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Reduction of Self-localization Errors in Multi-Agent Autonomous Formations ⋆

Iman Shames * **Barış Fidan** * **Brian D. O. Anderson** *

* *Australian National University and National ICT Australia*
Canberra, ACT 0200, Australia,
(e-mail: {Iman.Shames,Baris.Fidan,Brian.Anderson}@anu.edu.au)

Abstract: This paper considers the problem of reduction of self-localization errors in multi-agent autonomous formations when only distance measurement is available to the agents in a globally rigid formation. It is shown that there is a relationship between the singular values of a matrix called *reduced rigidity matrix* and the error induced by measurement error on localization solution. This fact is exploited to introduce an optimal selection of anchors, agents with exactly known positions, which results in a small induced error by measurement errors on localization solution. In the end, some simulation results are presented to demonstrate this optimal anchor selection in globally rigid formations.

Keywords: Localization; Multi-agent systems; Sensor networks; Networks of robots and intelligent sensors; Multi-vehicle systems

1. INTRODUCTION

The topic of multiagent formations has gained much attention in recent years. In order to be able to accomplish most of the tasks associated with multigent autonomous formations, such as reconnaissance and surveillance, the formation should be able to determine its position in a known global coordinate system. For example in surveying an unknown territory, the formation should be able to localize itself in a known global coordinate system in order to successfully record the acquired data in the mission (such as the location of emitters, sensed perhaps using angle of arrival or time-difference-of arrival, see Applewhite (2002),Oh et al. (2007) schemes) and make it possible for the data to be used when the formation has returned to the base. A trivial solution for this problem might be obtained by installing a Global Positioning System (GPS) sensor on each of the agents in the formation. But due to the fact that a precise GPS sensor is expensive and/or constitutes a weight burden, this solution may be impractical. In order to solve the aforementioned problem of determining position information of the agents within the formation, the tools used in the field of multiagent systems localization can be employed. Localization problems have been well-studied in the context of wireless sensor networks. In sensor network localization, it is typically assumed that a small fraction of sensors, called *anchors*, have *a priori* information about their global coordinates (Mao et al. (2006)). Exploiting the fact that the position of these anchor nodes are known in a global coordinate system and that a number of inter-node distances are known, all the other nodes in the network can be localized under a condition which will be discussed later in this paper, i.e. global rigidity of the underlying graph of the network. We can carry over this idea to a formation of mobile agents. We designate some agents as anchor agents, and use them (together with square inter-agent range measurements, typically obtained from timing information in inter-agent communications) to localize other agents in the formation. In this way one can perform localization with a smaller number of accurate GPS sensors.

While in principle, any choice of three noncollinear agents in a two-dimensional formation or four non-coplanar agents in a three-dimensional formation can be made for anchors (given also enough inter-agent distance measurements), in fact there is a nontrivial choice to be made. This is because, when agent distances are not exactly known but rather measured with some error, the localization algorithm for the non-anchor agents will inevitably give erroneous positions, with the error depending on the inter-agent distance errors *and also the choice of agents to serve as anchors.*

The main contributions of this paper are, firstly, to introduce a criterion to measure the effect of distance measurement error on the localization of agents , and, second, to introduce a methodology for selection of anchors among a formation of agents with a view to minimizing that error. For the time being only planar formations are taken into consideration. In addition, the current study only deals with errors originating from inter-agent distance measurements.

The paper is organized as follows. In the next section graph theoretic preliminaries relevant to the localization problem are described. In the third section calculation of error statistics given a set of anchors and a criterion for selection of anchors are presented. The fourth section contains a method for choosing of three anchors to minimize the

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errors. Simulation results are presented in the fifth section. In the last section some concluding remarks are presented.

2. RIGIDITY, GLOBAL RIGIDITY AND SENSOR NETWORK LOCALIZATION

In order to have a clear idea about the terms that are used in this paper, in the following some of the key tools used in establishing the results and the related graph theory concepts are briefly explained. For the sake of simplicity, we use sensor network terminology throughout this paper. Of course, this mild abuse of language will not affect the results produced for formations of autonomous agents. A network N is represented by a graph-coordinates set pair (G, Π) where the graph $G = (V, E)$ represents the inter-agent sensing topology of N, $V \triangleq \{v_i\}_{i=1}^{|V|}$ is the set of vertices and E is the set of edges in G, each vertex v_i representing a sensor node in N and $e_{ij} \in E$ denoting the edge connecting the vertices v_i and v_j, an edge being incident on vertices representing two nodes (agents) just when they can measure the Euclidean distance between each other or the distance between these two nodes is precisely known, $\Pi \triangleq \{\pi_i\}_{i=1}^{|V|}$ is the set of coordinates for the agents (nodes) in the network, each π_i being a 2-vector. The graph G is called the *underlying graph* of N.

Two networks (G, Π) and (G, Π'), where $G = (V, E)$, are *equivalent* if $\|\pi_i - \pi_j\| = \|\pi'_i - \pi'_j\|$ for any vertex pair $v_i, v_j \in V$, for which $e_{ij} \in E$. The two networks (G, Π) and (G, Π'), where $G = (V, E)$, are *congruent* if $\|\pi_i - \pi_j\| = \|\pi'_i - \pi'_j\|$ for any vertex pair $v_i, v_j \in V$, whether or not $e_{ij} \in E$. This means that if (G, π') and (G, Π) are congruent then (G, Π') can be obtained from (G, Π) applying a combination of translations, rotations and reflections only. A network (G, Π) is called rigid if there exists a sufficiently small positive constant ε such that if (G, Π') is equivalent to (G, Π) and $\|\pi_i - \pi'_i\| < \varepsilon$ for all $v_i \in V$ then (G, Π') is congruent to (G, Π). A network (G, Π) is *globally rigid* if every network which is equivalent to (G, Π) is congruent to (G, Π). It is easy to see if G is a complete graph then the network (G, Π) is necessarily globally rigid.

There are combinatorial tests for rigidity and generic global rigidity in $\mathbb{R}^2$. Rigidity can be tested using Laman's Theorem which says;

Theorem 1. (Laman (1970)). A graph $G = (V, E)$ modeling a framework in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges is generically rigid if and only if there exists a subgraph $G' = (V, E')$ with $2|V| - 3$ edges such that for any subset V'' of V, the induced subgraph $G'' = (V'', E'')$ of G' obeys $|E''| \leq 2|V''| - 3$.

To state a test for generic global rigidity in $\mathbb{R}^2$, we need two concepts, *3-connected* graph and *redundantly rigid* graph. The notion of 3-connected graph is standard, see Diestel (2005). A graph is termed generically redundantly rigid if with the removal of any edge, it remains generically rigid, see Jackson and Jordan (2005) and Aspnes et al. (2006). In $\mathbb{R}^2$, there exists a test for checking generic redundant rigidity which is a variant of Theorem 1. The explanation in detail is beyond the scope of this paper; the reader may refer to Mao et al. (2006) and references within for more details.

Based on these concepts, the following theorem states an elegant necessary and sufficient condition for generic global rigidity of a network in $\mathbb{R}^2$.

Theorem 2. (Jackson and Jordan (2005)). A graph $\mathcal{G}$ with $n \geq 4$ vertices is generically globally rigid in $\mathbb{R}^2$ if and only if it is 3-connected and redundantly rigid in $\mathbb{R}^2$.

Using the notions above, the network localization problem can be defined as follows,

Problem 1. (Network Localization Problem). Let N be a network in $\mathbb{R}^2$, consisting of $m \geq 3$ anchor nodes (beacon nodes) located at known positions $\pi_1, \pi_2, \cdots, \pi_m$ and $n - m > 0$ ordinary nodes located at unknown positions $\pi_{m+1}, \cdots, \pi_n$, and let $G = (V, E)$ be the underlying graph of N. For each $e_{ij} \in E$, let the distance between nodes i and j be given as $\|\pi_i - \pi_j\| = d_{ij}$ Find locations $p_1, \cdots, p_{m+n} \in \mathbb{R}^2$ satisfying $\|p_i - p_j\| = d_{ij}$, $\forall e_{ij} \in E$ with respect to the fact that $p_k = \pi_k$ for $k \in \{1, \cdots, m\}$.

For unique localization of a sensor network we have the following theorem,

Theorem 3. (Eren et al. (2004)). Problem 1 is uniquely solvable if and only if G is globally rigid.

An alternative approach to characterizing the rigidity and global rigidity of a formation uses the concept of the *rigidity matrix*. We will use this concept also to establish some of the main results of this paper. Consider a network (G, Π) in $\mathbb{R}^2$ with the underlying graph $G = (V, E)$. Let the coordinates of vertex v_j be $\pi_j = (x_j, y_j)^\top$. The rigidity matrix is defined with an arbitrary ordering of the vertices and edges, and has $2|V|$ columns and $|E|$ rows. Each edge gives rise to a row, and if the edge links vertices v_j and v_k, the nonzero entries of the row of the matrix are in columns $2j - 1$, $2j$, $2k - 1$ and $2k$, and are respectively $x_j - x_k$, $y_j - y_k$, $x_k - x_j$, $y_k - y_j$. For example, for the graph of Fig. 1, the rigidity matrix is

$$R = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 & 0 & 0 \end{bmatrix}$$

The rank of the matrix R contains information about the rigidity of the framework, or the generic rigidity of the underlying graph G. The key result is:

Theorem 4. (Tay and Whiteley (1985)). A graph $G = (V, E)$ modeling a network in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges is rigid if and only if for generic[1] vertex positions, the rigidity matrix has rank $2|V| - 3$.

It is easy to verify the claim of Theorem 4 for the rigid network presented in Fig. 1. The rank for the rigidity matrix R of Fig. 1 is 5 in almost all vertex coordinates. In more detail, suppose that $e_{ij} \in E$, so that the coordinates of vertices i, j in the network obey for all time

$$||\pi_i(t) - \pi_j(t)||^2 = d_{ij}^2 \qquad (1)$$

where d_{ij} is the actual (constant) distance between vertices i and j and $\|.\|$ denotes the 2-norm of a vector. Assuming motion is smooth, it follows that

$$[\pi_i(t) - \pi_j(t)]^\top [\dot{\pi}_i(t) - \dot{\pi}_j(t)] = 0 \qquad (2)$$

[1] Here, "generic" positions correspond to "almost all" arbitrary selections of positions. Some discussions on the need for using the qualifiers "generic" and "almost all" can be found in (Tay and Whiteley (1985))

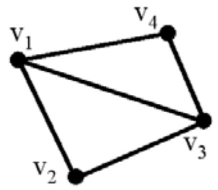

Fig. 1. An example of a rigid network.

and by stacking together $|E|$ such equations there results

$$R\frac{d}{dt}\pi(t) = 0 \tag{3}$$

where $\pi(t)$ denotes the $2|V|$-vector obtained by stacking the $\pi_i(t)$. There is a second useful consequence of (1). Suppose that vertex positions are initially fixed and satisfy distance constraints, but then a small displacement $\delta\pi$ is made to the vector of vertex positions, in a way allowing edge lengths to change. There will of course be a corresponding change in the lengths corresponding to the edges of the graph. *To first order*, this change is described by

$$R\delta\pi = \delta d/2 \tag{4}$$

where δd is the vector of changes in the squares of the lengths, ordered in the same way as the edges are ordered in defining R. In general R is not square, let alone invertible. Therefore, it does not immediately make sense to contemplate the change in vertex positions that would flow from a change in lengths, at least without some kind of constraint. One can however contemplate constraining some of the vertices not to move, and some of the lengths not to change. Then a submatrix of R will map small changes in some of the vertex positions to small changes in some of the lengths, and the inverse of this matrix (assuming of course that it is square and nonsingular) will map small changes in some of the lengths to small changes in some of the vertex positions. In the next section however, we shall consider a situation where this submatrix is not square, and the resulting effect on the posing and solution of a localization problem.

3. ANCHOR AGENTS AND LOCALIZATION ACCURACY IN AUTONOMOUS FORMATIONS

Consider a formation, $\mathcal{F}(G,\Pi)$, with the underlying sensing graph $G = (V,E)$. Suppose a subset of the agents are anchor agents, i.e. we know their global position. Denote the corresponding subset of V by V'. Adopt the standard convention that two anchor agents know their inter-agent distance, and let $E' \subset E$ denote the set of edges in G joining vertices in V'. The *formation localization problem* can be defined exactly as stated in Problem 1, with the only different that "N", "network", and "node" are replaced with $\mathcal{F}$, "formation", and "agent" respectively. [2]

We remark that the full determination of G requires nomination of anchors; before choice of anchors, not all agents which end up being nominated as anchors may be able to sense one another.

[2] In the sequel we will use the notion of "formation localization problem" as defined here.

In addition to above, suppose that the graph G is generically globally rigid. We would like to characterize errors in localization ($\delta p_i = p_i - \pi_i,\ i = 1,\cdots,|V|$) which occur when noise perturbs inter-agent distance measurements (d_{ij}), apart from distances between anchor agent pairs.

For a noiseless situation the equations which apply to the formation after using anchor node information include distance information and coordinate information, and are of the form

$$\begin{array}{c}\|p_i - p_j\| = d_{ij}\ \forall_{e_{ij}} \in E\setminus E' \\ p_i = \pi_i\ \forall v_i \in V'\end{array} \tag{5}$$

Let us identify the number of equations associated with the edge constraints in (5). Since the framework is globally rigid, we can drop any edge and it remains rigid. Suppose we choose the edge to be one of those in the edge constraint set $E\setminus E'$, call the edge e. Since the associated graph $(V, E\setminus\{e\})$ is now rigid, there exists by Laman's theorem a minimally rigid subgraph $G_1 = (V, E_1)$, and thus one with $2|V| - 3$ edges, such that any induced subgraph of G_1 defined using a subset V_2 of V has at most $2|V_2| - 3$ edges. Identify V_2 with $|V'|$. Then in G_1, there are at most $2|V'| - 3$ edges joining vertices both in V' and so at least $2|V \setminus V'|$ other edges. Hence, apart from the distance constraint for the edge e that was dropped from the edge constraint set before constructing G_1, there are necessarily at least $2|V\setminus V'|$ distance constraint equations in (5), i.e. at least $2|V\setminus V'|+1$ equations in all. There are also precisely $2|V \setminus V'|$ unknowns to be determined from (5) and (6), taking the coordinates of the anchor nodes as known. Therefore, the unknown coordinates are the solutions of an *overdetermined* set of equations. Typically, if an overdetermined equation set has a solution, it will be unique; that is so here because of the global rigidity, the true coordinate positions, π_i can be found by solving (5). Now in the presence of the typically small noise, δd_{ij}, perturbing the squares of the true distances (presumably due to measurement error), equation (5) would formally become,

$$\begin{array}{c}\|p_i - p_j\|^2 = d_{ij}^2 + \delta d_{ij}\ \forall_{e_{ij}} \in E\setminus E' \\ p_i = \pi_i\ \forall v_i \in V'\end{array} \tag{6}$$

which still results in an overdetermined system of simultaneous equations, though now there will generally be no solution. Nevertheless, the notion of approximate localization makes sense. Instead of solving (6), we seek those position values for p_i, call them π_i^*, for $v_i \in V\setminus V'$ which solve the following minimization problem:

$$\begin{array}{c}\displaystyle\min_{\{p_i, v_i\in V\setminus V'\}} \sum_{e_{ij}\in E\setminus E'} [\|p_i - p_j\|^2 - (d_{ij}^2 + \delta d_{ij})]^2 \\ \text{subject to} \\ p_i = \pi_i\ \forall v_i \in V'\end{array} \tag{7}$$

It is intuitively reasonable, but proven elsewhere (Anderson et al. (2007)), that if the noises are bounded by a suitably small constant, the solution of the minimization problem stated in (7), π_i^*, for $v_i \in V\setminus V'$, will be close to the solution of (5), and in fact the error in the position value will depend continuously on the error in the squared distances. We can use this fact and rewrite the minimisation problem in (7) in terms of rigidity matrix. The following theorem deals with this issue.

Theorem 5. Consider a formation $\mathcal{F}(G, \Pi)$ with the underlying graph $G = G(V, E)$ generically globally rigid. Suppose that a subset V' of V corresponds to vertices whose coordinates are precisely known. Let $E' \subset E$ correspond to edges joining vertices in V', so that the corresponding edge lengths in the formation are precisely known. Suppose that edge lengths d_{ij} corresponding to edges in $E \setminus E'$ are known only to within an error satisfying a bound, Δ say, and let δd denote the vector of errors in the squares of the inter-agent lengths, ordered in the same way as the edges are ordered in defining R, the rigidity matrix of G evaluated at Π, where π_i is the correct position of the i-th agent. Let π_i^* for $v_i \in V \setminus V'$ solve the approximate localization problem (7), and suppose that Δ is sufficiently small that there is a unique solution to (7) whose distance from π for $v_i \in V \setminus V'$ depends continuously on the δd_{ij}. Define $\delta\pi^* = \pi^* - \pi$. Then the vector $\delta\pi^*$ is the solution to the following minimization problem neglecting higher order terms in δd.

$$\begin{gathered} \min \|R\delta p - \delta d/2\|^2. \\ \text{subject to} \\ \delta p_i = 0 \; \forall v_i \in V' \end{gathered} \tag{8}$$

Proof. We only need to show that (7) and (8) are the same for the particular summand in (7) and the corresponding summand in the above minimization problem. (Obviously the result then holds after summation of all of these summands as well.) Replacing p_i and p_j by $\pi_i + \delta p_i$ and $\pi_j + \delta p_j$, where δp_i is the perturbation of the position of i-th node from its real positions, in (7) respectively, for the summand associated with the edge e_{ij}, s_{ij}, we obtain,

$$s_{ij} = \|\pi_i - \pi_j + \delta p_i - \delta p_j\|^2 - (d_{ij}^2 + \delta d_{ij}) \tag{9}$$

Replacing π_i, π_j, δp_i, and δp_j with $(x_i, y_i)^\top$, $(x_j, y_j)^\top$, $(\delta x_i, \delta y_i)^\top$ and $(\delta x_j, \delta y_j)^\top$ in (9) respectively, we obtain,

$$\begin{aligned} s_{ij} &= (x_i - x_j)^2 + (y_i - y_j)^2 - d_{ij}^2 + \\ &(\delta x_i - \delta x_j)^2 + (\delta y_i - \delta y_j)^2 - \delta d_{ij} + \\ &2(x_i - x_j)(\delta x_i - \delta x_j) + 2(y_i - y_j)(\delta y_i - \delta y_j) \end{aligned} \tag{10}$$

using $(x_i - x_j)^2 + (y_i - y_j)^2 = d_{ij}^2$ and neglecting the higher order terms of $(\delta x_i - \delta x_j)^2 + (\delta y_i - \delta y_j)^2$ and dividing by 2 we obtain

$$s_{ij} = (x_i - x_j)(\delta x_i - \delta x_j) + (y_i - y_j)(\delta y_i - \delta y_j) - \delta d_{ij}/2 \tag{11}$$

On the other hand, one can rewrite (8) as

$$\begin{gathered} \min_{\delta p_i, v_i \in V \setminus V'} \sum_{k=1}^{|E|} (R_k \delta p - \delta d_k/2)^2. \\ \text{subject to} \\ \delta p_i = 0 \; \forall v_i \in V' \end{gathered} \tag{12}$$

where R_k and δd_k are the k-th rows of R and δd, respectively. Cosidering the summand associated with the edge e_{ij}, s'_{ij} we obtain,

$$s'_{ij} = (x_i - x_j)(\delta x_i - \delta x_j) + (y_i - y_j)(\delta y_i - \delta y_j) - \delta d_{ij}/2 \tag{13}$$

It is immediate that the two summands ((11) and (13)) are the same for a particular i and j (neglecting the factor of 2 and $(\delta x_i - \delta x_j)^2 + (\delta y_i - \delta y_j)^2$). □

For considering the norm in (8), viz. $\|R\delta p - \delta d/2\|^2$, one can delete those columns from R relating to anchors to study the effect of perturbation, since these columns corresponds to positions of anchors that are already known so no perturbation may happen. We can also delete any row corresponding to an edge between two anchors (such an edge may or may not be present before designation of certain nodes as anchors ,but in any case the corresponding entry of $R\delta p - \delta d/2$ will be zero.). After doing this deletion process, the norm presented in (8) will transform to

$$\min \|R_{r,(|E \setminus E'|) \times (2|V \setminus V'|)} \delta p_r - \delta d_r/2\|^2 \tag{14}$$

Here R_r, the *reduced rigidity matrix*, is constructed by deletion of columns of R corresponding to anchor positions and rows of R corresponding to edges between two anchors, respectively. Furthermore, δp_r is the perturbation in non-anchor positions and δd_r is the error vector in the square of the length of edges connecting those edges with at least one non-anchor end.

The minimisation problem stated in (14) can be solved for deterministic values of δd. For deterministic values of δd, $\delta\pi^*$ is computed by,

$$\delta\pi^* = R_r^\dagger \delta d_r/2 \tag{15}$$

where $R_r^\dagger = (R_r^\top R_r)^{-1} R_r^\top$ is the Penrose pseudoinverse of R_r, see Horn and Johnson (1991). However, in general the error in the square of the length measurement, δd_r, is not known and it is not possible to accurately compute the error in the localization. The measurement error can be more realistically modeled by random variables, with specific covariance and mean value. One much more likely to be interested in translating statistics of δd_r to statistics of the agents position error,δp_r, when δd_r is a random variable. This can be done by the following equation, which holds between δd_r and δp_r,

$$cov(\delta\pi^*) = R_r^\dagger cov(\delta d_r)(R_r^\dagger)^\top/4. \tag{16}$$

where $\delta\pi^*$ is the solution to the minimization problem in (14). If δd_r is a normal random variable with zero mean and I as its covariance we have

$$cov(\delta\pi^*) = R_r^\dagger (R_r^\dagger)^\top/4. \tag{17}$$

One might also reasonably postulate that each distance measurement, rather than its square, is subject to additive zero mean Gaussian noise, of variance σ^2, say. Then δd_r will be zero mean, with covariance matrix $diag(d_{ij}^2)\sigma^2$, and (17) will be replaced by;

$$cov(\delta\pi^*) = R_r^\dagger (diag(d_{ij}^2)\sigma^2)(R_r^\dagger)^\top. \tag{18}$$

Equations (17) and (18) address the task of characterizing localization errors posed in the beginning of section 3. Note that, as is common in characterizing mean square errors arising out of algorithms, the characterization involves the solution of the corresponding noiseless problem–in this case R_r involves the true agent positions. We comment on this further in Section 6.

The following theorem guarantees that R_r is of full column rank and the minimization problem stated in (14) has a unique solution.

Theorem 6. Assume that R is the rigidity matrix of a graph such that there is at least one selection of three anchors with the property that the graph obtained by adding inter-anchors edges is globally rigid. Then the reduced rigidity matrix obtained by deletion of columns of R corresponding to anchor positions and rows of R corresponding to edges between two anchors, is of full column rank.

Proof. The matrix R_r has dimensions of $(|E \setminus E'| \times 2|V| - 2|V'|)$ (Note that $|V'|$ is theset of vertices selected as anchors.). We assume that R has an ordering such that, the last $2|V'|$ columns correspond to the anchors and last E' rows correspond to edges connecting the anchors in the original graph.

Suppose to obtain a contradiction to the assertion of the Theorem that, $\exists\alpha \neq 0$ such that $R_r\alpha = 0$. Define $\beta = \begin{bmatrix} \alpha \\ 0_{2|V'|} \end{bmatrix}$; then $R\beta = 0$. Now it is known from Tay and Whiteley (1985) that the $2|V|$-vectors, $\lambda_1 = [1, 0, 1, 0, \cdots]^\top$, $\lambda_2 = [0, 1, 0, 1, \cdots]^\top$, and $\lambda_3 = [y_1, -x_1, y_2, -x_2, \cdots]^\top$, are a basis of the three-dimensional null-space of R. Hence β must be a linear combination of λ_1, λ_2, and λ_3, for some scalar a, b, and c, not all zero,

$$\beta = a\lambda_1 + b\lambda_2 + c\lambda_3$$

The last $2|V'|$ rows give the equation

$$a\bar{\lambda}_1 + b\bar{\lambda}_2 + c\bar{\lambda}_3 = 0$$

where $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\lambda}_3$ are the last 6 rows of λ_1, λ_2, and λ_3, respectively. However, inspection of these vectors show they are independent. Hence nonzero a,b, and c cannot exist, i.e. there is no nonzero α that satisfies,

$$R_r\alpha = 0.$$

It results that R_r is of full column rank. □

In the next section we discuss which nodes are the best candidates for being selected as anchors.

4. SELECTION OF ANCHORS IN THE FORMATION

To start we formally define the following problem,

Problem 2. Consider a formation of the type described in the beginning of section 3, such that the reduced rigidity matrix, $R_{r,|E\setminus E'|\times 2(|V|-|V'|)}$, is obtained by discarding the rows and columns associated to the position of the anchor nodes and their interconnecting edges. What is the best selection of $|V'|$ nodes to be selected as anchors?

4.1 Selection of Three Anchors in the Formation

In this section we present an answer (in principle) Problem 2 for $|V'| = 3$. Note that the statement of Problem 2 fails to specify what is meant by best. We shall adopt the view that we need to mitigate the effect of random errors, and this is best done by seeking to minimize a scalar measure associated with $cov(\delta\pi^*)$. When (17) applies, one such scalar measure is $\lambda_{max}(R_r^\dagger(R_r^\dagger)^\top)$ [3].

Remark 1. Since, $\lambda_{max}(R_r^\dagger(R_r^\dagger)^\top) = \sigma_{min}(R_r)^{-2}$, the problem of minimizing $\lambda_{max}(R_r^\dagger(R_r^\dagger)^\top)$ is equivalent to the problem of maximizing $\sigma_{min}(R_r)$.

Here we formally define optimal anchor selection.

Definition 1. [4] [Optimal Anchor Selection] The selection of triple (v_i, v_j, v_k) $(|V'| = 3)$ is termed *σ-optimal*, if the minimum singular value of R_r associated with this selection is the largest among other minimum singular values associated with other possible selections.

[3] Another measure that can be used for reducing the measurement error is the trace of $R_r^\dagger(R_r^\dagger)^\top$. Since, one would be interested not just in keeping down the maximum eigenvalue, but in keeping down all the eigenvalues.

[4] Optimal anchor selection definition should be modified when one wants to use (18).

Selection	Nodes Selected	Selection	Nodes Selected
1	(1, 2, 3)	19	(2, 3, 7)
2	(1, 2, 4)	20	(2, 4, 5)
3	(1, 2, 5)	21	(2, 4, 6)
4	(1, 2, 6)	22	(2, 4, 7)
5	(1, 2, 7)	23	(2, 5, 6)
6	(1, 3, 4)	24	(2, 5, 7)
7	(1, 3, 5)	25	(2, 6, 7)
8	(1, 3, 6)	26	(3, 4, 5)
9	(1, 3, 7)	27	(3, 4, 6)
10	(1, 4, 5)	28	(3, 4, 7)
11	(1, 4, 6)	29	(3, 5, 6)
12	(1, 4, 7)	30	(3, 5, 7)
13	(1, 5, 6)	31	(3, 6, 7)
14	(1, 5, 7)	32	(4, 5, 6)
15	(1, 6, 7)	33	(4, 5, 7)
16	(2, 3, 4)	34	(4, 6, 7)
17	(2, 3, 5)	35	(5, 6, 7)
18	(2, 3, 6)		

Table 1. Selection cases and the nodes selected in each case.

Using Definition 4, we are able to answer, at least in principle, the question mentioned in Problem 2, when $|V'| = 3$. To do so, we can use an exhaustive search through all possible selections of anchors to find the largest possible minimum singular value, or use another combinatorial search algorithm, such as a genetic algorithm. Simulation results appear in the next section.

5. SIMULATION RESULTS

In this section, we have studied three formation localization scenarios and found the best selection of nodes to choose as anchors. In the first scenario, a formation of 7 agents with a globally rigid underlying graph of the formation is studied. Figure 2 shows the formation. In Figure 3 the minimum singular value of the reduced rigidity matrix associated with each of the possible 35 anchor selections is presented, In Table 1 the possible anchor selections are presented. It is evident that selection of nodes $(1, 4, 7)$ and $(2, 3, 7)$ results in having the largest minimum singular value. Selections $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, $(2, 3, 4)$, $(2, 6, 7)$ yield the 5 lowest singular values.

In the second simulation we used the same graph that was used in the first simulation but with different geometric positions for the agents; the formation is depicted in Figure 4. The best choice of anchors in this case is $(1, 4, 7)$. The minimum singular value associated with each selection is presented in Figure 5.

As a rule of thumb one can say that a triple of agents that already form a triangle in the formation prior to anchor selection are not good candidates for being selected as anchors. As can be seen from the simulation results, such selections are never among those selections that yields a larger minimum singular value for the reduced rigidity matrix. In many cases (that cannot be presented here due to the lack of space) the σ-optimal anchor selection for $|V'| = 3$ is the selection of those anchors such that the triangle formed by these three points has the largest area among all other possible triangles defined by any other three nodes in the graph. Furthermore, in some other cases σ-optimal anchor selection for $|V'| = 3$ happens when the smallest loop connecting the three anchors is the longest.

Most of the time in real-life scenarios we can not evaluate the rigidity matrix with real values (nominal) of positions of agents, so we have to use the values obtained from

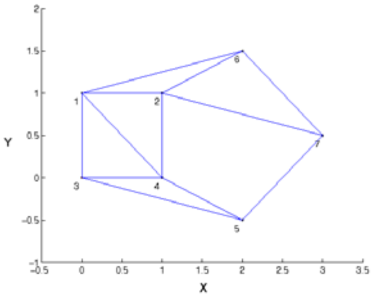

Fig. 2. The globally rigid formation used in the first simulation.

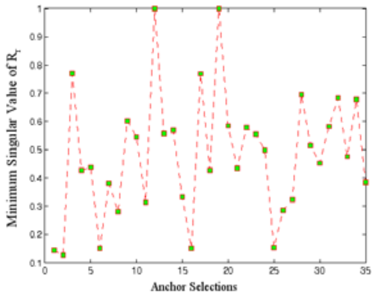

Fig. 3. σ_{min} value of reduced rigidity matrix for all possible 35 anchor selections for the formation depicted in Figure 2.

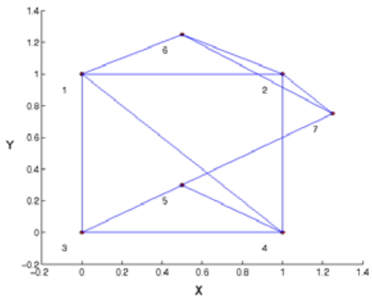

Fig. 4. The globally rigid formation used in the second simulation.

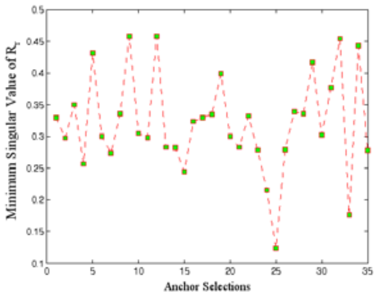

Fig. 5. σ_{min} value of reduced rigidity matrix for all possible 35 anchor selections for the formation depicted in Figure 4.

a minimization problem like the one presented in (6). If we use non-nominal postion values for the agents in the formation, i.e. using π^* instead of π, in evaluating rigidity matrix the best anchor selection may differ from the case where the nominal position values, π are used in evaluating R.

6. CONCLUSIONS AND FUTURE WORKS

In this paper, we have studied the noises arising in self-localization of mobile formations. The paper has postulated a statistical measure for the effect of these noises, and derived an intuitively pleasing result that the degree to which the noises will be a problem when a certain rigidity matrix associated with the formation (but not the usual one, rather a 'reduced one') corresponds to the closeness of a singular value of that rigidity matrix to zero. Since the rigidity matrix in question depends on which particular nodes are nominated as anchors, we also studied the question of what anchor choices were likely to be good, and obtained some loose guidelines. Different anchor selection problems can actually be formulated, when more than three anchors are permitted.

In future work, we intend to study the problem of anchor selection in large sensor networks, and proposing a methodology for selecting the anchors when checking all the possible selections is not feasible.

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