent value V^* of the Lyapunov function V in (28). Since $\lim_{t\to\infty} V = 0$ is equivalent to (9), we analyze two cases of $V^* = 0$ and $V^* \neq 0$, separately.

Case 1: $V^* = 0$

From (28) and (24), we have $E^* = E_r$, $\overline{q}_a^* = 0$, and $q_a^* = 0$. Thus, from (40), we obtain

$$\dot{q}_1^2 = \frac{2E_r}{M_{11}(0)}(1 - \sin q_1). \tag{41}$$

Therefore, the closed-loop solution $(q(t), \dot{q}(t))$ approaches the following invariant set as $t \to \infty$:

$$W_r = \{(q,\dot{q}) \mid (q_1, \dot{q}_1) \text{ satisfies (41)};\; q_a \equiv 0\}. \tag{42}$$

Since (41) is a homoclinic orbit (see the definition in Sastry [1999], p.44) which converges to the equilibrium point $(q_1, \dot{q}_1) = (\pi/2, 0)$ as $t \to \infty$,$(q_1(t), \dot{q}_1(t))$ will have $(\pi/2, 0)$ as an ω-limit point, that is, there exists a sequence of times t_n $(n = 1, \ldots, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ for which $\lim_{n\to\infty}(q_1(t_n), \dot{q}_1(t_n)) = (\pi/2, 0)$. Hence, the robot can enter any arbitrarily small neighborhood of the upright equilibrium point as $t \to \infty$.

Case 2: $V^* \neq 0$

Using $E \equiv E^*$, $q_a \equiv q_a^*$, $\overline{q}_a \equiv \overline{q}_a^*$, and (29), we can show that $E^* \neq E_r$ and τ is a constant vector τ^* satisfying:

$$k_P \overline{q}_a^* + (E^* - E_r)\Phi^{\mathrm{T}}(\overline{q}_a^*)\tau^* = 0, \text{with } E^* \neq E_r. \tag{43}$$

We are ready to present the following lemma.

LEMMA 1. Consider the invariant set W defined in (40). Let $(q(t), \dot{q}(t)) \in W$. If $V^* \neq 0$ holds, then $q_1(t)$ is a constant in the invariant set W. In other words, $q(t) \equiv q^*$ holds in the invariant set W.

Proof. See Appendix B. ∎

Putting $q \equiv q^*$, into (1), (2) and (4), we obtain

$$G_1(q^*) = 0, \quad \tau^* = G_a(q^*), \quad E^* = P(q^*). \tag{44}$$

Define the following equilibrium set:

$$\Omega = \{(q^*, 0) \mid q^* \text{ satisfies (43) and (44) }\}. \tag{45}$$

We are ready to present the following theorem.

Theorem 2. Consider the robot in (1) and (2). Suppose that k_D satisfies (36), $k_P > 0$ and $k_V > 0$ hold. Then under the controller (35), as $t \to \infty$, the closed-loop solution $(q(t), \dot{q}(t))$ approaches

$$W = W_r \cup \Omega, \quad \textit{with } W_r \cap \Omega = \emptyset, \tag{46}$$

where W_r is defined in (42), and Ω is the set of equilibrium points defined in (45).

6. ON CLOSED-LOOP EQUILIBRIUM POINTS

Note that all closed-loop equilibrium points are those in the set Ω and the upright equilibrium point. If the set Ω contains a stable equilibrium point in the sense of the Lyapunov stability, then the robot can not be swung up arbitrarily close to the upright equilibrium point from some neighborhoods close to the stable equilibrium point.

Since for any given k_P the set Ω contains at least one element of the downward equilibrium point, where all links remain in their downward positions, we aim at providing some conditions on k_P such that the set Ω does not contain any other equilibrium point. We present a main result of this paper by the following theorem with its proof given in Appendix C.

Theorem 3. Consider the robot in (1) and (2). Suppose that k_D satisfies (36), and $k_V > 0$ holds. If k_P satisfies

$$k_P > \max_{2 \le i \le n} k_{mi}, \tag{47}$$

where

$$k_{mi} := 2E_r \beta_{i-1} \left(\sum_{j=i}^{n} \beta_j \right) \Big/ \left(\sum_{j=i-1}^{n} \beta_j \right), \tag{48}$$

then under the controller (35),

1) *Ω in (45) contains only the downward equilibrium point;*
2) *the downward equilibrium point is unstable in the closed-loop system;*
3) *the closed-loop solution $(q(t), \dot{q}(t))$ approaches*
$$W = W_r \cup \{(-\pi/2, 0, \cdots, 0, 0, \cdots, 0)\} \tag{49}$$
as $t \to \infty$, where W_r is defined in (42).

Note that conditions on k_p in (47) for $n = 2$ and $n = 3$ coincide with that for the Acrobot in Xin and Kaneda [2007a] and that for the 3-link robot in Xin and Kaneda [2007b], respectively.

7. CONCLUSIONS

The control objective of this paper is to swing up an n-link planar robot in a vertical plane with a passive first joint to any arbitrarily small neighborhood of the upright equilibrium point. In this paper, first, we addressed how to devise a series of virtually composite links in an iterative way for designing a coordinate transformation on the angles of all active joints. Second, we constructed a Lyapunov function based on the transformation for designing a swing-up controller. We also provided the necessary and sufficient condition for nonexistence of any singular points in the controller for the robot starting from any initial state. Third, we attained a global analysis of the motion of the robot under the devised controller. Indeed, we showed that starting from any initial state, the state of the robot will eventually approach either any arbitrarily small neighborhood of the upright equilibrium point, or a certain equilibrium point belonging to the set Ω in (45). With the advantage of the coordinate transformation, this paper attained the condition on the control parameter k_P such that the set Ω contains only the downward equilibrium point, which is shown to be unstable under the proposed controller.

This paper provided insight into the energy or passivity based control to underactuated multi-DOF systems. Moreover, the design and analysis in this paper can be extended to the swing-up control problem of an n-link robot in a vertical plane with any single passive joint among all joints. Thus, this paper contributes to fault tolerance of fully-actuated n-DOF manipulators when a joint actuator fails.

Appendix A. PROOF OF THEOREM 1

First, we analyze (34) in what follows. Using $E(q, \dot{q}) \ge P(q)$, $E_r \ge P(q)$, and $\widehat{M}_{aa}(q_a) > 0$, we obtain

$$\Lambda(q, \dot{q}) \ge \big(k_D - (E_r - P(q))\lambda_{\max}(\widehat{M}_{aa})\big)I.$$

where I is an $(n-1) \times (n-1)$ identity matrix. Therefore, a *sufficient* condition such that (34) holds is

$$k_D > \max_q f(q); \; f(q) = (E_r - P(q))\lambda_{\max}\big(\widehat{M}_{aa}(q_a)\big). \tag{A1}$$

We show that (A1) is also a *necessary condition* such that (34) holds. To this end, for any given k_D satisfying $0 < k_D \le \max_q f(q)$, we just need to show that there exists an initial state $(q(0), \dot{q}(0))$ at which $\Lambda(q, \dot{q})$ is singular. Let $b \in \mathbb{R}^n$ be a value of q which maximizes $f(q)$, that is, $b = \arg\max_q f(q)$. Owing to $k_D \le f(b)$, there exists $d \in \mathbb{R}^n$ such that

$$\frac{1}{2} d^{\mathrm{T}} M(b_a) d = \frac{f(b) - k_D}{\lambda_{\max}\big(\widehat{M}_{aa}(b_a)\big)} \ge 0,$$

where $b_a = [\,0 \;\, I\,]\, b$. Thus, for an initial state $(q(0), \dot{q}(0)) = (b, d)$, we have

$$\begin{aligned} & k_D + \Big(E\big(q(0), \dot{q}(0)\big) - E_r\Big)\lambda_{\max}\Big(\widehat{M}_{aa}\big(q_a(0)\big)\Big) \\ = {} & k_D + \Big(\frac{1}{2} d^{\mathrm{T}} M(b_a) d + P(b) - E_r\Big)\lambda_{\max}\Big(\widehat{M}_{aa}(b_a)\Big) = 0. \end{aligned}$$

This yields

$$\det\bigg(k_D I + \Big(E\big(q(0), \dot{q}(0)\big) - E_r\Big)\widehat{M}_{aa}\big(q_a(0)\big)\bigg) = 0.$$

Thus, (34) has singular points if $k_D \le \max_q f(q)$.

In what follows, we show that (36) is equivalent to (A1). Note that for any q which maximizes $f(q)$ in (A1),

$$\frac{\partial f(q)}{\partial q_1} = -\frac{\partial P(q)}{\partial q_1} \lambda_{\max}\big(\widehat{M}_{aa}(q_a)\big) = 0 \tag{A2}$$

must hold. This yields $\partial P(q)/\partial q_1 = G_1(q) = 0$. Under the condition $G_1(q) = 0$, using $P(q)$ in (5) and $G_1(q) = 0$ in

(7), computing $P^2(q) = P^2(q) + G_1^2(q)$, we can express $P(q)$ in terms of q_a as follows:

$$P(q) = \pm\psi(q_a), \quad \text{when } G_1(q) = 0, \tag{A3}$$

where $\psi(q_a)$ is defined in (37). Therefore, substituting $P(q) = -\psi(q_a)$ into (A1), we can see that (36) is a necessary and sufficient condition such that (34) holds.

Second, let W be the largest invariant set in $\Gamma = \{(q, \dot{q}) \mid \dot{V} = 0\}$. From $\dot{V} \leq 0$ in (30), using LaSalle's theorem, we know that every $(q(t), \dot{q}(t))$ approaches W as $t \to \infty$. Since $\dot{V} = 0$ holds identically in W, owing to (30), V and q_a are constant in W. Moreover, from (11) and (28), we know that $\overline{q}_a$ and E are also constant in W. This proves (38) and (39). Substituting $q_a = q_a^*$ and $E = E^*$ into (4) yields W expressed in (40). ∎

Appendix B. PROOF OF LEMMA 1

Note that $q_a(t) \equiv q_a^*$ holds in the invariant set W in (40). According to Theorem 1, we just need to show that if $V^* \neq 0$ holds, then $q_1(t)$ is constant in the invariant set W.

Since $q_a(t) \equiv q_a^*$, there exists no relative motion among the links 2 to n, in this case, we can we consider links 2 to n as a composite link, that is, the robot can be treated as the Acrobot (a 2-link robot with pass first joint). Moreover, the relative angle between link 1 and the composite link is $\overline{q}_2^*$ and the torque on joint 2 is τ_2^*.

We proceed the proof *similarly* as we did for the Acrobot in Xin and Kaneda [2007a] . The detail is omitted due to space limitations. ∎

Appendix C. PROOF OF THEOREM 3

Consider an equilibrium point $(q^*, 0)$ of Ω in (45). Using $\overline{q}_2^*$ and $\Phi^{\mathrm{T}}(\overline{q}_a^*)$, we can rewrite (44) and (43) as

$$\beta_1 \cos q_1^* + \overline{\beta}_2^* \cos(q_1^* + \overline{q}_2^*) = 0, \tag{C1}$$

$$k_P \overline{q}_2^* + (P(q^*) - E_r)\tau_2^* = 0, \tag{C2}$$

$$k_P \overline{q}_i^* + (P(q^*) - E_r)\Big(\tau_i^* + \sum_{j=2}^{i-1} \phi_{ji}(\overline{q}_a^*)\tau_j^*\Big) = 0, \tag{C3}$$

$$P(q^*) \neq E_r, \tag{C4}$$

where i in (C3) satisfies $3 \leq i \leq n$.

As to Statement 1), we carry out the proof by induction via the following two steps:

Step 1: For $i = 2$, we show that if

$$k_P > k_{m2} = \beta_1 \sum_{j=2}^{n} \beta_j, \tag{C5}$$

then only $(q_1^*, \overline{q}_2^*) = (\pi/2, 0)$ and $(q_1^*, \overline{q}_2^*) = (-\pi/2, 0)$ satisfy (C1) and (C2).

Step 2: Suppose that we have proved the following statement: For $i = k - 1$, if $k_P > \max_{2 \leq j \leq k-1} k_{mj}$, where k_{mj} is defined in (48), then

$$\begin{cases} q_1^* = \pi/2, \quad \text{or} \quad q_1^* = -\pi/2, \\ \overline{q}_j^* = 0, \quad \text{for} \;\; 2 \leq j \leq k-1. \end{cases} \tag{C6}$$

We show that for $i = k$, if $k_P > k_{mk}$ also holds, then $\overline{q}_k^* = 0$ also holds.

The detailed description of these two steps are omitted due to space limitations. ∎

As to Statement 2), we omit its proof due to page limitation. **As to Statement** 3), it is direct consequence of Statement 1) and Theorem 2. ∎

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