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Adaptive Control Design based on Adaptive Optimization Principles

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Abstract: Recently, we introduced an adaptive control design for linearly parameterized multi-input nonlinear systems admitting a known Control Lyapunov Function (CLF) that depends on the unknown system parameters. The main advantage of that design is that it overcomes the problem where the estimation model becomes uncontrollable although the actual system is controllable. However, the resulted adaptive control design is quite complicated and, moreover, it exhibited poor transient behaviour in various applications. In this paper, we propose and analyze a new computationally efficient adaptive control design that overcomes the aforementioned shortcomings. The proposed design is based on an adaptive optimization algorithm proposed recently by the author, which makes sure that the parameters to be optimized (which correspond to the controller parameters in this paper) are modified so as to both lead to a decrease of the function to be minimized and satisfy a persistence of excitation condition. The main advantage of the proposed adaptive control design is that it can produce arbitrarily good transient performance outside the regions of the state space where the system becomes uncontrollable. It is also worth noticing that the class of systems where the proposed algorithm is applicable is more general than that of our previous work.

Keywords: Adaptive Optimization, Multivariable Adaptive Control, Control Lyapunov Functions, Persistence of Excitation

1. INTRODUCTION

Despite the recent advances in the theory of adaptive control of nonlinear systems, the problem of designing an efficient adaptive controller for general nonlinear systems remains an open issue. Probably the main problem in adaptive control design of nonlinear systems and especially in the case of multi-input systems, is the problem where the estimation of the input vector-field (or of a transformation of it) becomes non-invertible (in which case the estimation model becomes uncontrollable) although the inverse of the input-vector field or its transformation does exist. For this reason, most adaptive control designs impose strict assumptions on the controlled system such as that the sign of the input-vector field (which corresponds to the sign of the high-frequency gain in the case of linear systems) is known [7, 9, 3]; we note that the extension of such an assumption to multi-input systems even in the case where the system is linear is a very complicated issue (see e.g. [13] and section 9.7 of [2] for a discussion on the extension of the assumption on knowledge of the sign of the high-frequency gain of SISO linear systems to MIMO linear systems). Moreover, even in the case of single-input linear or nonlinear systems the removal of the assumption on the knowledge of the sign of the input vector-field leads in most cases to the deployment of Nussbaum-type adaptive controllers which may produce very poor transient performance (see e.g. [15]).

In [4] we proposed an adaptive control design that removed the assumption on the knowledge of the sign of the input vector-field. The approach of [4] was applicable to systems satisfying

$$\begin{aligned}
&\dot{x} = \bar{\vartheta}\bar{f}(x) + \bar{\vartheta}\bar{g}(x)u \\
&\frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{g}(x) = 0 \Longrightarrow \frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{f}(x) < 0, \ \forall x \neq 0 \\
&V(x) = a(\bar{\vartheta})^{\tau} v(x) \qquad (1)
\end{aligned}$$

where $x \in \Re^{n_x}, u \in \Re^{n_u}$ denote the vectors of system states and control inputs, respectively, $\bar{\vartheta} \in \Re^{n \times \bar{L}}$ is a matrix of *unknown* constant parameters and $\bar{f} : \Re^{n_x} \mapsto \Re^{\bar{L}}, \bar{g} : \Re^{n_x} \mapsto \Re^{\bar{L} \times n_u}$ denote known, at-least C^1 functions; the function V denotes a smooth, positive definite, radially unbounded, Control Lyapunov Function (CLF) for the system that is calculated as the vector product of the known smooth vector function $v(x)$ and the vector function $a(\bar{\vartheta})$ that depends on the vector $\bar{\vartheta}$ of unknown parameters. It is worth noticing that multi-input linearly parameterized feedback linearizable systems belong to the family (1) as shown in [4]. Moreover, as shown in [11], the family (1) also includes stabilizable systems with polynomial vector fields, in which case the CLF function $V(x) = a(\bar{\vartheta})^{\tau} v(x)$ can be constructed by using Sums-of-Squares optimization methods.

It is worth noticing that no assumption about the sign(s) of the input-vector field $\bar{\vartheta}\bar{g}(x)$ was made in [4]. However,

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the adaptive control design of [4] results even in the case where it is applied to linear MIMO systems into a very complicated controller; moreover, the controller of [4] exhibited poor transient behavior in various applications tested in our lab, which was mainly due to the rapid switching of the controller.

In this paper we present a computationally efficient adaptive control design that overcomes the above mentioned shortcomings of the control design of [4]. The proposed design is applicable to – the more general than (2) – class of systems, described as follows:

$$\dot{x} = \bar{\vartheta}\bar{f}(x) + \bar{\vartheta}\bar{g}(x)u \quad (2)$$

$$\frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{g}(x) = 0 \Longrightarrow \frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{f}(x) < 0, \quad \forall x \neq 0 \quad (3)$$

$$V(x) = v(\bar{\vartheta}, x)$$

where V denotes a smooth, positive definite, radially unbounded CLF for the system and v denotes a smooth known function that depends nonlinearly on the unknown parameters $\bar{\vartheta}$. The proposed design is based on an adaptive optimization algorithm proposed recently by the author (see [6, 5]) which makes sure that the parameters to be optimized (which correspond to the controller parameters in this paper) are modified so as to both lead to a decrease of the function to be minimized (which corresponds to the CLF V in this paper) and satisfy a persistence of excitation condition. The main advantage of the proposed adaptive control design is that it can produce arbitrarily good transient performance outside the regions of the state space where the system becomes uncontrollable.

1.1 Notation

$\mathcal{Z}, \Re$ denote the sets of nonnegative integers and real numbers, respectively. For a vector $x \in \Re^n$, $|x|$ denotes the Euclidean norm of x (i.e., $|x| = \sqrt{x^\tau x}$), while for a matrix $A \in \Re^{n^2}$, $|A|$ denotes the induced matrix norm of A. A function f is said to be C^m, where m is a positive integer, if it is uniformly continuous and its first m derivatives are uniformly continuous. I_n denotes the n-dimensional identity matrix.

2. THE PROPOSED ALGORITHM

Since – by (3) – system (2) is stabilizable it makes sense to assume that there exists a known controller – parameterized by a vector of unknown controller parameters – that solves the stabilization problem. More precisely, we will assume the following:

(A1) For any positive constant η and any compact subset $\mathcal{X} \subset \Re^{n_x}$, there exists an, at-least C^1, bounded for bounded x, *known* vector function $\pi : \Re^{n_x} \mapsto \Re^{n_u \times n_\theta}$ satisfying the following: there exists a vector $\theta^* \in \Re^{n_\theta}$ such that the closed-loop system (2) solutions with $u = \pi(x)\theta^*$ and initial state $x(0) = x_0$ satisfy $\sup_{t\in[0,\infty)} |x(t)| < \infty$ and $\limsup_{t\to\infty} |x(t)| < \eta$ for any $x_0 \in \mathcal{X}$.

It is not difficult for someone to see that (3) implies (A1). To see this note that from (3) we have that there exists a – probably discontinuous at a set of zero Lebesque measure – controller $u = k(x)$ such that the closed-loop system (2) with $u = k(x)$ is stable for any bounded initial condition $x(0) = x_0$ and, moreover, its solutions converge to zero asymptotically. Then, if the function π is chosen so that it belongs to a Universal Approximation (UA) family (e.g. π can be a polynomial function of x), standard results based on approximation theory (see e.g. [8] and the references therein) can be used to establish that, for any positive constant c_η there exists a vector θ^* such that $\sup_{x\in\bar{\mathcal{X}}} |k(x) - \pi(x)\theta^*| < c_\eta$ for any compact subset $\bar{\mathcal{X}} \subset \Re^{n_x}$. Control Lyapunov function arguments [12, 10] can be then applied to establish that, if c_η is sufficiently small, the controller $u = \pi(x)\theta^*$ satisfies (A1) for η depending on the approximation constant c_η, provided that $\mathcal{X} \subset \bar{\mathcal{X}}$ and the distance between the boundaries of $\mathcal{X}$ and $\bar{\mathcal{X}}$ is sufficiently large, so that the closed-loop solutions under $u = \pi(x)\theta^*$ satisfy $x(t) \in \bar{\mathcal{X}}, \ \forall t$. Note also that (A1) reduces the problem of finding a controller stabilizing system (2) into the problem of finding a vector θ or a vector sequence $\theta(t)$ such that the controller $u(t) = \pi(x(t))\theta(t)$ guarantees $\sup_{t\in[0,\infty)} |x(t)| < \infty$ and $\limsup_{t\to\infty} |x(t)| < \eta$.

Note that contrary to the controller $u = k(x)$ mentioned above which provides with globally stable closed-loop solutions, the controller $u = \pi(x)\theta^*$ of (A1) provides with semi-global closed-loop stability. In the sequel we will assume that all initial conditions $x(0) = x_0$ belong to the compact subset $\mathcal{X}$. Moreover, we will assume that the subset $\bar{\mathcal{X}}$ over which condition $\sup_{x\in\bar{\mathcal{X}}} |k(x) - \pi(x)\theta^*| < c_\eta$ holds, is sufficiently large so that it contains not only the solutions $x(t)$ under the controller $u = \pi(x)\theta^*$ but also the the solutions $x(t)$ under the proposed adaptive controller.

Since V is smooth and $\bar{f}, \bar{g}$ are at-least C^1, it can be seen that following lemma holds:

Lemma 1. Assumption (A1) implies the existence of positive constants $\epsilon_i, \ i = 1, 2, 3$ such that the following condition holds, for all $x \in \bar{\mathcal{X}}$,

$$\left|\frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{\bar{g}}(x)\right| < \epsilon_1 \text{ and } |x| > \epsilon_3 > \eta \Rightarrow \frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{f}(x) < -\epsilon_2 \quad (1)$$

where $\bar{\bar{g}}(x) = \bar{g}(x)\pi(x)$.

Proof. The proof is a directly corollary of Lemma 1 of [4].

Let us define the *uncontrollable region* of (2) to be the subset $\mathcal{U}$ defined according to

$$\mathcal{U} = \left\{x \in \Re^{n_x} : |x| > \epsilon_3 \text{ and } \left|\frac{\partial V}{\partial x}^{\tau}(x)\bar{\vartheta}\bar{\bar{g}}(x)\right| < \epsilon_1\right\}$$

From condition (1) we have that that as long as $x(t) \in \mathcal{U}$, the choice $\theta(t) = 0$ (or $\theta(t)$ being small enough) guarantees that $V(t)$ is decreasing.

We will also need the following assumption regarding system (2).

(A2) $\bar{f}, \bar{g}$ are bounded for bounded x.

2.1 Input vector-field preprocessing

It is no loss of generality to assume that the constant ϵ_1 in (1) is as large as desired. If this is not the case we can always employ a precompensator of the form $u = K\bar{u}$ where K is a user-defined large positive constant and $\bar{u}$ is the "new" control input; then the proposed design can be applied to the transformed – after applying the precompensator $u = K\bar{u}$ – system (2) with $\bar{g} \mapsto \bar{g}K, u \mapsto \bar{u}$ and $\epsilon_1 \mapsto K\epsilon_1$.

Also, for reasons that will be made clear later on, we wish the entries $\bar{\bar{g}}_{ij}(x)$ of the matrix function $\bar{\bar{g}}(x)$ to be bounded away from zero. Since this will not be always the case, we transform the system (2) under the controller $u = \pi(x)\theta$ into the equivalent system

$$\begin{aligned}\dot{x} &= \vartheta f(x) + \vartheta g(x)\theta + \varepsilon(x, \theta) \\ &= \vartheta\phi(x, \theta) + \varepsilon(x, \theta)\end{aligned} \tag{2}$$

where the entries $g_{ij}(x)$ of $g(x)$ satisfy $|g_{ij}(x)| \geq \bar{\epsilon}, \forall x$, the function $f(x)$ is defined according to $\vartheta f(x) = \bar{\vartheta}\bar{f}(x)$ and $|\varepsilon(x, \theta)| < c_1\bar{\epsilon}\,|\theta|$, where $\bar{\epsilon}$ is a user-defined positive constant and c_1 is a finite positive constant independent of x, θ and $\bar{\epsilon}$. The above transformation can be easily performed as follows: for all entries $\bar{\bar{g}}_{ij}(x)$ that satisfy $|\bar{\bar{g}}_{ij}(x)| \geq \bar{\epsilon}, \forall x$, we set $g_{ij}(x) = \bar{\bar{g}}_{ij}(x)$, while for those entries $\bar{\bar{g}}_{ij}(x)$ that cross zero, we correspond two entries $g_{ij}(x)$ and $g_{ij'}(x)$ such that $g_{ij}(x) = \bar{\bar{g}}_{ij}(x)$ if $\bar{\bar{g}}_{ij}(x) \geq \bar{\epsilon}$ and $g_{ij}(x) = \bar{\epsilon}$, otherwise, and similarly, $g_{ij'}(x) = \bar{\bar{g}}_{ij}(x)$ if $\bar{\bar{g}}_{ij}(x) \leq -\bar{\epsilon}$ and $g_{ij'}(x) = -\bar{\epsilon}$, otherwise. Obviously the new parameter matrix ϑ satisfies $\vartheta \in \Re^{n\times L}$, where $L \leq 2\bar{L}$ and, moreover, we can easily construct n_x matrices $\Pi_i \in \{0,1\}^{\bar{L}\times L}$ (note that the matrices Π_i are not uniquely defined) such that $\bar{\vartheta}^i = \Pi_i\vartheta^i$ and, moreover, for any vector $\hat{\vartheta}^i$ satisfying $\left|\hat{\vartheta}^i - \vartheta^i\right| < \delta$ then, we have that $\left|\Pi_i\hat{\vartheta}^i - \bar{\vartheta}^i\right| < \bar{c}\delta$ for some finite positive constant $\bar{c}$ independent of δ; here $\vartheta^i, \bar{\vartheta}^i$ denote the column vectors that correspond to the i-th row of ϑ and $\bar{\vartheta}$, respectively.

2.2 Adaptive estimator

The proposed adaptive control scheme updates the control vector θ every Δt time-units; in other words, if $t_k = t_{k-1} + \Delta t, t_0 = 0,\ k \in \mathcal{Z}$ denote the time-instances at which the controller vector is updated, then we have that $\theta(t)$ remains constant in the intervals $t \in [t_{\ell-1}^+, t_\ell)$. In order to calculate the updates of the control vector $\theta(t)$ the proposed adaptive control design makes use of an adaptive estimator described as follows: Let $\theta_{\ell-1} = \theta(t), t \in [t_{\ell-1}^+, t_\ell)$ denote the updates of $\theta(t)$ at $t = t_{\ell-1}^+$ and $\psi_\ell^i, \zeta_\ell^i$ be defined as follows:

$$\psi_\ell^i = \int\limits_{t=t_{\ell-1}}^{t_\ell} f_i(x(s))ds, \quad \zeta_\ell^i = \int\limits_{t=t_{\ell-1}}^{t_\ell} g_i(x(s))ds$$

where $f_i(x), g_i(x)$ denote the i-th column of $f(x), g(x)$, respectively. Then, at each time-instant t_k an estimate $\hat{\vartheta}(t_k^+) \equiv \hat{\vartheta}_k$ is calculated so as to satisfy for $i \in \{1, \ldots, n_x\}$,

$$\hat{\vartheta}_k^i = \arg\min_{\hat{\vartheta}^i} \sum_{\ell=\ell_k}^{k} \left(x_i(t_\ell) - x_i(t_{\ell-1}) - \hat{\vartheta}^i\left(\psi_\ell^i + \zeta_\ell^i\theta_{\ell-1}\right)\right)^2 \tag{3}$$

where $\hat{\vartheta}_k^i$ denotes the column vector that corresponds to the i-th row of $\hat{\vartheta}_k$ and

$$\ell_k = \min\{1, k - L - T_h\}$$

with T_h being a nonnegative user-defined constant. The next lemma establishes convergence of $\hat{\vartheta}_k - \vartheta$ under a *persistence of excitation* condition.

Lemma 2. If the matrix Ψ_k^i defined as

$$\Psi_k^i = \left[\psi_{k-L+1}^i + \zeta_{k-L+1}^i\theta_{k-L+1}, \ldots, \psi_k^i + \zeta_k^i\theta_k\right] \tag{4}$$

satisfies

$$\text{rank}\left(\Psi_k^i\right) = L,\ \forall k \geq L,\ \forall i \in \{1, \ldots, n_x\} \tag{5}$$

and, moreover, $\sup_{\ell\in\{\ell_k-1,\ldots,k\}} |x(t_\ell)| < \infty$ and $\sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell|$ then, the parameter estimation error $\tilde{\vartheta}_k = \vartheta - \hat{\vartheta}_k$ satisfies

$$\left|\tilde{\vartheta}_k\right| \leq c_2\bar{\epsilon} \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell| \tag{6}$$

for some finite positive constant c_2 that depends on $\sup_{\ell\in\{\ell_k-1,\ldots,k\}} |x(t_\ell)|$.

Proof. Let $\Phi_k = \sum_{\ell=\ell_k}^{k}\left(\psi_\ell^i + \zeta_\ell^i\theta_\ell\right)\left(\psi_\ell^i + \zeta_\ell^i\theta_\ell\right)^\tau$. It can be easily seen that since Ψ_k^i is full rank for $k \geq L$ then Φ_k^{-1} exists for $k \geq L$ and, moreover, the solution of (3) satisfies

$$\begin{aligned}\hat{\vartheta}_k^i &= \Phi_k^{-1}\Phi_k\vartheta^i + \Phi_k^{-1}\sum_{\ell=\ell_k}^{k}\bar{\varepsilon}_\ell^i\left(\psi_\ell^i + \zeta_\ell^i\theta_\ell\right)^\tau \\ &= \vartheta^i + \Phi_k^{-1}\sum_{\ell=\ell_k}^{k}\bar{\varepsilon}_\ell^i\left(\psi_\ell^i + \zeta_\ell^i\theta_\ell\right)^\tau, \\ \bar{\varepsilon}_\ell^i &= \int\limits_{t=t_{\ell-1}}^{t_\ell}\varepsilon_i(x(s), \theta_{\ell-1})ds\end{aligned}$$

from which we readily obtain (6), by using $|\varepsilon(x, \theta)| < c_1\bar{\epsilon}\,|\theta|$.

2.3 The proposed controller

The proposed controller update scheme is as follows:

$$\begin{aligned}\theta(t_k^+) &= \arg\min_{\pm\theta_k^{(j)}, j\in\{1,\ldots,n_\theta\}} \dot{V}_k^{(\pm j)} \\ \dot{V}_k^{(\pm j)} &= \frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k^\tau\phi(x_k, \pm\theta_k^{(j)})\end{aligned} \tag{7}$$

where $x_k = x(t_k)$, the rows $\hat{\bar{\vartheta}}_k^i$ of $\hat{\bar{\vartheta}}_k$ are obtained by using $\hat{\bar{\vartheta}}_k^i = \Pi_i\hat{\vartheta}_k^i$ and $\theta_k^{(j)}$ are n_θ zero-mean random vectors in $\{-\alpha_k, \alpha_k\}^{n_\theta}$ satisfying

$$\left|\left[\theta_k^{(1)}, \ldots, \theta_k^{(n_\theta)}\right]\right|^{-1} < \frac{\Xi}{\alpha_k} \tag{8}$$

where α_k is user-defined positive sequence and Ξ is a finite positive number independent of α_k. It can be seen [1, 14, 6] that a choice $\theta_k^{(j)} = \alpha_k\Delta_k^{(j)}$, where $\Delta_k^{(j)}$ are Bernoulli random or Bernoulli-like vectors in $\{-1,+1\}^{n_\theta}$ satisfies (8).

To understand the rationale behind the proposed algorithm note that

$$\dot{V} = \frac{\partial V}{\partial x}^\tau(x)(\vartheta f(x) + \vartheta g(x)\theta + \varepsilon(x,\theta)) \\ = \frac{\partial V}{\partial x}^\tau(x)\left(\vartheta\phi(x,\theta) + \varepsilon(x,\theta)\right) \quad (9)$$

Apparently, the variables $\dot{\hat{V}}_k^{(\pm j)}$ denote the estimates – produced using the adaptive estimator (3) – of $\dot{V}(t_k^+)$ under the choice $\theta(t_k^+) = \pm\theta_k^{(j)}$. In other words, the choice for $\theta(t_k^+)$ according to (7) corresponds to the one, among all $\pm\theta_k^{(j)}$, that leads to the maximum estimated decrease of V. As it will be seen in the proof of the main result of this paper, condition (8) is crucial to make sure that this aforementioned maximum estimated decrease is non-negligible; moreover, as it will be seen in the next Lemma, the random choice for $\theta_k^{(j)}$, in combination with the fact that by design $g_{ij}(x)$ are bounded away from zero, guarantee that the regressor matrix $\phi(x,u)$ is persistently exciting.

Lemma 3. If $\alpha_k > 0, \forall k \in \mathcal{Z}$ then condition (5) of Lemma 2 holds with probability 1.

Proof. For the sake of contradiction let us assume that the matrix Ψ_k^i is not full-rank for $k \geq L$. Then there exists a non-zero vector b such that

$$b^\tau\left(\psi_\ell^i + \zeta_\ell^i\theta_\ell\right) = 0, \ \forall \ell \in \{k-L+1,\ldots,k\}$$

It is not difficult for someone to see that since θ_ℓ are random vectors in $\{-\alpha_\ell,\alpha_\ell\}^{n_\theta}$ and by design (see subsection II.A) the entries of the vector ζ_ℓ are bounded away from zero, the term $\psi_{\ell,1}^i + \zeta_{\ell,1}^i\theta_{\ell,1} \neq 0$ for all $\ell \in \{k-L+1,\ldots,k\}$ with probability 1; then the above equality implies that, $\forall \ell \in \{k-L+1,\ldots,k\}$,

$$b_1 = -\frac{1}{\psi_{\ell,1}^i + \zeta_{\ell,1}^i\theta_{\ell,1}}\sum_{j=2}^{n_\theta} b_j\left(\psi_{\ell,j}^i + \zeta_{\ell,j}^i\theta_{\ell,j}\right) \ \text{ w. p. 1}$$

Since θ_ℓ are random vectors the probability of b_1 to satisfy all k above equations is zero, which concludes the proof.

We are ready to establish the main result of this paper.

Theorem 4. Let (A1), (A2) hold and assume that the design constant $\bar{\epsilon}$ as well as the sampling interval Δt are sufficiently small and ϵ_1 is sufficiently large. Then, for arbitrary $\bar{\alpha} > 0$, there exist finite positive constants $\beta_0, \beta_1, \beta_2$ with β_1, β_2 satisfying

$$\frac{1}{4n_\theta}\epsilon_1\beta_1 > \lambda_1 + \lambda_2\bar{\epsilon}\beta_2 + \lambda_3\bar{\epsilon}\beta_2^2 + \bar{\alpha} \quad (10)$$

for some finite positive constants $\lambda_1, \lambda_2, \lambda_3$ independent of $\bar{\epsilon}$, such that if α_k satisfies

$$\begin{array}{ll} 0 < \alpha_k \leq \beta_0 & \text{if } k < L \\ \beta_1 \leq \alpha_k \leq \beta_2 & \text{if } \left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \geq \hat{\epsilon}_1 \\ & \text{and } k \geq L \\ 0 < \alpha_k \leq \dfrac{-\bar{\bar{\alpha}} + \epsilon_2}{\sqrt{n_\theta}\epsilon_1} & \text{if } \left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| < \hat{\epsilon}_1 \\ & \text{and } k \geq L \end{array} \quad (11)$$

where $\bar{\bar{\alpha}}$ is a positive design constant satisfying $\bar{\bar{\alpha}} < \epsilon_2$, and $\hat{\epsilon}_1$ is a positive design constant satisfying

$$\frac{1}{4}\epsilon_1 \leq \hat{\epsilon}_1 \leq \frac{1}{2}\epsilon_1 \quad (12)$$

then, the proposed adaptive control scheme (3), (7), (8) guarantees that the closed-loop solutions are bounded and, moreover,

$$\limsup_{t\to\infty}|x(t)| \leq \epsilon_3, \text{ with probability 1} \quad (13)$$

and, moreover,

$$\dot{V}(t_k^+) < \begin{cases} -\bar{\alpha} & \text{if } x_k \notin \mathcal{U} \\ -\bar{\bar{\alpha}} & \text{if } x_k \in \mathcal{U} \end{cases} \quad (14)$$

Proof. Note that the solutions of the closed-loop system are continuous in the intervals $[t_k^+, t_{k+1})$ and thus – since these intervals have non-negligible length – it can be established (similar to e.g. theorem 2 of [4]) that there exists a time-interval $[0,\omega)$ of maximal length on which the closed-loop system possesses a unique Caratheodory solution. Note also that since the functions $\bar{f}(x), \bar{g}(x), \pi(x)$ are at least C^1 and – from (A1), (A2) – bounded for bounded x, it is not difficult for someone to see that there exists a positive constant β_0 such that if $\alpha_k \leq \beta_0, k < L$ then $x(t) \in \bar{\mathcal{X}}, t \in [0, t_L)$ and, moreover, the distance between $x(t_L)$ and the boundary of $\bar{\mathcal{X}}$ is sufficiently large.

We now concentrate on the case where $k \geq L$. We consider the following cases for all $|x| > \epsilon_3, x \in \bar{\mathcal{X}}$:

(C1) $x_k \notin \mathcal{U}$ and $\left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \geq \hat{\epsilon}_1$. Let

$$\dot{V}_k^{(\pm j)} = \frac{\partial V}{\partial x}^\tau(x_k)\left[\bar{\vartheta}\bar{f}(x_k) \pm \bar{\vartheta}\bar{g}(x_k)\theta_k^{(j)}\right]$$

By using (8) and $x_k \notin \mathcal{U}$, it is not difficult for someone to see that

$$\min_{j\in\{1,\ldots,n_\theta\}}\frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{g}(x_k)\left[\pm\theta_k^{(1)},\ldots,\pm\theta_k^{(n_\theta)}\right] < -\frac{1}{n_\theta}\epsilon_1\alpha_k$$

Therefore,

$$\min_{j\in\{1,\ldots,n_\theta\}}\dot{V}_k^{(\pm j)} < \frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{f}(x_k) - \frac{1}{n_\theta}\epsilon_1\alpha_k \quad (15)$$

Note also that

$$\begin{aligned} \dot{V}_k^{(\pm j)} - \dot{\hat{V}}_k^{(\pm j)} &= \frac{\partial v}{\partial x}^\tau(\bar{\vartheta}, x_k) \times \\ &\quad \left(\vartheta\phi(x_k, \pm\theta_k^{(j)}) + \varepsilon(x_k, \pm\theta_k^{(j)})\right) \\ &\quad - \frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k\phi(x_k, \pm\theta_k^{(j)}) \\ &= \left(\frac{\partial v}{\partial x}^\tau(\bar{\vartheta}, x_k) - \frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\right)\vartheta\phi(x_k, \pm\theta_k^{(j)}) \\ &\quad + \frac{\partial v}{\partial x}^\tau(\bar{\vartheta}, x_k)\varepsilon(x_k, \pm\theta_k^{(j)}) \\ &\quad + \frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\tilde{\vartheta}_k\phi(x_k, \pm\theta_k^{(j)}) \end{aligned}$$

Using lemmas 2 and 3 and $|\varepsilon(x,\theta)| < c_1\bar{\epsilon}\,|\theta|$ we directly obtain that the above inequality may be rewritten as

$$\dot{V}_k^{(\pm j)} - \dot{\hat{V}}_k^{(\pm j)} \leq c_3\bar{\epsilon} \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell| a_1(x_{[t_{\ell_k-1},t_k)}) + c_4\bar{\epsilon}a_2(x_{[t_{\ell_k-1},t_k)}) \times \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell| \left|\theta_k^{(j)}\right| \quad (16)$$

where $a_i(x_{[t_{\ell_k-1},t_k)})$, $i = 1,2$ are nonnegative terms satisfying $a_i(x_{[t_{\ell_k-1},t_k)}) < \sup_{t\in[t_{\ell_k-1},t_k)} |k_i(x(t))|$ with k_i, $i = 1,2$ being radially unbounded functions wrt their arguments and $c_i, i = 3,4$ are finite positive constants independent of $x(t)$ and $\theta(t)$.

Combining (15) and (16) and using the fact that $\theta_\ell \in \{-\alpha_\ell, \alpha_\ell\}^{n_\theta} \Rightarrow |\theta_\ell| = \sqrt{n_\theta}\alpha_\ell$, we readily obtain that

$$\begin{aligned}\frac{1}{n_\theta}\epsilon_1\alpha_k > & \frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{f}(x_k) \quad (17)\\ & + c_3\bar{\epsilon}a_1(x_{[t_{\ell_k-1},t_k)})\sqrt{n_\theta} \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} \alpha_\ell \\ & + c_4\bar{\epsilon}a_2(x_{[t_{\ell_k-1},t_k)})n_\theta\alpha_k \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} \alpha_\ell \\ & + \bar{\alpha} \Longrightarrow \\ & \arg\min_{j\in\{1,\ldots,n_\theta\}} \dot{V}_k^{(\pm j)} \equiv \arg\min_{j\in\{1,\ldots,n_\theta\}} \dot{\hat{V}}_k^{(\pm j)} \\ & \text{and } \dot{V}(t_k^+) < -\bar{\alpha} \quad (18)\end{aligned}$$

where $\bar{\alpha}$ is a positive constant.

(C2) $x_k \in \mathcal{U}$ and $\left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| < \hat{\epsilon}_1$. By using Lemma 1 we obtain

$$\dot{V}(t_k^+) < -\epsilon_2 + \sqrt{n_\theta}\epsilon_1\alpha_k$$

from which we have that

$$0 < \alpha_k < \frac{-\bar{\bar{\alpha}} + \epsilon_2}{\sqrt{n_\theta}\epsilon_1} \Longrightarrow \quad (19)$$

$$-\epsilon_2 < \dot{V}(t_k^+) < -\bar{\bar{\alpha}} < 0 \quad (20)$$

(C3) $x_k \notin \mathcal{U}$ and $\left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| < \hat{\epsilon}_1$. By using lemmas 2 and 3 it can be seen that,

$$\left|\frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{\bar{g}}(x_k) - \frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \leq c_5\bar{\epsilon} \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell| \quad (21)$$

where c_5 is a positive constant that depends on $\sup_{\ell\in\{\ell_k-1,\ldots,k\}} |x_\ell|$. Therefore, if (12) holds and $\bar{\epsilon}$ is sufficiently small we have that (C3) never takes place as long as and $\sup_{\ell\in\{\ell_k-1,\ldots,k\}} |x_\ell| < \infty, \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} |\theta_\ell| < \infty$.

(C4) $x_k \in \mathcal{U}$ and $\left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \geq \hat{\epsilon}_1$. By using (21) it can be readily seen that if $\bar{\epsilon}$ is sufficiently small and (12) holds, then

$$\left|\frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{\bar{g}}(x_k)\right| > \frac{1}{4}\epsilon_1 \quad (22)$$

Then, by replacing in (C1) the constant ϵ_1 by $1/4\epsilon_1$ and using the same arguments as in (C1), we obtain

$$\begin{aligned}\frac{1}{4n_\theta}\epsilon_1\alpha_k > & \frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{f}(x_k) \\ & + c_3\bar{\epsilon}a_1(x_{[t_{\ell_k-1},t_k)})\sqrt{n_\theta} \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} \alpha_\ell \\ & + c_4\bar{\epsilon}a_2(x_{[t_{\ell_k-1},t_k)})n_\theta\alpha_k \sup_{\ell\in\{\ell_k-1,\ldots,k-1\}} \alpha_\ell \\ & + \bar{\alpha} \Longrightarrow \quad (23)\\ & \dot{V}(t_k^+) < -\bar{\alpha} \quad (24)\end{aligned}$$

In conclusion, from the analysis in (C1)-(C4) above we have that, if (23) and (19) hold for finite α_k (note that (23) implies that the weaker condition (17) also holds) and $\bar{\epsilon}$ is sufficiently small, then either $\dot{V}(t_k^+) < -\bar{\alpha}$ or $\dot{V}(t_k^+) < -\bar{\bar{\alpha}}$ for all $t_k \in [0,\omega), |x(t_k)| \geq \epsilon_3$; therefore if (23) and (19) hold for finite α_k and Δt is sufficiently small (wrt $\bar{\alpha}, \bar{\bar{\alpha}}$) we have – since V is smooth and $\bar{f}, \bar{g}, \pi$ are at least C^1 – that $\dot{V}(t)$ is negative for all $t \in [0,\omega), |x(t_k)| \geq \epsilon_3$. On the other hand, as long as $x(t), t \in [0, t_k)$ is bounded, it is easy to see that there exist a sequence of finite α_k satisfying conditions (23) and (19) (provided that $\bar{\epsilon}$ is sufficiently small wrt ϵ_1). Therefore, we can establish there exists a sequence of finite α_k such that all closed-loop signals are bounded and (13) holds by using a standard cyclic argument, i.e., the fact that $x(t_L^+)$ is bounded leads to $\dot{V}(t)$ being negative for a time-interval $[t_L^+, T_1)$ of non-negligible length (unless $|x(t)| < \epsilon_3$ in $[t_L^+, T_1)$) which in turns implies that $x(T_1^+)$ is bounded and, so on. The proof is concluded by noticing that (14) implies (23) and (19) provided that β_1, β_2 satisfy the following:

$$\frac{1}{4n_\theta}\epsilon_1\beta_1 > \frac{\partial V}{\partial x}^\tau(x_k)\bar{\vartheta}\bar{f}(x_k) + c_3\bar{\epsilon}a_1(x_{[t_{\ell_k-1},t_k)})\sqrt{n_\theta}\beta_2 + c_4\bar{\epsilon}a_2(x_{[t_{\ell_k-1},t_k)})n_\theta\beta_2^2 + \bar{\alpha}, \forall k \in \mathcal{Z}$$

which, in turn, is satisfied[1] if (10) is satisfied for appropriately defined $\lambda_1, \lambda_2, \lambda_3$.

Some remarks are in order:

- The condition "with probability 1" in (13) can be removed if a rank test is performed at each t_k to the matrices Ψ_k^i (defined in (4)); in the zero-probability case where one of this matrices has rank less than $\min\{k, L\}$ a different choice of random vectors $\theta_k^{(j)}$ should be generated until $\text{rank}(\Psi_k^i) = \min(k, L)$ for all i. However, it has to be emphasized that a rank test is computationally expensive since it requires $\mathcal{O}(L^3)$ computations.
- The proposed scheme guarantees arbitrarily good transient performance outside the regions of the state space where the system becomes uncontrollable (i.e. for $x_k \notin \mathcal{U}$). To see this, notice that the larger are the design terms α_k for $t_k : \left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \geq \hat{\epsilon}_1$ and $k \geq L$, the larger is the constant $\bar{\alpha}$ in (14), and thus the "more negative" is the time-derivative $\dot{V}(t_k^+)$. On the other hand, the terms α_k for $t_k : \left|\frac{\partial v}{\partial x}^\tau(\hat{\bar{\vartheta}}_k, x_k)\hat{\vartheta}_k g(x_k)\right| \geq \hat{\epsilon}_1$ and $k \geq L$ can be made arbitrarily large, since from (10) we have that the

[1] It is not difficult for someone to see that an upper bound of the terms $a_1(x_{[t_{\ell_k-1},t_k)})$ and $a_2(x_{[t_{\ell_k-1},t_k)})$ can be found that it is independent of $\bar{\epsilon}$; to see this, notice that $\dot{V}(t_k^+) < 0$ for all $k \geq L : x(t_k) \neq 0$ and therefore we have that $V(t) \leq \bar{V}, \forall t \geq t_k$ where $\bar{V}$ denotes a finite positive constant that depends on β_0, L and Δt and thus $x(t) \leq \bar{X}, \forall t \geq t_k$ where, similarly to $\bar{V}$, $\bar{X}$ is a finite positive constant that depends on β_0, L and Δt.

bounds β_1, β_2 – and thus α_k – can be made arbitrarily large by choosing $\bar{\epsilon}$ sufficiently small.

3. CONCLUSIONS

In this note a computationally efficient adaptive control design for a general class of nonlinear multi-input systems has been proposed and analyzed. Among the advantages of the proposed design is that it can guarantee arbitrarily good transient performance outside the regions of the state space where the system becomes uncontrollable, by increasing the magnitude of the constant $\bar{\alpha}$ defined in the proof of Theorem 4.

It is worth noticing that the analysis concentrates in the ideal case of parametric uncertainties. Our current research involves the extension of the results of this paper to the case of non-parametric uncertainties such as unmodeled nonlinearities/dynamics and exogenous disturbances, as well as the on-line calculation of the constants β_i of Theorem 4.

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