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Generalized Coupled Algebraic Riccati Equations for Discrete-Time Markov Jump with Multiplicative Noise Systems ⋆

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Abstract: In this paper we consider the existence of the maximal and mean square stabilizing solutions for a set of generalized coupled algebraic Riccati equations (GCARE for short) associated to the infinite-horizon stochastic quadratic optimal control problem of discrete-time Markov jump with multiplicative noise linear systems. The weighting matrices of the state and control for the quadratic part are allowed to be indefinite. We present a sufficient condition under which there exists the maximal solution and a necessary and sufficient condition under which there exists the mean square stabilizing solution for the GCARE.

1. INTRODUCTION

The indefinite stochastic linear control with multiplicative noise has been intensively studied lately (see, for instance, Ait Rami and Zhou (2000), Chen et al. (1998), Lim and Zhou (1999), Wu and Zhou (2002)). In Costa and de Paulo (2007) the finite-horizon stochastic optimal control problem of discrete-time Markov jump with multiplicative noise linear systems, with the performance criterion formed by a quadratic part and a linear part in the state and control variables is considered, with the weighting matrices of the state and control for the quadratic part allowed to be indefinite. The optimal control law is written in terms of a set of coupled generalized Riccati difference equations interconnected with a set of coupled linear recursive equations. In this paper we analyze the generalized coupled algebraic Riccati equations (GCARE for short) associated to this problem. Our main results are to derive sufficient conditions for the existence of the maximal solution, and necessary and sufficient conditions for the existence of the mean square stabilizing solution for the GCARE. To the best of our knowledge there is no other work handling this kind of problem in the literature. Indeed previous works on the coupled algebraic Riccati equation for the discrete-time case, as in Abou-Kandil et al. (1995), Ait Rami et al. (2001), Costa and Marques (1999), Czornik and Swierniak (2001), Ji et al. (1991), Morozan (1995), Morozan (1998), considered only positive semi-definite weighting matrices of the state and control and/or didn't consider the multiplicative noise.

This paper is organized in the following way. Section 2 presents the notation and some definitions that will be used throughout the work, and the formulation of the problem. Section 3 presents some auxiliary results which are crucial for the development of our results. Section 4 presents the main results regarding the GCARE, which consist of providing a sufficient condition for the existence of the maximal solution and a necessary and sufficient condition for the existence of the mean square stabilizing solution. Section 5 presents a numerical example. The paper is concluded with some final remarks.

2. PRELIMINARIES

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces we set $\mathbb{B}(\mathbb{X},\mathbb{Y})$ the Banach space of all bounded linear operators of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|.\|$. For simplicity we shall set $\mathbb{B}(\mathbb{X}) := \mathbb{B}(\mathbb{X},\mathbb{X})$. The spectral radius of an operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ will be denoted by $r_\sigma(\mathcal{T})$. If $\mathbb{X}$ is a Hilbert space then the inner product will be denoted by $\langle .;. \rangle$, and for $\mathcal{T} \in \mathbb{B}(\mathbb{X})$, $\mathcal{T}^*$ will denote the adjoint operator of $\mathcal{T}$. As usual, $\mathcal{T} \geq 0$ ($\mathcal{T} > 0$ respectively) will denote that the operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ will be positive semi-definite (positive definite). In particular we shall denote by $\mathbb{C}^n$ the n dimensional complex Euclidean spaces and by $\mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$ the normed bounded linear space of all $m \times n$ complex matrices, with $\mathbb{B}(\mathbb{C}^n) := \mathbb{B}(\mathbb{C}^n,\mathbb{C}^n)$.

Set $\mathbb{H}^{n,m}$ the linear space made up of all N-sequences of complex matrices $V = (V_1, ..., V_N)$ with $V_i \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $i = 1,\dots,N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. For $V = (V_1, ..., V_N) \in \mathbb{H}^{n,m}$, we consider the following norms in $\mathbb{H}^{n,m}$: $\|V\|_1 := \sum_{i=1}^N \|V_i\|$, $\|V\|_2 := \Big(\sum_{i=1}^N tr\big(V_i^* V_i\big)\Big)^{\frac{1}{2}}$. It is easy to verify that $\mathbb{H}^{n,m}$ equipped with any of the above norms is a Banach space and, in fact, $\big(\|.\|_2,\mathbb{H}^{n,m}\big)$ is a Hilbert space, with the inner product given, for $V = (V_1, ..., V_N)$ and $S = (S_1, ..., S_N)$ in $\mathbb{H}^{n,m}$, by $\langle V; S\rangle = \sum_{i=1}^N tr\big(V_i^* S_i\big)$. We shall say that $V = (V_1, ..., V_N) \in \mathbb{H}^n$ is hermitian if $V_i = V_i^*$ for $i = 1,\dots,N$, and denote this set by $\mathbb{H}^{n*}$. We shall write $\mathbb{H}^{n+} := \big\{V = (V_1, ..., V_N) \in \mathbb{H}^{n*}; V_i \geq 0,\ i = 1,\dots,N\big\}$ and for $V \in \mathbb{H}^n$, $S \in \mathbb{H}^n$, we write that $V \geq S$ if $V - S = (V_1 - S_1, ..., V_N - S_N) \in \mathbb{H}^{n+}$, and that $V > S$ if $V_i - S_i > 0$ for each $i = 1,\dots,N$.

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On a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$ we consider the following Markov Jump Linear System with multiplicative noise:

$$\begin{aligned} x(k+1) =& \Big(\bar{A}_{\theta(k)} + \sum_{s=1}^{\nu^x} \widetilde{A}_{\theta(k),s} w_s^x(k)\Big) x(k) \\ &+ \Big(\bar{B}_{\theta(k)} + \sum_{s=1}^{\nu^u} \widetilde{B}_{\theta(k),s} w_s^u(k)\Big) u(k) \\ x(0) =& x_0, \theta(0) = \theta_0, \end{aligned} \tag{1}$$

where $\theta(k)$ denotes a time-invariant Markov chain taking values in $\{1, ..., N\}$ with transition probability matrix $\mathbb{P} = [p_{ij}]$, $\{w_s^x(k); s = 1, \ldots \nu^x, k = 0, 1, \ldots\}$ and $\{w_s^u(k); s = 1, \ldots \nu^u, k = 0, 1, \ldots\}$ are zero-mean random variables with variance equal to 1, $E(w_i^x(k) w_j^x(k)) = 0$, $E(w_i^u(k) w_j^u(k)) = 0$, for all k and $i \neq j$, and independent of the Markov chain $\{\theta(k)\}$. The initial conditions θ_0 and x_0 are assumed to be independent of $\{w_s^x(k)\}$, $\{w_s^u(k)\}$, with x_0 an n-dimensional random vector with finite second moments. The mutual correlation between $w_{s_1}^x(k)$ and $w_{s_2}^u(k)$ is denoted by $E(w_{s_1}^x(k) w_{s_2}^u(k)) = \rho_{s_1,s_2}$. Without loss of generality, we assume that $\nu = \nu^x = \nu^u$. We also have $\bar{A} = (\bar{A}_1, \ldots, \bar{A}_N) \in \mathbb{H}^n$, $\widetilde{A}_s = (\widetilde{A}_{1,s}, \ldots, \widetilde{A}_{N,s}) \in \mathbb{H}^n$, $s = 1, \ldots, \nu$, $\bar{B} = (\bar{B}_1, \ldots, \bar{B}_N) \in \mathbb{H}^{m,n}$, $\widetilde{B}_s = (\widetilde{B}_{1,s}, \ldots, \widetilde{B}_{N,s}) \in \mathbb{H}^{m,n}$ $s = 1, \ldots, \nu$. We set $\mathcal{F}_\tau$ the σ-field generated by $\{(\theta(t), x(t)); t = 0, \ldots, \tau\}$, and the set of admissible controllers $\mathbb{U}$ is defined as $\mathbb{U} = \{u = (u(0), \ldots); u(k)$ is an m-dimensional random vector with finite second moments, $\mathcal{F}_k$-measurable for each $k = 0, \ldots$ and yields in (1) $E(\|x(T)\|^2\|) \to 0$ as $k \to \infty\}$. Consider $Q = (Q_1, \ldots, Q_N) \in \mathbb{H}^{n*}$, $L = (L_1, \ldots, L_N) \in \mathbb{H}^{n,1}$, $M = (M_1, \ldots, M_N) \in \mathbb{H}^{m*}$, $H = (H_1, \ldots, H_N) \in \mathbb{H}^{m,1}$. The infinite-horizon indefinite quadratic optimal control problems associated to (1) is defined as

$$J(x_0, \theta_0) = \inf_{u \in \mathbb{U}} \sum_{k=0}^{T-1} E\Big(x(k)^* Q_{\theta(k)} x(k) + u(k)^* M_{\theta(k)} u(k)\Big). \tag{2}$$

Notice that the quadratic cost matrices Q_i and M_i are just assumed to be hermitian. We define next the following operators $\mathcal{E} \in \mathbb{B}(\mathbb{H}^n)$, $\mathcal{A} \in \mathbb{B}(\mathbb{H}^n)$, $\mathcal{G} \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^{n,m})$, $\mathcal{R} \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^m)$,

$$\mathcal{E}_i(X) = \sum_{j=1}^{N} p_{ij} X_j,$$

$$\mathcal{A}_i(X) = Q_i + \bar{A}_i^* \mathcal{E}_i(X) \bar{A}_i + \sum_{s=1}^{\nu} \widetilde{A}_{i,s}^* \mathcal{E}_i(X) \widetilde{A}_{i,s}, \tag{3}$$

$$\mathcal{G}_i(X) = \Big(\bar{A}_i^* \mathcal{E}_i(X) \bar{B}_i + \sum_{s_1=1}^{\nu} \sum_{s_2=1}^{\nu} \rho_{s_1,s_2} \widetilde{A}_{i,s_1}^* \mathcal{E}_i(X) \widetilde{B}_{i,s_2}\Big)^*, \tag{4}$$

$$\mathcal{R}_i(X) = \bar{B}_i^* \mathcal{E}_i(X) \bar{B}_i + \sum_{s=1}^{\nu} \widetilde{B}_{i,s}^* \mathcal{E}_i(X) \widetilde{B}_{i,s} + M_i. \tag{5}$$

Set $\mathbb{L} := \{X \in \mathbb{H}^{n*}; \mathcal{R}_i(X)^{-1}$ exists for each $i = 1, ..., N\}$ and define $\mathcal{S} \in \mathbb{B}(\mathbb{L}, \mathbb{H}^n)$ and $\mathcal{K} \in \mathbb{B}(\mathbb{L}, \mathbb{H}^{m,n})$ as follows. For $X \in \mathbb{L}$ and $i = 1, \ldots, N$,

$$\mathcal{S}_i(X) = -X_i + \mathcal{A}_i(X) - \mathcal{G}_i(X)^* \mathcal{R}_i(X)^{-1} \mathcal{G}_i(X), \tag{6}$$

$$\mathcal{K}_i(X) = -\mathcal{R}_i(X)^{-1} \mathcal{G}_i(X). \tag{7}$$

We will study the following set of generalized coupled algebraic Riccati equations (GCARE) associated to problem (2) (see Costa and de Paulo (2007)):

$$\mathcal{S}(X) = 0. \tag{8}$$

We introduce the following notation: $\mathbb{N} := \{X \in \mathbb{L}; \mathcal{R}(X) > 0\}$, $\mathbb{M} := \{X \in \mathbb{N}; \mathcal{S}(X) \geq 0\}$, $\widehat{\mathbb{M}} := \{X \in \mathbb{N}; \mathcal{S}(X) = 0\}$. For $K = (K_1, ..., K_N) \in \mathbb{H}^{n,m}$ we define the following operators $\mathcal{L}_K(.) = (\mathcal{L}_{K,1}(.), \ldots, \mathcal{L}_{K,N}(.)) \in \mathbb{B}(\mathbb{H}^n)$ and $\mathcal{T}_K(.) = (\mathcal{T}_{K,1}(.), \ldots, \mathcal{T}_{K,N}(.)) \in \mathbb{B}(\mathbb{H}^n)$:

$$\begin{aligned} &\mathcal{L}_{K,i}(V) = (\bar{A}_i + \bar{B}_i K_i)^* \mathcal{E}_i(V) (\bar{A}_i + \bar{B}_i K_i) + \\ &\sum_{s=1}^{\nu} \widetilde{A}_{i,s}^* \mathcal{E}_i(V) \widetilde{A}_{i,s} + \sum_{s_1=1}^{\nu} \sum_{s_2=1}^{\nu} \rho_{s_1,s_2} \big(\widetilde{A}_{i,s_1}^* \mathcal{E}_i(V) \widetilde{B}_{i,s_2} K_i + \\ &K_i^* \widetilde{B}_{i,s_2}^* \mathcal{E}_i(V) \widetilde{A}_{i,s_1}\big) + \sum_{s=1}^{\nu} K_i^* \widetilde{B}_{i,s}^* \mathcal{E}_i(V) \widetilde{B}_{i,s} K_i, \end{aligned} \tag{9}$$

$$\begin{aligned} \mathcal{T}_{K,j}(V) =& \sum_{i=1}^{N} p_{ij} \big[(\bar{A}_i + \bar{B}_i K_i) V_i (\bar{A}_i + \bar{B}_i K_i)^* \\ &+ \sum_{s=1}^{\nu} \widetilde{A}_{i,s} V_i \widetilde{A}_{i,s}^* + \sum_{s_1=1}^{\nu} \sum_{s_2=1}^{\nu} \rho_{s_1,s_2} \big(\widetilde{A}_{i,s_1} V_i K_i^* \widetilde{B}_{i,s_2}^* \\ &+ \widetilde{B}_{i,s_2} K_i V_i \widetilde{A}_{i,s_1}^*\big) + \sum_{s=1}^{\nu} \widetilde{B}_{i,s} K_i V_i K_i^* \widetilde{B}_{i,s}^*\big], \end{aligned} \tag{10}$$

where $V = (V_1, ..., V_N) \in \mathbb{H}^n$. It is easy to verify that with the inner product defined above we have that $\mathcal{T}_K = \mathcal{L}_K^*$. It is also easy to check that the operators $\mathcal{L}_K$, and $\mathcal{T}_K$ map $\mathbb{H}^{n*}$ into $\mathbb{H}^{n*}$.

Consider model (1) with $u(k) = K_{\theta(k)} x(k)$, where $K = (K_1, ..., K_N) \in \mathbb{H}^{n,m}$, and $w(k) = 0$. Using the same arguments as in Proposition 3.1 of Costa et al. (2005), page 32, it follows that, for $U_i(k) = E(x(k) x(k)^* 1_{\{\theta(k)=i\}})$, $U(k) = (U_1(k), \ldots, U_N(k)) \in \mathbb{H}^{n+}$, we have that

$$U(k+1) = \mathcal{T}_K(U(k)),\ k = 0, 1, \ldots \tag{11}$$

where $\mathcal{T}_K$ is as in (10). Similarly, for $P = (P_1, \ldots, P_N) \in \mathbb{H}^{n+}$, we have that

$$\begin{aligned} E(x(k+1)^* P_{\theta(k+1)} x(k+1) | \theta(k), x(k)) = \\ x(k)^* \mathcal{L}_{K,\theta(k)}(P) x(k). \end{aligned} \tag{12}$$

¿From (11) and (12) it follows that $\mathcal{T}_K$ and $\mathcal{L}_K$ map $\mathbb{H}^{n+}$ into $\mathbb{H}^{n+}$. We define next the stability and stabilizability concepts that we shall consider in the following sections.

Definition 1. We say that $K = (K_1, ..., K_N) \in \mathbb{H}^{n,m}$ stabilizes (1) in the mean square sense if, when we make $u(k) = K_{\theta(k)} x(k)$ in system (1) with $w(k) = 0$, we have that $E(\|x(k)\|^2) \to 0$ as $k \to \infty$ for any initial condition $x(0)$ and $\theta(0)$. We say that (1) is mean square stabilizable if for some $K = (K_1, \ldots, K_N) \in \mathbb{H}^{n,m}$ we have that K stabilizes (1) in the mean square sense.

Definition 2. We say that $X = (X_1, ..., X_N) \in \mathbb{H}^{n*}$ is a hermitian solution for the GCARE if $X \in \mathbb{L}$ and satisfies (8). We say that X is a maximal solution if it is an hermitian solution for the GCARE and $X \geq Y$ for any $Y \in \mathbb{M}$. We say that X is a mean square stabilizing solution if it is an hermitian solution for the GCARE and $\mathcal{K}(X)$ stabilizes (1) in the mean square sense.

Using the same arguments as in Proposition 3.25 of Costa et al. (2005), page 44, or in Dragan and Morozan

(2006), we have the following result showing that $K = (K_1, ..., K_N)$ stabilizes system (1) in the mean square sense if and only if the expectral radius of the operator (10) is less than one.

Lemma 1. $K = (K_1, ..., K_N) \in \mathbb{H}^{n,m}$ stabilizes (1) in the mean square sense if and only if $r_\sigma(\mathcal{T}_K) < 1$, where $\mathcal{T}_K$ is as in (10).

3. AUXILIARY RESULTS

The next lemmas will be crucial for the development of the main results of this paper.

Lemma 2. Suppose that $r_\sigma(\mathcal{L}_F) < 1$ for some $F = (F_1, ..., F_N) \in \mathbb{H}^{n,m}$, where $\mathcal{L}_F$ is defined as in (9) and consider $G = (G_1, \ldots, G_N) \in \mathbb{H}^{n,m}$. If for some $P = (P_1, \ldots, P_N) \in \mathbb{H}^{n+}$ and $\delta > 0$,

$$P_i - \mathcal{L}_{G,i}(P) \geq \delta(G_i - F_i)^*(G_i - F_i), \ \ i = 1, \ldots, N, \quad (13)$$

then $r_\sigma(\mathcal{L}_G) < 1$.

Proof. Note that for arbitrary $\epsilon > 0$, $V = (V_1, \ldots, V_N) \in \mathbb{H}^{n+}$, and any $k \geq 0$ and $i = 1, \ldots, N$,

$$0 \leq E\Big[\Big\{\epsilon\Big(\bar{A}_i + \bar{B}_i F_i + \sum_{s=1}^{\nu} \widetilde{A}_{i,s} w_s^x(k) + \sum_{s=1}^{\nu} \widetilde{B}_{i,s} F_i w_s^u(k)\Big) - \frac{1}{\epsilon}\Big(\bar{B}_i(G_i - F_i) + \sum_{s=1}^{\nu} \widetilde{B}_{i,s}(G_i - F_i) w_s^u(k)\Big)\Big\} V_i \times \Big\{\epsilon\Big(\bar{A}_i + \bar{B}_i F_i + \sum_{s=1}^{\nu} \widetilde{A}_{i,s} w_s^x(k) + \sum_{s=1}^{\nu} \widetilde{B}_{i,s} F_i w_s^u(k)\Big) - \frac{1}{\epsilon}\Big(\bar{B}_i(G_i - F_i) + \sum_{s=1}^{\nu} \widetilde{B}_{i,s}(G_i - F_i) w_s^u(k)\Big)\Big\}^*\Big]. \quad (14)$$

Defining $\mathcal{Q}(V) \in \mathbb{H}^{n+}$ as

$$\mathcal{Q}_j(V) = \sum_{i=1}^{N} p_{ij}\Big[\bar{B}_i(G_i - F_i)V_i(G_i - F_i)^*\bar{B}_i^* + \sum_{s=1}^{\nu} \widetilde{B}_{i,s}(G_i - F_i)V_i(G_i - F_i)^*\widetilde{B}_{i,s}^*\Big]. \quad (15)$$

we have that (14) yields

$$0 \leq \mathcal{T}_{G,j}(V) \leq (1 + \epsilon^2)\mathcal{T}_{F,j}(V) + (1 + \frac{1}{\epsilon^2})\mathcal{Q}_j(V) \quad (16)$$

where we recall that $\mathcal{T}_G = \mathcal{L}_G^*$ and $\mathcal{T}_F = \mathcal{L}_F^*$. We choose now $\epsilon > 0$ such that $(1 + \epsilon^2)r_\sigma(\mathcal{T}_F) < 1$. This is possible since by assumption $r_\sigma(\mathcal{L}_F) < 1$ and $r_\sigma(\mathcal{T}_F) = r_\sigma(\mathcal{L}_F)$ since $\mathcal{T}_F = \mathcal{L}_F^*$. Set $\widehat{\mathcal{T}} = (1 + \epsilon^2)\mathcal{T}_F$. Define for $t = 0, 1, \ldots$ the sequences

$$X(t+1) = \mathcal{T}_G(X(t)), \quad X(0) \in \mathbb{H}^{n+}, \quad (17)$$

$$Y(t+1) = \widehat{\mathcal{T}}(Y(t)) + \widehat{\mathcal{Q}}(X(t)), \quad Y(0) = X(0) \quad (18)$$

with $\widehat{\mathcal{Q}}(X(t)) = (1 + \frac{1}{\epsilon^2})\mathcal{Q}(.)$. Then for $t = 0, 1, 2, \ldots$

$$Y(t) \geq X(t) \geq 0. \quad (19)$$

Indeed, (19) is immediate from (17), (18), for $t = 0$. Suppose (19) holds for t. Then from (16) and recalling that $\mathcal{T}_G$ and $\mathcal{T}_F$ map $\mathbb{H}^{n+}$ into $\mathbb{H}^{n+}$, we have that

$$Y(t+1) = \widehat{\mathcal{T}}(Y(t)) + \widehat{\mathcal{Q}}(X(t)) \geq X(t+1) \geq 0,$$

showing the result for $t + 1$. From (18) it follows that

$$Y(t) = \widehat{\mathcal{T}}^t(X(0)) + \sum_{s=0}^{t-1} \widehat{\mathcal{T}}^{t-1-s}\widehat{\mathcal{Q}}(X(s))$$

and taking the 1-norm of the above expression, we have that

$$\|Y(t)\|_1 \leq \Big\|\widehat{\mathcal{T}}^t\Big\| \|X(0)\|_1 + \sum_{s=0}^{t-1} \Big\|\widehat{\mathcal{T}}^{t-1-s}\Big\| \Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1.$$

Since $r_\sigma(\widehat{\mathcal{T}}) < 1$, it is possible to find $a > 0$, $0 < b < 1$, such that $\Big\|\widehat{\mathcal{T}}^s\Big\| \leq ab^s$, $s = 0, 1, \ldots$ (see, for instance, Kubrusly (1985)), and thus,

$$\|Y(t)\|_1 \leq ab^t \|X(0)\|_1 + a\sum_{s=0}^{t-1} b^{t-1-s} \Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1.$$

Suppose for the moment that $\sum_{s=0}^{\infty} \Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1 < \infty$. Then from (17) and (19), for any $X(0) = (X_1(0), \ldots, X_N(0)) \in \mathbb{H}^{n+}$

$$0 \leq \sum_{t=0}^{\infty} \Big\|\mathcal{T}_G^t(X(0))\Big\|_1 = \sum_{t=0}^{\infty} \|X(t)\|_1 \leq \sum_{t=0}^{\infty} \|Y(t)\|_1 < \infty.$$

and thus (see Proposition 2.5 in Costa et al. (2005)) $r_\sigma(\mathcal{T}_G) < 1$, and since $\mathcal{T}_G = \mathcal{L}_G^*$, $r_\sigma(\mathcal{L}_G) < 1$. Remains to prove that $\sum_{s=0}^{\infty} \Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1 < \infty$. Indeed, from (13) we obtain, for an appropriate positive constant c_0, that

$$\Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1 \leq c_0 \Big\{\Big\langle X(s); P\Big\rangle - \Big\langle X(s+1); P\Big\rangle\Big\}.$$

Taking the sum for $s = 0$ to r, we get that

$$\sum_{s=0}^{r} \Big\|\widehat{\mathcal{Q}}(X(s))\Big\|_1 \leq c_0\Big\langle X(0); P\Big\rangle,$$

since that $P \in \mathbb{H}^{n+}$ and $X(r+1) \in \mathbb{H}^{n+}$. Taking the limit as $r \to \infty$, we obtain the desired result. ■

Lemma 3. For some $K = (K_1, ..., K_N) \in \mathbb{H}^{n,m}$, let $\mathcal{L}_K$ be as defined in (9). If $r_\sigma(\mathcal{L}_K) < 1$ then for any $S = (S_1, \ldots, S_N) \in \mathbb{H}^n$ there exists a unique solution $Y = (Y_1, \ldots, Y_N) \in \mathbb{H}^n$ which satisfies

$$Y_i - \mathcal{L}_{K,i}(Y) = S_i, \ i = 1, \ldots, N. \quad (20)$$

Moreover if S is hermitian (≥ 0, > 0 respectively) then Y is hermitian (≥ 0, > 0). Conversely if for some $S > 0$ there exists $Y > 0$ satisfying equation (20) then $r_\sigma(\mathcal{L}_K) < 1$.

Proof. If $r_\sigma(\mathcal{L}_K) < 1$, we have that (see Weidmann (1980), page 102), $(\mathcal{I} - \mathcal{L}_K)^{-1}(.) = \sum_{j=0}^{\infty} \mathcal{L}_K^j(.)$ where $\mathcal{I}$ represents the identity operator. Therefore the unique solution Y of (20) is given by $Y = \sum_{j=0}^{\infty} \mathcal{L}^j(S)$ and if $S \in \mathbb{H}^{n*}$ ($S \in \mathbb{H}^{n+}$, $S > 0$ respectively) then $Y \in \mathbb{H}^{n*}$ ($Y \in \mathbb{H}^{n+}$, $Y > 0$). For the remaining of the proof, see Theorem 3.19 of Costa et al. (2005), page 41. ■

Finally we conclude this section with the following lemma (see Oostveen and Zwart (1996) for similar results).

Lemma 4. Suppose that $X \in \mathbb{L}$ and for some $\widehat{F} \in \mathbb{H}^{n,m}$ we have that $\widehat{X} \in \mathbb{H}^{n*}$ satisfies for $i = 1, \ldots, N$

$$\widehat{X}_i - \mathcal{L}_{\widehat{F},i}(\widehat{X}) = Q_i + \widehat{F}_i^* M_i \widehat{F}_i. \quad (21)$$

Then, for $i = 1, \ldots, N$,

$$(\widehat{X}_i - X_i) - \mathcal{L}_{\widehat{F},i}(\widehat{X} - X) = \mathcal{S}_i(X) + (\widehat{F}_i - \mathcal{K}_i(X))^*\mathcal{R}_i(X)(\widehat{F}_i - \mathcal{K}_i(X)). \quad (22)$$

Moreover, if $\widehat{X} \in \mathbb{L}$, then for $i = 1, \ldots, N$

$$(\widehat{X}_i - X_i) - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{X} - X) = \mathcal{S}_i(X) + (\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X))^*\mathcal{R}_i(X)(\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X)) + (\widehat{F}_i - \mathcal{K}_i(\widehat{X}))^*\mathcal{R}_i(\widehat{X})(\widehat{F}_i - \mathcal{K}_i(\widehat{X})). \quad (23)$$

Furthermore, if $\widehat{Y} \in \mathbb{H}^{n*}$ and satisfies, for $i = 1, \dots, N$

$$\widehat{Y}_i - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{Y}) = Q_i + \mathcal{K}(\widehat{X})^* M_i \mathcal{K}(\widehat{X}). \quad (24)$$

then for $i = 1, \dots, N$,

$$(\widehat{X}_i - \widehat{Y}_i) - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{X} - \widehat{Y}) = (\widehat{F}_i - \mathcal{K}_i(\widehat{X}))^*\mathcal{R}_i(\widehat{X})(\widehat{F}_i - \mathcal{K}_i(\widehat{X})). \quad (25)$$

Proof. After some algebraic manipulations, we have for $i = 1, \dots, N$ that

$$X_i - \mathcal{L}_{\widehat{F},i}(X) = Q_i + \widehat{F}_i^* M_i \widehat{F}_i - (\mathcal{K}_i(X) - \widehat{F}_i)^*\mathcal{R}_i(X)(\mathcal{K}_i(X) - \widehat{F}_i) - \mathcal{S}_i(X), \quad (26)$$

$$X_i - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(X) = Q_i + \mathcal{K}_i(\widehat{X})^* M_i \mathcal{K}_i(\widehat{X}) - \mathcal{S}_i(X) - (\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X))^*\mathcal{R}_i(X)(\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X)), \quad (27)$$

$$\widehat{X}_i - \mathcal{L}_{\widehat{F},i}(\widehat{X}) = Q_i + \widehat{F}_i^* M_i \widehat{F}_i - (\widehat{F}_i - \mathcal{K}_i(\widehat{X}))^*\mathcal{R}_i(\widehat{X})(\widehat{F}_i - \mathcal{K}_i(\widehat{X})) - \mathcal{S}_i(\widehat{X}). \quad (28)$$

¿From (21) and (28) it follows that

$$\widehat{X}_i - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{X}) = Q_i + \mathcal{K}_i(\widehat{X})^* M_i \mathcal{K}_i(\widehat{X}) + (\widehat{F}_i - \mathcal{K}_i(\widehat{X}))^*\mathcal{R}_i(\widehat{X})(\widehat{F}_i - \mathcal{K}_i(\widehat{X})). \quad (29)$$

Thus, (22) follows by taking (21) minus (26), (23) follows by subtracting (27) from (29) and (25) follows subtracting (24) from (29). ■

4. MAXIMAL AND STABILIZING SOLUTIONS

The following theorem provides a sufficient condition for the existence of the maximal hermitian solution of (8).

Theorem 1. Suppose that (1) is mean square stabilizable and $\mathbb{M} \neq \emptyset$. Then for $\ell = 0, 1, 2, \dots$, there exists $X^\ell \in \mathbb{N}$ and $F^\ell \in \mathbb{H}^{n,m}$ satisfying the following properties:

a) $X^0 \geq X^1 \geq \cdots \geq X^\ell \geq X$, for arbitrary $X \in \mathbb{M}$;
b) $r_\sigma(\mathcal{L}_{F^\ell}) < 1$;
c) For $i = 1, \dots, N$,

$$X_i^\ell - \mathcal{L}_{F^\ell,i}(X^\ell) = Q_i + F_i^{\ell*} M_i F_i^\ell. \quad (30)$$

Moreover there exists $X^+ \in \widehat{\mathbb{M}}$ such that $X^+ \geq X$ for any $X \in \mathbb{M}$, $r_\sigma(\mathcal{L}_{\mathcal{K}(X^+)}) \leq 1$, and $X^\ell \to X^+$ as $\ell \to \infty$.

Proof. Let us apply induction on ℓ to show the result. Consider an arbitrary $X \in \mathbb{M}$ (thus $\mathcal{S}(X) \geq 0$) and $F = \mathcal{K}(X)$. Since that (1) is mean square stabilizable we can find $F^0 \in \mathbb{H}^{n,m}$ such that $r_\sigma(\mathcal{L}_{F^0}) < 1$ (see Definition 1 and Lemma 1). Thus, from Lemma 3, there exists a unique $X^0 \in \mathbb{H}^{n*}$ satisfying (30) for $\ell = 0$. We have from (22) that for $i = 1, \dots, N$

$$(X_i^0 - X_i) - \mathcal{L}_{F^0,i}(X^0 - X) = \mathcal{S}_i(X) + (F_i^0 - F_i)^*\mathcal{R}_i(X)(F_i^0 - F_i)$$

and since $\mathcal{S}_i(X) + (F_i^0 - F_i)^*\mathcal{R}_i(X)(F_i^0 - F_i) \geq 0$ and $r_\sigma(\mathcal{L}_{F^0}) < 1$ we have from Lemma 3 again that $X^0 - X \geq 0$. This also shows that $X^0 \in \mathbb{N}$, since that for each $i = 1, \dots, N$, $\mathcal{R}_i(X^0) \geq \mathcal{R}_i(X) > 0$ and thus the result is proved for $\ell = 0$. Suppose now that the result holds for $\ell - 1$. Set $F^\ell = \mathcal{K}(X^{\ell-1})$. From equation (23) we get that

$$(X_i^{\ell-1} - X_i) - \mathcal{L}_{F^\ell,i}(X^{\ell-1} - X) = \mathcal{S}_i(X) + (F_i^\ell - F_i)^* \times \mathcal{R}_i(X)(F_i^\ell - F_i) + (F_i^\ell - F_i^{\ell-1})^*\mathcal{R}_i(X^{\ell-1})(F_i^\ell - F_i^{\ell-1}) \geq \delta(F_i^\ell - F_i^{\ell-1})^*(F_i^\ell - F_i^{\ell-1})$$

for some $\delta > 0$ since by the induction hypothesis, $\mathcal{R}_i(X^{\ell-1}) > 0$ for $i = 1, \dots, N$. Thus from Lemma 2, $r_\sigma(\mathcal{L}_{F^\ell}) < 1$. Let $X^\ell \in \mathbb{H}^{n*}$ be the unique solution satisfying (30) (see Lemma 3). Equation (22) yields, for $i = 1, \dots, N$,

$$(X_i^\ell - X_i) - \mathcal{L}_{F^\ell,i}(X^\ell - X) = \mathcal{S}_i(X) + (F_i^\ell - F_i)^*\mathcal{R}_i(X)(F_i^\ell - F_i)$$

and since $r_\sigma(\mathcal{L}_{F^\ell}) < 1$, we get from Lemma 3 that $X^\ell \geq X$. Thus $\mathcal{R}(X^\ell) \geq \mathcal{R}(X) > 0$, which shows that $X^\ell \in \mathbb{N}$. Equation (25) yields for $i = 1, \dots, N$

$$(X_i^{\ell-1} - X_i^\ell) - \mathcal{L}_{F^\ell,i}(X^{\ell-1} - X^\ell) = (F_i^\ell - F_i^{\ell-1})^*\mathcal{R}_i(X^{\ell-1})(F_i^\ell - F_i^{\ell-1})$$

which shows, from the fact that $r_\sigma(\mathcal{L}_{F^\ell}) < 1$, $(F_i^\ell - F_i^{\ell-1})^*\mathcal{R}_i(X^{\ell-1})(F_i^\ell - F_i^{\ell-1}) \geq 0$ for each $i = 1, \dots, N$, and Lemma 3, that $X^{\ell-1} \geq X^\ell \geq X$. This completes the induction argument. Since $\{X^\ell\}_{\ell=0}^\infty$ is a decreasing sequence with $X^\ell \geq X$ for all $\ell = 0, 1, \dots$, we get that there exists X^+ hermitian such that (see Sontag (1990), page 79) $X^\ell \downarrow X^+$ as $\ell \to \infty$. Clearly, $X^+ \geq X$, and thus $\mathcal{R}(X^+) \geq \mathcal{R}(X) > 0$, showing that $X^+ \in \mathbb{N}$. Moreover, substituting $F_i^\ell = \mathcal{K}_i(X^{\ell-1})$ into (30) and taking the limit as $\ell \to \infty$, we get, after rearranging the terms, that $\mathcal{S}(X^+) = 0$, showing the desired result. Since X is arbitrary in $\mathbb{M}$, it follows that $X^+ \geq X$ for all $X \in \mathbb{M}$. Finally notice that since $r_\sigma(\mathcal{L}^k) < 1$ we get that (see Sontag (1990), p. 328 for continuity of the eigenvalues on finite dimensional linear operator entries) $r_\sigma(\mathcal{L}_{F^+}) \leq 1$, where $F^+ = \mathcal{K}(X^+)$. ■

We show next that there exists at most one mean square stabilizing solution for (8).

Lemma 5. If $\mathbb{M} \neq \emptyset$ then there exists at most one mean square stabilizing solution for the GCARE (8), which will coincide with the maximal solution.

Proof. Suppose that $\widehat{X}$ is a mean square stabilizing solution for the GCARE (8). Clearly (1) is mean square stabilizable and since $\mathbb{M} \neq \emptyset$ we get from Theorem 1 that there exists the maximal solution $X^+ \in \widehat{\mathbb{M}}$. We have that

$$\widehat{X}_i - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{X}) = Q_i + \mathcal{K}_i(\widehat{X})^* M_i \mathcal{K}_i(\widehat{X})$$

so that (22) yields

$$(\widehat{X}_i - X_i^+) - \mathcal{L}_{\mathcal{K}(\widehat{X}),i}(\widehat{X} - X^+) = (\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X^+))^*\mathcal{R}_i(X^+)(\mathcal{K}_i(\widehat{X}) - \mathcal{K}_i(X^+)) \geq 0 \quad (31)$$

since $\mathcal{R}_i(X^+) > 0$. Recalling that $\widehat{X}$ is mean square stabilizing, we have from (31) and Lemma 3 that $\widehat{X} - X^+ \geq 0$. But this also implies that $\mathcal{R}(\widehat{X}) \geq \mathcal{R}(X^+) > 0$ and consequently $\widehat{X} \in \mathbb{M}$. From Theorem 1 it follows that $\widehat{X} - X^+ \leq 0$, completing the proof. ■

Our next result provides necessary and sufficient conditions for the existence of the mean square stabilizing solution. We need to define, for $K \in \mathbb{H}^{n,m}$ and $\Gamma \in \mathbb{H}^n$, the following operator $\mathcal{V}_{\Gamma,K} \in \mathbb{B}(\mathbb{H}^n)$: for $V \in \mathbb{H}^n$,

$$\mathcal{V}_{\Gamma,K,i}(V) = (\Gamma_i + \bar{B}_i K_i)^* \mathcal{E}_i(V)(\Gamma_i + \bar{B}_i K_i) + \sum_{s=1}^{\nu} \widetilde{A}_{i,s}^* \mathcal{E}_i(V) \widetilde{A}_{i,s} + \sum_{s_1=1}^{\nu}\sum_{s_2=1}^{\nu} \rho_{s_1,s_2}\big(\widetilde{A}_{i,s_1}^* \mathcal{E}_i(V)\widetilde{B}_{i,s_2} K_i + K_i^* \widetilde{B}_{i,s_2}^* \mathcal{E}_i(V)\widetilde{A}_{i,s_1}\big) + \sum_{s=1}^{\nu} K_i^* \widetilde{B}_{i,s}^* \mathcal{E}_i(V) \widetilde{B}_{i,s} K_i.$$

Clearly we have that $\mathcal{L}_K = \mathcal{V}_{\bar{A},K}$.

Theorem 2. Suppose that $\mathbb{M} \neq \emptyset$. The following assertions are equivalent:

i) system (1) is mean square stabilizable and for some $X \in \mathbb{M}$ there exists $T \in \mathbb{H}^n$ such that $r_\sigma(\mathcal{V}_{\Gamma(X),\mathcal{K}(X)}) < 1$ where $\Gamma_i(X) = \bar{A}_i + T_i \mathcal{S}_i(X)^{\frac{1}{2}}$ for $i = 1, \ldots, N$.
ii) there exists the mean square stabilizing solution to the GCARE (8).

Moreover if $X \in \widehat{\mathbb{M}}$ is the mean square stabilizing solution to the GCARE (8) then an optimal control law for problem (2) is given by

$$\hat{u}(k) = \mathcal{K}_{\theta(k)}(X)x(k). \tag{32}$$

Proof. Let us show first that i) implies ii). From Theorem 1 and the hypothesis that system (1) is mean square stabilizable and $\mathbb{M} \neq \emptyset$ we conclude that there exists the maximal solution $X^+ \in \widehat{\mathbb{M}}$. Consider $X \in \mathbb{M}$ and $T \in \mathbb{H}^n$ satisfying i). Set $F^+ = \mathcal{K}(X^+)$ and $F = \mathcal{K}(X)$. Since

$$X_i^+ - \mathcal{L}_{F^+,i}(X^+) = Q_i + F_i^{+*} M_i F_i^+$$

we have that (22) yields for $i = 1, \ldots, N$,

$$(X_i^+ - X_i) - \mathcal{L}_{F^+,i}(X^+ - X) = \mathcal{S}_i(X) + (F_i^+ - F_i)^* \mathcal{R}_i(X)(F_i^+ - F_i).$$

Since $\mathcal{S}_i(X) \geq 0$ and $\mathcal{R}_i(X) > 0$, $i = 1, \ldots, N$, we get that we can find $\delta > 0$ such that for $i = 1, \ldots, N$,

$$(X_i^+ - X_i) - \mathcal{L}_{F^+,i}(X^+ - X) \geq \delta(\mathcal{S}_i(X) + (F_i^+ - F_i)^*(F_i^+ - F_i)). \tag{33}$$

Define $\widehat{F}^+ \in \mathbb{H}^{n,n+m}$, $\widehat{F} \in \mathbb{H}^{n,n+m}$, $\widehat{\bar{B}} \in \mathbb{H}^{n+m,n}$ and $\widehat{\widetilde{B}} \in \mathbb{H}^{n+m,n}$ as follows: $\widehat{\bar{B}}_i := \begin{pmatrix} T_i & \bar{B}_i \end{pmatrix}$, $\widehat{\widetilde{B}}_i := \begin{pmatrix} 0 & \widetilde{B}_i \end{pmatrix}$ and

$$\widehat{F}_i^+ := \begin{pmatrix} 0 & F_i^+ \end{pmatrix}', \ \ \widehat{F}_i := \begin{pmatrix} \mathcal{S}_i(X)^{\frac{1}{2}} & F_i \end{pmatrix}'.$$

Consider the operator $\widehat{\mathcal{L}}_{\widehat{K}}$ as in (9) replacing $\bar{B}$, $\widetilde{B}$ by respectively $\widehat{\bar{B}}$, $\widehat{\widetilde{B}}$, and K by $\widehat{K} \in \mathbb{H}^{n,n+m}$. Then it is easy to verify that $\widehat{\mathcal{L}}_{\widehat{F}^+} = \mathcal{L}_{F^+}$ and $\widehat{\mathcal{L}}_{\widehat{F}} = \mathcal{V}_{\Gamma,F}$. Thus (33) can be re-written as

$$(X_i^+ - X_i) - \widehat{\mathcal{L}}_{\widehat{F}^+,i}(X^+ - X) \geq \delta(\widehat{F}_i^+ - \widehat{F}_i)^*(\widehat{F}_i^+ - \widehat{F}_i)$$

and recalling that $X^+ - X \geq 0$ and $r_\sigma(\widehat{\mathcal{L}}_{\widehat{F}}) = r_\sigma(\mathcal{V}_{\Gamma,F}) < 1$ we can conclude from Lemma 2 that $r_\sigma(\widehat{\mathcal{L}}_{\widehat{F}^+}) = r_\sigma(\mathcal{L}_{F^+}) < 1$, showing the first part. Let us show now that ii) implies i). Suppose that $X \in \widehat{\mathbb{M}}$ is the mean square stabilizing solution for the GCARE (8). Then clearly (1) will be mean square stabilizable and $\Gamma_i(X) = \bar{A}_i$ (since $\mathcal{S}_i(X) = 0$) so that $\mathcal{V}_{\bar{A},\mathcal{K}(X)} = \mathcal{L}_{\mathcal{K}(X)}$ and the result follows since $r_\sigma(\mathcal{L}_{\mathcal{K}(X)}) < 1$.

Consider now that $X \in \widehat{\mathbb{M}}$ is the mean square stabilizing solution to the GCARE (8) and set $\Lambda(x, i) = x^* X_i x$. From Proposition 2 in Costa and de Paulo (2007) we have that

$$x(k)^* Q_{\theta(k)} x(k) + u(k)^* M_{\theta(k)} u(k) + E\Big(x(k+1)^* X_{\theta(k+1)} x(k+1) | \mathcal{F}_k\Big) = x(k)^* X_{\theta(k)} x(k) + \Big(u(k) - \mathcal{K}_{\theta(k)}(X)x(k)\Big)^* \mathcal{R}_{\theta(k)}(X)\Big(u(k) - \mathcal{K}_{\theta(k)}(X)x(k)\Big)$$

and since $\mathcal{R}(X) > 0$, we get for any $u = (u(0), \ldots) \in \mathbb{U}$ that

$$E(\Lambda(x(T), \theta(T))) - E(\Lambda(x(0), \theta(0))) \geq -E\Big(\sum_{k=0}^{T-1}(x(k)^* Q_{\theta(k)} x(k) + u(k)^* M_{\theta(k)} u(k))\Big) \tag{34}$$

with equality when $u = \hat{u}$ as in (32). From (34) and recalling that $E(\|x(T)\|^2) \to 0$ as $T \to \infty$ we have that

$$E(\Lambda(x(0),\theta(0))) \leq E\Big(\sum_{k=0}^{\infty}(x(k)^* Q_{\theta(k)} x(k) + u(k)^* M_{\theta(k)} u(k))\Big)$$

with equality when $u = \hat{u}$ as in (32), showing the result. ■

We conclude this section establishing a link between a LMI (linear matrix inequality) optimization problem and the maximal solution X^+ in $\mathbb{M}$. Suppose that all matrices involved below are real. Consider the following convex optimization programming problem:

$$\max \ \operatorname{tr}\Big(\sum_{i=1}^{N} X_i\Big)$$
$$\text{subject, for } i = 1, \ldots, N, \text{ to}$$
$$\begin{bmatrix} -X_i + \mathcal{A}_i(X) & \mathcal{G}_i(X)^* \\ \mathcal{G}_i(X) & \mathcal{R}_i(X) \end{bmatrix} \geq 0$$
$$\mathcal{R}_i(X) > 0, X_i = X_i^* \tag{35}$$

Lemma 6. Suppose that (1) is mean square stabilizable. Then there exists $X^+ \in \hat{\mathbb{M}}$ such that $X^+ \geq X$ for all $X \in \mathbb{M}$ if and only if there exists a solution $\hat{X}$ for the above convex programming problem (35). Moreover, $\hat{X} = X^+$.

Proof. First of all notice that, from Schur's complement, $X = (X_1, \ldots, X_N)$ satisfies the restrictions (35) if and only if $-X_i + \mathcal{A}_i(X) - \mathcal{G}_i(X)^* \mathcal{R}_i(X)^{-1} \mathcal{G}_i(X) \geq 0$ and $\mathcal{R}_i(X) > 0$, $X_i = X_i^*$ for $i = 1, \ldots, N$, that is, if and only if $X \in \mathbb{M}$. Thus if $X^+ \in \mathbb{M}$ is such that $X^+ \geq X$ for all $X \in \mathbb{M}$, clearly $tr(X_1^+ + \ldots + X_N^+) \geq tr(X_1 + \cdots + X_N)$ for all $X \in \mathbb{M}$ and since $X^+ \in \widehat{\mathbb{M}} \subset \mathbb{M}$, it follows that X^+ is the solution of the convex programming problem (35). On the other hand, suppose that $\hat{X}$ is a solution of the convex programming problem (35). Thus $\hat{X} \in \mathbb{M} \neq \emptyset$ and from Theorem 1, there exists $X^+ \in \mathbb{M}$ such that $X^+ \geq \hat{X}$. But from the optimality of $\hat{X}$ and the fact that $\widehat{\mathbb{M}} \subset \mathbb{M}$, $tr(X_1^+ - \hat{X}_1) + \ldots + tr(X_N^+ - \hat{X}_N) \leq 0$. Since $X_1^+ - \hat{X}_1 \geq 0, \ldots, X_N^+ - \hat{X}_N \geq 0$, we have $X_1^+ = \hat{X}_1, \ldots, X_N^+ = \hat{X}_N$. ■

5. NUMERICAL EXAMPLE

Consider a system with three operation modes, $i = 1$, $i = 2$, $i = 3$, where the transition probability matrix is given by

$$\mathbb{P} = \begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.3 & 0.47 & 0.23 \\ 0.26 & 0.1 & 0.64 \end{pmatrix}.$$

The Table 1 presents the parameters of the system (1) and of the cost function in (2). From (35), the maximal solution for each mode $i = 1, 2, 3$, is given by

$$X_1 = \begin{pmatrix} 18.9992 & -19.2374 \\ -19.2374 & 28.6941 \end{pmatrix}$$
$$X_2 = \begin{pmatrix} 33.0131 & -23.2143 \\ -23.2143 & 38.0363 \end{pmatrix}$$
$$X_3 = \begin{pmatrix} 36.4552 & -39.8795 \\ -39.8795 & 51.5007 \end{pmatrix}. \quad (36)$$

Considering $K_i = \mathcal{K}_i(X_i)$, with $\mathcal{K}_i(X_i)$ as in (7), we have that $r_\sigma(\mathcal{T}) = 0.6970$. Thus, the system (1) is mean square stabilizable and (36) are mean square stabilizing solutions. The optimal control law (7), for each $i = 1, 2, 3$, is given by

$$\mathcal{K}_1 = (2.3186 \ -2.3342)$$
$$\mathcal{K}_2 = (4.1608 \ -3.7034)$$
$$\mathcal{K}_3 = (-5.1661 \ 5.7921).$$

Table 1. Parameters of the system and of the cost function for each operation mode.

	operation modes		
Parameters	$i = 1$	$i = 2$	$i = 3$
Q_i	$\begin{bmatrix} 3,6 & -3,8 \\ 3,8 & 4,87 \end{bmatrix}$	$\begin{bmatrix} 10 & -3 \\ -3 & 8 \end{bmatrix}$	$\begin{bmatrix} 5 & -4,5 \\ -4,5 & 4,5 \end{bmatrix}$
M_i	$\begin{bmatrix} 2,6 \end{bmatrix}$	$\begin{bmatrix} 1,165 \end{bmatrix}$	$\begin{bmatrix} 1,111 \end{bmatrix}$
$\bar{A}_i$	$\begin{bmatrix} 0 & 1 \\ -2,5 & 3,2 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -4,3 & 4,5 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 5,3 & -5,2 \end{bmatrix}$
$\bar{B}_i$	$\begin{bmatrix} 0 & 1 \end{bmatrix}'$	$\begin{bmatrix} 0 & 1 \end{bmatrix}'$	$\begin{bmatrix} 0 & 1 \end{bmatrix}'$
$\widetilde{A}_i$	$\begin{bmatrix} 0.042 & 0 \\ 00 & 0.065 \end{bmatrix}$	$\begin{bmatrix} 0.065 & 0 \\ 0 & 0.085 \end{bmatrix}$	$\begin{bmatrix} 0.021 & 0 \\ 0 & 0.042 \end{bmatrix}$
$\widetilde{B}_i$	$\begin{bmatrix} 0.042 & 0.065 \end{bmatrix}'$	$\begin{bmatrix} 0.064 & 0.086 \end{bmatrix}'$	$\begin{bmatrix} 0.021 & 0.042 \end{bmatrix}'$
$\rho_{x,u}$	$\begin{bmatrix} 0.58 \end{bmatrix}$	$\begin{bmatrix} 0.58 \end{bmatrix}$	$\begin{bmatrix} 0.58 \end{bmatrix}$

6. FINAL REMARKS

In this paper we have considered the infinite horizon stochastic optimal control problems of discrete-time Markov jump with multiplicative noise linear systems, with indefinite quadratic matrices on the state and control variables. We presented a sufficient condition for the existence of a maximal solution for the set of generalized coupled algebraic Riccati equations (GCARE) that arise from these problems, as well as a necessary and sufficient condition for the existence of the mean square stabilizing solution, and derived an optimal control law whenever this solution exists.

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