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Novel Delay-Dependent Exponential Stability of a Class of Fuzzy Cellular Neural Networks with Time-Varying Delays ⋆

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Abstract: The global exponential stability of the neural networks is investigated for a new fuzzy cellular neural networks with time-varying delays. A novel delay-dependent stability criterion is derived based on Lyapunov stability theory and the linear matrix inequality. By transforming the fuzzy logic terms with time-delay, our criteria are less conservative than existing results. Two examples are provided to verify the effectiveness of the proposed results.

1. INTRODUCTION

Since fuzzy cellular neural networks (FCNN) are first introduced in 1996, it has become a very useful tool for image processing problems, which is a cornerstone in image processing and pattern recognition; see Yang et al. (1996a), Yang et al. (1996b) and Yang et al. (1996c). An FCNN is a fuzzy neural network which integrates fuzzy logic into the structure of a traditional cellular neural network (CNN). Generally speaking, there are two kinds of FCNN, the Takagi-Sugeno (T–S) FCNN and the common FCNN. The T–S one is an FCNN based on the T–S fuzzy model; see Hou et al. (2007). And the common one combines fuzzy logic with traditional CNN; see Huang (2006), Huang (2007), Liu et al. (2004), Yuan et al. (2006) and Zhong et al. (2006). However, in hardware implementation, time-delays inevitably occur due to the finite switching speed of amplifiers and the communication time. Time-delays can be harmful since they can destroy a stable network and cause sustained oscillations, bifurcation or chaos.

So far, some results on stability have been obtained for the FCNN with time-delays. Some sufficient conditions based on M-matrix were given to check the global exponential stability of FCNN with constant and time-varying delays, distributed delays and diffusion, respectively; see Liu et al. (2004), Huang (2006) and Huang (2007). And a sufficient condition of the global exponential stability based on algebraic inequality for FCNN with time-varying delays was proposed by Zhong et al. (2006). The LMI is first used to guarantee the global exponential stability of FCNN with time-varying delays by Yuan et al. (2006). But these schemes employed a similar transformation for the non-fuzzy terms, i.e., taking the absolute value, which led to the increase of conservativeness in their results. To the best of our knowledge, no results have overcome this problem thus far.

In this paper, a new FCNN with time-varying delays is proposed. A novel sufficient condition is derived to guarantee the global exponential stability of the new FCNN with time-varying delays. It is obtained based on Lyapunov stability theory and linear matrix inequality (LMI) technique. Different from the previous methods to transform non-fuzzy terms as in Liu et al. (2004), Zhong et al. (2006) and Yuan et al. (2006), the transformation of fuzzy logic terms with time-delay is carried out in this paper. Thus, the sign of the non-fuzzy weight matrices is considered, which makes our results less conservative. Finally, two illustrative examples prove that our criteria are less conservative than existing results in previously repotred literature.

In the following, $B = [b_{ij}]_{n\times n}$ denotes an $n \times n$ real matrix. B^T, $\lambda_M(B)$, $\lambda_m(B)$ and $\|B\|$ represent the transpose, the maximum eigenvalue, the minimum eigenvalue and norm of matrix, respectively, where $\|\cdot\|$ is Euclidean norm. $B > 0$ ($B < 0$) denotes that B is a positive (negative) definite matrix. $B \geqslant 0$ ($B \leqslant 0$) denotes that B is a positive (negative) semidefinite matrix. $|B| = [|b_{ij}|]_{n\times n}$ denotes a matrix formed from B by taking absolute value componentwise.

2. SYSTEM DESCRIPTION

The common FCNN with time-varying delays can be described by following

$$\begin{aligned}\dot{z}_i(t) = &-d_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} c_{ij} u_j + I_i \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z_j(t-\tau(t))) + \bigwedge_{j=1}^{n} T_{ij} u_j \\ &+ \bigvee_{j=1}^{n} \beta_{ij} f_j(z_j(t-\tau(t))) + \bigvee_{j=1}^{n} H_{ij} u_j\end{aligned} \quad (1)$$

where $\alpha_{ij}, \beta_{ij}, T_{ij}$ and H_{ij} are elements of fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feedforward MIN template and fuzzy feedforward MAX template, respectively. a_{ij} and c_{ij} are elements of feedback and feedforward

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template. $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively. z_i, u_i and I_i denote state, input and bias of the ith neurons, respectively. f_i is the activation function, $\tau(t)$ is transmission delay, and τ is the upper bound of the delay. The initial conditions of (1) are given by $z_i(t) = \varphi_i(t) \in ([-\tau, 0], \Re)$.

The delays are not only in the fuzzy feedback template, but also in the common feedback template. According to the principle above, the new FCNN with delays will be derived.

The new FCNN with time-varying delays can be described by following:

$$\begin{aligned}\dot{z}_i(t) &= -d_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} c_{ij} u_j \\ &+ \sum_{j=1}^{n} b_{ij} f_j(z_j(t-h(t))) + I_i \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z_j(t-\tau(t))) + \bigwedge_{j=1}^{n} T_{ij} u_j \\ &+ \bigvee_{j=1}^{n} \beta_{ij} f_j(z_j(t-\tau(t))) + \bigvee_{j=1}^{n} H_{ij} u_j\end{aligned} \tag{2}$$

where b_{ij} is elements of delay feedback template, $h(t)$ is transmission delay of delay feedback template, h is the upper bound of delay, and the definitions of other parameters are the same as the common FCNN. The $|\alpha|$ and $|\beta|$ are the symmetric matrices, which are the absolute value of the fuzzy weight matrices α and β. The initial conditions of (2) are given by $z_i(t) = \varphi_i(t) \in ([-\max(\tau, h), 0], \Re)$, and $\varphi_i(t)$ are continuous and bounded functions.

It is obvious that the new FCNN model is proposed in this paper include two terms with delays, delay feedback and fuzzy feedback term. When the delay $h(t) = 0$, the new FCNN (2) will be reduced to the FCNN (1). Therefore, the FCNN are the special case of the new FCNN.

Assumption 1. For each f_i, satisfies the global Lipschitz condition, i.e., there exist positive finite constants l_i such that

$$|f_i(u) - f_i(v)| \leqslant l_i |u - v| \tag{3}$$

for all $u, v \in \Re$, and $i = 1, 2, \cdots, n$.

*Assumption 1**. For each f_i, there exist positive finite constants l_i such that the following condition is satisfied

$$0 \leqslant \frac{f_i(u) - f_i(v)}{u - v} \leqslant l_i \tag{4}$$

for all $u, v \in \Re$, $u \neq v$, and $i = 1, 2, \cdots, n$.

Remark 1. By observation, (3) is rather general as it merely requires the neuronal nonlinearity to be Lipschitz, whereas (4) requires a monotone increasing feature of the neuronal activation. Clearly, the former includes the latter as a special case.

Assumption 2. For the delays $\tau(t)$ and $h(t)$, the following conditions are satisfied,

$$0 \leqslant \tau(t) \leqslant \tau, 0 \leqslant h(t) \leqslant h, \dot{\tau}(t) \leqslant \mu < 1, \dot{h}(t) \leqslant \gamma < 1.$$

where τ and h are the upper bound of the delays, and μ and γ are the upper bound of the delay's derivatives, respectively.

By Theorem 1 in Yang et al. (1996a) and the boundedness of functions f_i, it follows readily from Browner's fixpoint theorem that the network (1) has at least one equilibrium point $z^* = (z_1^*, z_2^*, \cdots, z_n^*)^T$. Then, define $x_i(t) = z_i(t) - z_i^*$, (2) is transformed into the following form,

$$\begin{aligned}\dot{x}_i(t) &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-h(t))) \\ &+ \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau(t)) + z_j^*) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z_j^*) \\ &+ \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau(t)) + z_j^*) - \bigvee_{j=1}^{n} \beta_{ij} f_j(z_j^*) \\ x_i(t) &= \hat{\varphi}_i(t),\ t \in ([-\max(\tau, h), 0], \Re).\end{aligned} \tag{5}$$

where $g_i(x_i(t)) = f_i(x_i(t) + z_i^*) - f_i(z_i^*)$ with $g_i(0) = 0$, and $g_i(x_i(t - h(t))) = f_i(x_i(t - h(t)) + z_i^*) - f_i(z_i^*)$. Let $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$. If the origin of the system (5) is globally exponentially stable, then the equilibrium point z^* is also globally exponentially stable. And it means that the equilibrium point of (2) is globally exponentially stable.

The following definition and lemmas will be used for deriving our main result.

Definition 1. (Global Exponential Stability) The system (5) is said to be globally exponentially stable with the convergence rate k, if there exist constants $k > 0$ and $M > 1$, such that

$$\|x(t)\| \leqslant M\|\phi\| e^{-kt}$$

where $\|\phi\| = \sup_{-\max(h,\tau) \leqslant \theta \leqslant 0} \|x(\theta)\|$.

Lemma 1. (Yang et al. (1996a)) Let z and z' be two states of system (1) or (2), then we have

$$\left| \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z'_j) \right| \leqslant \sum_{j=1}^{n} |\alpha_{ij}| \left| f_j(z_j) - f_j(z'_j) \right|,$$

$$\left| \bigvee_{j=1}^{n} \beta_{ij} f_j(z_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(z'_j) \right| \leqslant \sum_{j=1}^{n} |\beta_{ij}| \left| f_j(z_j) - f_j(z'_j) \right|.$$

Lemma 2. (Boyd et al. (1994)) (Schur complement) For any given symmetric matrices $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0$, where S_{ii} is $r_i \times r_i$ nonsingular matrix, $i = 1, 2, \cdots, n$, and $S_{21} = S_{12}^T$.The following conditions are equivalent:
1) $S < 0$;
2) $S_{11} < 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} < 0$;
3) $S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{21} < 0$.

Lemma 3. (Wang et al. (2007)) For any two vectors $a, b \in \Re^n$, any matrix A, any positive definte symmetric matrix B with the same dimensions, and any two positive constants m, n, the following inequality holds,

$$-ma^T Ba + 2na^T Ab \leqslant n^2 b^T A^T (mB)^{-1} Ab$$

Lemma 4. (He et al. (2006)) For any two vectors $a, b \in \Re^n$, the inequality

$$2a^T b \leqslant a^T Xa + b^T X^{-1} b$$

holds, where X is any positive matrix (i.e., $X > 0$).

Lemma 5. For any symmetric constant matrix $A = [a_{ij}]_{n \times n}$ with $a_{ij} \geqslant 0$, the following inequality holds

$$A \leqslant A_S$$

where $A_S = \text{diag}\left(\sum_{i=1}^{n} a_{i1}, \sum_{i=1}^{n} a_{i2}, \cdots, \sum_{i=1}^{n} a_{in} \right)$, $i, j = 1, 2, \cdots, n$.

Proof: Let $B = A_S - A$, we can obtain

$$B = \begin{bmatrix} \sum_{i=2}^{n} a_{i1} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{i=1,i\neq 2}^{n} a_{i2} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{i=1}^{n-1} a_{in} \end{bmatrix}$$

where $a_{ij} = a_{ji}$.

Obviously, B is a diagonal dominant matrix with positive elements in main diagonal. According to the definition of positive semidefinite matrix, we can get $B \geqslant 0$, i.e., $A \leqslant A_S$. Thus, the lemma is proved.

3. MAIN RESULTS

The following delay-dependent exponential stability criterion is obtained.

Theorem 1. The equilibrium point of system (5) is globally exponentially stable with the convergence rate $k > 0$, if Assumptions 1 and 2 are satisfied, and if there exist some positive definite diagonal matrices P, Q and Y, a positive definite symmetric matrices W, such that the following LMI is feasible:

$$\Xi = \begin{bmatrix} \omega & PA & PB & P(|\alpha|_S + |\beta|_S) \\ * & Q+W-Y & 0 & 0 \\ * & * & -e^{-2kh}(1-\gamma)W & 0 \\ * & * & * & -e^{-2k\tau}(1-\mu)Q \end{bmatrix} < 0 \quad (6)$$

where $\omega = 2kP - PD - DP + LYL$, $L = \text{diag}(l_i)$, $D = \text{diag}(d_i)$, $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $|\alpha| = [|\alpha_{ij}|]_{n\times n}$, $|\beta| = [|\beta_{ij}|]_{n\times n}$, $|\alpha|_S = \text{diag}(\sum_{i=1}^{n} |\alpha_{i1}|, \sum_{i=1}^{n} |\alpha_{i2}|, \cdots, \sum_{i=1}^{n} |\alpha_{in}|)$, $|\beta|_S = \text{diag}(\sum_{i=1}^{n} |\beta_{i1}|, \sum_{i=1}^{n} |\beta_{i2}|, \cdots, \sum_{i=1}^{n} |\beta_{in}|)$, and $i, j = 1, 2, \cdots, n$.

Proof: Construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x(t)) &= e^{2kt} \sum_{i=1}^{n} p_i x_i^2(t) + \int_{t-\tau(t)}^{t} e^{2ks} g^T(x(s)) Q g(x(s)) ds \\ &\quad + \int_{t-h(t)}^{t} e^{2ks} g^T(x(s)) W g(x(s)) ds \end{aligned} \quad (7)$$

where $P = \text{diag}(p_i)$ with $p_i > 0$, $Q = \text{diag}(q_i) > 0$, $W = W^T > 0$, and $i, j = 1, 2, \cdots, n$. Obviously, the $V(x(t))$ is the nonnegative and radially unbounded function. From system (5), Assumption 2 and Lemma 1, calculating the time derivatives of $V(x(t))$ along the trajectories of system (5) yields

$$\begin{aligned} \dot{V}(x(t)) &= 2ke^{2kt} \sum_{i=1}^{n} p_i x_i^2(t) + 2e^{2kt} \sum_{i=1}^{n} p_i x_i(t)\dot{x}_i(t) \\ &\quad + e^{2kt} g^T(x(t))(Q+W)g(x(t)) \\ &\quad - e^{2k(t-\tau(t))}(1-\dot{\tau}(t)) g^T(x(t-\tau(t))) Q g(x(t-\tau(t))) \\ &\quad - e^{2k(t-h(t))}(1-\dot{h}(t)) g^T(x(t-h(t))) W g(x(t-h(t))) \\ &\leqslant 2ke^{2kt} x^T(t) P x(t) - 2e^{2kt} x^T(t) PD x(t) \\ &\quad + 2e^{2kt} x^T(t) PA g(x(t)) \\ &\quad + 2e^{2kt} x^T(t) PB g(x(t-h(t))) \\ &\quad + 2e^{2kt} |x(t)|^T P(|\alpha| + |\beta|) |g(x(t-\tau(t)))| \\ &\quad + e^{2kt} g^T(x(t))(Q+W)g(x(t)) \\ &\quad - e^{2kt} e^{-2k\tau}(1-\mu) g^T(x(t-\tau(t))) Q g(x(t-\tau(t))) \\ &\quad - e^{2kt} e^{-2kh}(1-\gamma) g^T(x(t-h(t))) W g(x(t-h(t))). \end{aligned} \quad (8)$$

According to Lemma 5, we have $|\alpha| + |\beta| \leqslant |\alpha|_S + |\beta|_S$, where $|\alpha|_S$ is diagonal dominance in column of $|\alpha|$, i.e., $|\alpha|_S = \text{diag}(\sum_{i=1}^{n} |\alpha_{i1}|, \sum_{i=1}^{n} |\alpha_{i2}|, \cdots, \sum_{i=1}^{n} |\alpha_{in}|)$. Similarly, $|\beta|_S = \text{diag}(\sum_{i=1}^{n} |\beta_{i1}|, \sum_{i=1}^{n} |\beta_{i2}|, \cdots, \sum_{i=1}^{n} |\beta_{in}|)$. Thus, for (8), we have

$$\begin{aligned} \dot{V}(x(t)) &\leqslant 2ke^{2kt} x^T(t) P x(t) - 2e^{2kt} x^T(t) PD x(t) \\ &\quad + 2e^{2kt} x^T(t) PA g(x(t)) \\ &\quad + 2e^{2kt} x^T(t) PB g(x(t-h(t))) \\ &\quad + 2e^{2kt} |x(t)|^T P(|\alpha|_S + |\beta|_S) |g(x(t-\tau(t)))| \\ &\quad + e^{2kt} g^T(x(t))(Q+W)g(x(t)) \\ &\quad - e^{2kt} e^{-2k\tau}(1-\mu) g^T(x(t-\tau(t))) Q g(x(t-\tau(t))) \\ &\quad - e^{2kt} e^{-2kh}(1-\gamma) g^T(x(t-h(t))) W g(x(t-h(t))) \end{aligned} \quad (9)$$

According to Assumption 1, we have

$$g_i^2(x_i(t)) \leqslant l_i^2 x_i^2(t), i = 1, 2, \cdots, n. \quad (10)$$

Therefore, for any matrix $Y = \text{diag}(y_1, y_2, \cdots, y_n) > 0$, it follows that

$$\begin{aligned} 0 &\leqslant -e^{2kt} \sum_{i=1}^{n} y_i (g_i^2(x_i(t)) - l_i^2 x_i^2(t)), \\ &= -e^{2kt} \left(g^T(x(t)) Y g(x(t)) - x^T(t) LYL x(t) \right). \end{aligned} \quad (11)$$

By Lemma 3, transforming the terms with absolute value in (9), it follows that

$$\begin{aligned} &-e^{2kt} e^{-2k\tau}(1-\mu) g^T(x(t-\tau(t))) Q g(x(t-\tau(t))) \\ &+2e^{2kt} |x(t)|^T P(|\alpha|_S + |\beta|_S) |g(x(t-\tau(t)))| \\ \leqslant\ & e^{2kt} e^{2k\tau}(1-\mu)^{-1} |x(t)|^T P(|\alpha|_S + |\beta|_S) Q^{-1} (|\alpha|_S + |\beta|_S)^T \\ &\times P|x(t)| \\ =\ & e^{2kt} e^{2k\tau}(1-\mu)^{-1} x^T(t) P(|\alpha|_S + |\beta|_S) Q^{-1} (|\alpha|_S + |\beta|_S)^T P x(t) \end{aligned} \quad (12)$$

From (9), (11) and (12), let $\zeta^T(t) = [x^T(t) \;\; g^T(x(t)) \;\; g^T(x(t-h(t)))]$, we obtain the following inequality

$$\dot{V}(x(t)) \leqslant e^{2kt} \zeta^T(t) \Phi \zeta(t). \quad (13)$$

where

$$\Phi = \begin{bmatrix} \omega & PA & PB \\ * & Q+W-Y & 0 \\ * & * & -e^{-2kh}(1-\gamma)W \end{bmatrix}, \quad (14)$$

$\omega = 2kP - PD - DP + LYL + e^{2k\tau}(1-\mu)^{-1} P(|\alpha|_S + |\beta|_S) Q^{-1} (|\alpha|_S + |\beta|_S)^T P$.

Since $\Phi < 0$ in (13) implies the $\dot{V}(x(t)) < 0$ for any $\zeta(t) \neq 0$. According to the Schur complement, $\Phi < 0$ is equivalent to (6), i.e., if $\Xi < 0$, then $\dot{V}(x(t)) < 0$.

It follows from $\dot{V}(x(t)) < 0$ that

$$V(x(t)) \leqslant V(x(0)). \tag{15}$$

We can obtain

$$\begin{aligned} V(x(0)) &= x^T(0)Px(0) \\ &\quad + \int_{-\tau(0)}^{0} e^{2ks} g^T(x(s))Qg(x(s))ds \\ &\quad + \int_{-h(0)}^{0} e^{2ks} g^T(x(s))Wg(x(s))ds \\ &\leqslant \lambda_M(P)\|\phi\|^2 + \tau\lambda_M(Q)\lambda_M(L^2)\|\phi\|^2 \\ &\quad h\lambda_M(W)\lambda_M(L^2)\|\phi\|^2 \\ &= \Upsilon\|\phi\|^2 \end{aligned} \tag{16}$$

where $\Upsilon = \lambda_M(P) + \tau\lambda_M(Q)\lambda_M(L^2) + h\lambda_M(W)\lambda_M(L^2)$. On the other hand,

$$\begin{aligned} V(x(t)) &\geqslant e^{2kt} x^T(t)Px(t) \\ &\geqslant e^{2kt}\lambda_m(P)\|x(t)\|^2. \end{aligned} \tag{17}$$

Therefore,

$$\|x(t)\| \leqslant \sqrt{\frac{\Upsilon}{\lambda_m(P)}}\|\phi\|e^{-kt}. \tag{18}$$

From Definition 1, the equilibrium point of (5) is globally exponentially stable and has the exponential convergence rate k. It means that the equilibrium point of system (2) is globally exponentially stable, too. The proof is completed.

Now, we consider our result under Assumption 1^*.

Theorem 2. The equilibrium point of system (2) is globally exponentially stable with the convergence rate $k > 0$, if Assumptions 1^* and 2 are satisfied, and if there exist some positive definite diagonal matrices P, Λ, Q and Y, a positive definite symmetric matrices W, such that the following LMI is feasible:

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & 0 & \omega_{15} \\ * & \omega_{22} & \omega_{23} & \omega_{24} & 0 \\ * & * & \omega_{33} & 0 & 0 \\ * & * & * & \omega_{44} & 0 \\ * & * & * & * & \omega_{55} \end{bmatrix} < 0 \tag{19}$$

where

$$\omega_{11} = 2kP - PD - DP + 2k\Lambda L,$$

$$\omega_{12} = PA + LY - D\Lambda,$$

$$\omega_{13} = PB,$$

$$\omega_{15} = P(|\alpha|_S + |\beta|_S),$$

$$\omega_{22} = \Lambda A + A^T\Lambda + Q + W - 2Y,$$

$$\omega_{23} = \Lambda B,$$

$$\omega_{24} = \Lambda(|\alpha|_S + |\beta|_S),$$

$$\omega_{33} = -e^{-2kh}(1-\gamma)W,$$

$$\omega_{44} = -\frac{1}{2}e^{-2k\tau}(1-\mu)Q,$$

$$\omega_{55} = -\frac{1}{2}e^{-2k\tau}(1-\mu)Q,$$

$L = \text{diag}(l_i)$, $D = \text{diag}(d_i)$, $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $|\alpha| = [|\alpha_{ij}|]_{n\times n}$, $|\alpha|_S = \text{diag}(\sum_{i=1}^{n}|\alpha_{i1}|, \sum_{i=1}^{n}|\alpha_{i2}|, \cdots, \sum_{i=1}^{n}|\alpha_{in}|)$, $|\beta| = [|\beta_{ij}|]_{n\times n}$, $|\beta|_S = \text{diag}(\sum_{i=1}^{n}|\beta_{i1}|, \sum_{i=1}^{n}|\beta_{i2}|, \cdots, \sum_{i=1}^{n}|\beta_{in}|)$, and $i, j = 1, 2, \cdots, n$.

Proof: Construct the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x(t)) &= e^{2kt}\sum_{i=1}^{n} p_i x_i^2(t) + 2\sum_{i=1}^{n}\lambda_i e^{2kt}\int_0^{x_i(t)} g_i(s)ds \\ &\quad + \int_{t-\tau(t)}^{t} e^{2ks} g^T(x(s))Qg(x(s))ds \\ &\quad + \int_{t-h(t)}^{t} e^{2ks} g^T(x(s))Wg(x(s))ds \end{aligned} \tag{20}$$

where $P = \text{diag}(p_i)$ with $p_i > 0$, $Q = \text{diag}(q_i) > 0$, $\Lambda = \text{diag}(\lambda_i) > 0$, $W = W^T > 0$, and $i, j = 1, 2, \cdots, n$.

Obviously, the $V(x(t))$ is the nonnegative and radially unbounded function. By system (5), Assumption 2, Lemma 1, inequality $\int_0^{x_i(t)} g_i(s)ds \leqslant \frac{1}{2}l_i x_i^2(t)$, and the same way as Theorem 1, the time derivatives of $V(x(t))$ along the trajectories of system (5) can be obtained as

$$\begin{aligned} \dot{V}(x(t)) &\leqslant 2ke^{2kt}x^T(t)Px(t) - 2e^{2kt}x^T(t)PDx(t) \\ &\quad +2e^{2kt}x^T(t)PAg(x(t)) \\ &\quad +2e^{2kt}x^T(t)PBg(x(t-h(t))) \\ &\quad +2e^{2kt}|x(t)|^T P(|\alpha|_S + |\beta|_S)|g(x(t-\tau(t)))| \\ &\quad +2ke^{2kt}x^T(t)\Lambda Lx(t) - 2e^{2kt}g^T(x(t))\Lambda Dx(t) \\ &\quad +2e^{2kt}g^T(x(t))\Lambda Ag(x(t)) \\ &\quad +2e^{2kt}g^T(x(t))\Lambda Bg(x(t-h(t))) \\ &\quad +2e^{2kt}|g(x(t))|^T\Lambda(|\alpha|_S + |\beta|_S)|g(x(t-\tau(t)))| \\ &\quad +e^{2kt}g^T(x(t))(Q+W)g(x(t)) \\ &\quad -e^{2kt}e^{-2k\tau}(1-\mu)g^T(x(t-\tau(t)))Qg(x(t-\tau(t))) \\ &\quad -e^{2kt}e^{-2kh}(1-\gamma)g^T(x(t-h(t)))Wg(x(t-h(t))) \end{aligned} \tag{21}$$

Similar to (12) in the proof of Theorem 1, transforming the terms with absolute value in (21), we obtain

$$\begin{aligned} \dot{V}(x(t)) &\leqslant 2ke^{2kt}x^T(t)Px(t) - 2e^{2kt}x^T(t)PDx(t) \\ &\quad +2e^{2kt}x^T(t)PAg(x(t)) \\ &\quad +2e^{2kt}x^T(t)PBg(x(t-h(t))) \\ &\quad +2e^{2kt}e^{2k\tau}(1-\mu)^{-1}x^T(t)P(|\alpha|_S + |\beta|_S)Q^{-1} \\ &\quad \times(|\alpha|_S + |\beta|_S)^T Px(t) \\ &\quad +2ke^{2kt}x^T(t)\Lambda Lx(t) - 2e^{2kt}g^T(x(t))\Lambda Dx(t) \\ &\quad +2e^{2kt}g^T(x(t))\Lambda Ag(x(t)) \\ &\quad +2e^{2kt}g^T(x(t))\Lambda Bg(x(t-h(t))) \\ &\quad +2e^{2kt}e^{2k\tau}(1-\mu)^{-1}g^T(x(t))\Lambda(|\alpha|_S + |\beta|_S)Q^{-1} \\ &\quad \times(|\alpha|_S + |\beta|_S)^T\Lambda g(x(t)) \\ &\quad +e^{2kt}g^T(x(t))(Q+W)g(x(t)) \\ &\quad -e^{2kt}e^{-2kh}(1-\gamma)g^T(x(t-h(t)))Wg(x(t-h(t))). \end{aligned} \tag{22}$$

From Assumption 1^*, for any matrix $Y = \text{diag}(y_1, y_2, \cdots, y_n) > 0$, it follows that

$$
\begin{aligned}
0 &\leqslant -2e^{2kt}\sum_{i=1}^{n} y_i(g_i^2(x_i(t)) - l_i g_i(x_i(t))x_i(t)), \\
&= -2e^{2kt}\left(g^T(x(t))Yg(x(t)) - g^T(x(t))YLx(t)\right).
\end{aligned} \tag{23}
$$

Then, from (22) and (23), let $\zeta^T(t) = [x^T(t)\;\; g^T(x(t))\;\; g^T(x(t-h(t)))]$, $\dot{V}(x(t))$ is given by

$$
\dot{V}(x(t)) \leqslant e^{2kt}\zeta^T(t)\Psi\zeta(t) \tag{24}
$$

where

$$
\Psi = \begin{bmatrix} \omega_1 & PA + LY - D\Lambda & PB \\ * & \omega_2 & \Lambda B \\ * & * & -e^{-2kh}(1-\gamma)W \end{bmatrix}, \tag{25}
$$

$\omega_1 = 2kP - PD - DP + 2k\Lambda L + 2e^{2k\tau}(1-\mu)^{-1}P(|\alpha|_S + |\beta|_S)Q^{-1}(|\alpha|_S + |\beta|_S)^T P$, $\omega_2 = \Lambda A + A^T\Lambda + Q + W - 2Y + 2e^{2k\tau}(1-\mu)^{-1}\Lambda(|\alpha|_S + |\beta|_S)Q^{-1}(|\alpha|_S + |\beta|_S)^T\Lambda$.

Since $\Psi < 0$ in (24) implies that $\dot{V}(x(t)) < 0$ for any $\zeta(t) \neq 0$, according to the Schur complement, $\Psi < 0$ is equivalent to (19), i.e., if $\Omega < 0$, then $\dot{V}(x(t)) < 0$.

Similar to Theorem 1, we can obtain the following inequality

$$
\|x(t)\| \leqslant \sqrt{\frac{\Upsilon}{\lambda_m(P)}}\|\phi\|e^{-kt} \tag{26}
$$

where $\Upsilon = \lambda_M(P) + \lambda_M(L\Lambda L) + \tau\lambda_M(Q)\lambda_M(L^2) + h\lambda_M(W)\lambda_M(L^2)$.

From Definition 1, the equilibrium point of (5) is globally exponentially stable and has the exponential convergence rate k. It means that the equilibrium point of system (2) is globally exponentially stable, too. The proof is completed.

When $B = 0$, the system (5) is reduced to the common FCNN,

$$
\begin{aligned}
\dot{x}_i(t) &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) \\
&\quad + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau(t)) + z_j^*) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z_j^*) \\
&\quad + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau(t)) + z_j^*) - \bigvee_{j=1}^{n} \beta_{ij} f_j(z_j^*) \\
x_i(t) &= \bar{\varphi}_i(t),\; t \in ([-\tau, 0], \Re).
\end{aligned} \tag{27}
$$

where the definition of parameters is the same as system (5). Therefore, we can extend the result to the global exponential stability of the common FCNN that is stated in the following corollaries.

Corollary 1. The equilibrium point of system (27) is globally exponentially stable with the convergence rate $k > 0$, if Assumptions 1 and 2 are satisfied, and if there exist some positive definite diagonal matrices P, Q and Y, such that the following LMI is feasible:

$$
\Xi = \begin{bmatrix} \omega & PA & e^{k\tau}P(|\alpha|_S + |\beta|_S) \\ * & Q - Y & 0 \\ * & * & -(1-\mu)Q \end{bmatrix} < 0 \tag{28}
$$

where $\omega = 2kP - PD - DP + LYL$, $L = \text{diag}(l_i)$, $D = \text{diag}(d_i)$, $A = [a_{ij}]_{n\times n}$, $|\alpha|_S = \text{diag}(\sum_{i=1}^{n}|\alpha_{i1}|, \sum_{i=1}^{n}|\alpha_{i2}|, \cdots, \sum_{i=1}^{n}|\alpha_{in}|)$, $|\beta|_S = \text{diag}(\sum_{i=1}^{n}|\beta_{i1}|, \sum_{i=1}^{n}|\beta_{i2}|, \cdots, \sum_{i=1}^{n}|\beta_{in}|)$, $|\alpha| = [|\alpha_{ij}|]_{n\times n}$, $|\beta| = [|\beta_{ij}|]_{n\times n}$, and $i, j = 1, 2, \cdots, n$.

Proof: The proof of the corollary is similar to the proof of Theorem 1.

Corollary 2. The equilibrium point of system (27) is globally exponentially stable with the convergence rate $k > 0$, if Assumptions 1^* and 2 are satisfied, and if there exist some positive definite diagonal matrices P, Λ, Q and Y, such that the following LMI is feasible:

$$
\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & 0 & \omega_{14} \\ * & \omega_{22} & \omega_{23} & 0 \\ * & * & \omega_{33} & 0 \\ * & * & * & \omega_{44} \end{bmatrix} < 0 \tag{29}
$$

where

$$
\begin{aligned}
\omega_{11} &= 2kP - PD - DP + 2k\Lambda L, \\
\omega_{12} &= PA + LY - D\Lambda, \\
\omega_{14} &= e^{k\tau}P(|\alpha|_S + |\beta|_S), \\
\omega_{22} &= \Lambda A + A^T\Lambda + Q - 2Y, \\
\omega_{23} &= e^{k\tau}\Lambda(|\alpha|_S + |\beta|_S), \\
\omega_{33} &= -\frac{1}{2}(1-\mu)Q, \\
\omega_{44} &= -\frac{1}{2}(1-\mu)Q,
\end{aligned}
$$

$L = \text{diag}(l_i)$, $D = \text{diag}(d_i)$, $A = [a_{ij}]_{n\times n}$, $|\alpha| = [|\alpha_{ij}|]_{n\times n}$, $|\beta| = [|\beta_{ij}|]_{n\times n}$, $|\alpha|_S = \text{diag}(\sum_{i=1}^{n}|\alpha_{i1}|, \sum_{i=1}^{n}|\alpha_{i2}|, \cdots, \sum_{i=1}^{n}|\alpha_{in}|)$, $|\beta|_S = \text{diag}(\sum_{i=1}^{n}|\beta_{i1}|, \sum_{i=1}^{n}|\beta_{i2}|, \cdots, \sum_{i=1}^{n}|\beta_{in}|)$, and $i, j = 1, 2, \cdots, n$.

Proof: The proof of the corollary is similar to the proof of Theorem 2.

Corollary 3. The equilibrium point of system (27) is globally exponentially stable with the convergence rate $k > 0$, if Assumptions 1^* and 2 are satisfied, and if there exist some positive definite diagonal matrices P and Q, such that the following equality holds

$$
\begin{aligned}
\Omega &= 2kP - PD - DP + PAL + LA^TP + LQL \\
&\quad + e^{2k\tau}(1-\mu)^{-1}P(|\alpha|_S + |\beta|_S)Q^{-1}(|\alpha|_S + |\beta|_S)^T P \\
&< 0
\end{aligned} \tag{30}
$$

or the following LMI is feasible:

$$
\Omega = \begin{bmatrix} \omega & e^{k\tau}P(|\alpha|_S + |\beta|_S) \\ * & -(1-\mu)Q \end{bmatrix} < 0 \tag{31}
$$

where $\omega = 2kP - PD - DP + PAL + LA^TP + LQL$, $L = \text{diag}(l_i)$, $D = \text{diag}(d_i)$, $A = [a_{ij}]_{n\times n}$, $|\alpha| = [|\alpha_{ij}|]_{n\times n}$, $|\beta| = [|\beta_{ij}|]_{n\times n}$, $|\alpha|_S = \text{diag}(\sum_{i=1}^{n}|\alpha_{i1}|, \sum_{i=1}^{n}|\alpha_{i2}|, \cdots, \sum_{i=1}^{n}|\alpha_{in}|)$, $|\beta|_S = \text{diag}(\sum_{i=1}^{n}|\beta_{i1}|, \sum_{i=1}^{n}|\beta_{i2}|, \cdots, \sum_{i=1}^{n}|\beta_{in}|)$, and $i, j = 1, 2, \cdots, n$.

Proof: Construct the following Lyapunov-Krasovskii functional:

$$
V(x(t)) = e^{2kt}\sum_{i=1}^{n} p_i x_i^2(t) + \int_{t-\tau(t)}^{t} e^{2ks} g^T(x(s))Qg(x(s))ds. \tag{32}
$$

And then, from Assumption 1^*, the result above can be derived.

4. NUMERICAL EXAMPLES

In this section, two examples are given to show the effectiveness of the proposed results.

Example 1. Consider the common FCNN (1) with the following parameters (see Yuan et al. (2006)),

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}, \alpha = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}, \beta = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} \end{bmatrix},$$

$\tau(t) = 0.1 + 0.025\sin(5t)$, and $f_i(x_i) = (|x_i + 1| + |x_i - 1|)/2$, $i = 1, 2$.

Clearly, from the activation function, the Lipschitz constant matrix is $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\tau = 0.125$, $\mu = 0.125$. $|\alpha|$ and $|\beta|$ are symmetric matrices.

Applying the Corollary 1, and with the exponential convergence rate $k = 0.1$, the LMI is satisfied by using LMI Toolbox in MATLAB. And the detailed parameters are as following:

$$P = \begin{bmatrix} 4.0261 & 0 \\ 0 & 4.0261 \end{bmatrix}, Q = \begin{bmatrix} 2.2459 & 0 \\ 0 & 2.2459 \end{bmatrix},$$

$$Y = \begin{bmatrix} 3.6460 & 0 \\ 0 & 3.6460 \end{bmatrix}.$$

Therefore, the FCNN with parameters in Example 1 is globally exponentially stable. Also, from the LMI Toolbox, when the convergence rate $k = 0.1$, the permissible upper bound on time delay that guarantees the exponential stability of system is calculated as $\tau = 0.2206$. Existing criterion such as those in Liu et al. (2004), Yuan et al. (2006) and Zhong et al. (2006) cannot be applied to this example. Obviously, the result in our paper is less conservative than those in the literature. When Corollary 3 is used, we can obtain even less conservative upper bound.

Example 2. Consider the new FCNN (2) with the following parameters:

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}, B = \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ -\frac{3}{20} & -\frac{1}{4} \end{bmatrix},$$

$$\alpha = \begin{bmatrix} \frac{1}{32} & -\frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} \end{bmatrix}, \beta = \begin{bmatrix} \frac{1}{32} & \frac{1}{32} \\ -\frac{1}{32} & \frac{1}{32} \end{bmatrix},$$

$\tau(t) = 0.4 + 0.1\sin(3t)$, and $f_i(x_i) = (|x_i+1|+|x_i-1|)/2$, $i = 1, 2$.

Similarly, we can obtain the Lipschitz constant matrix $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\tau = h = 0.5$, $\mu = \gamma = 0.3$. $|\alpha|$ and $|\beta|$ are symmetric matrices.

Applying Theorem 1 with convergence rate $k = 0.1$, the LMI is satisfied by using LMI Toolbox in MATLAB. And the detailed parameters are as following:

$$P = \begin{bmatrix} 3.5226 & 0 \\ 0 & 3.3863 \end{bmatrix}, Q = \begin{bmatrix} 0.8673 & 0 \\ 0 & 0.8464 \end{bmatrix},$$

$$W = \begin{bmatrix} 1.3675 & 0.0193 \\ 0.0193 & 1.2724 \end{bmatrix}, Y = \begin{bmatrix} 3.3305 & 0 \\ 0 & 3.1552 \end{bmatrix}.$$

Hence, the new FCNN with parameters in Example 2 is globally exponentially stable. And by calculation using the LMI Toolbox, when the convergence rate $k = 0.1$, the permissible upper bound on time delay that guarantees the exponential stability of the system is $\tau = h < 1.1$.

5. CONCLUSION

Based on Lyapunov stability theory and LMI, a novel delay-dependent global exponential stability criterion is derived for a new FCNN with time-varying delays. Compared with the method in existing literature, we only transform the fuzzy logic term with time-delays, and make our stability criteria less conservative.

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