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Saturated Root Locus: Theory and Application

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Abstract: This paper extends the standard root locus technique to systems with saturating actuators. This is accomplished by introducing the notion of S-poles, which are the poles of the quasilinear system obtained by applying the method of stochastic linearization to the system with saturation. The path traced by the S-poles on the complex plane when the gain of the controller changes from 0 to ∞ is the S-root locus. We show that the S-root locus is a subset of the standard root locus, which may terminate prematurely at the so-called termination points. A method for calculating these points is presented. In addition, the issue of amplitude truncation in terms of the S-root locus is investigated. Finally, an application of the S-root locus to hard disk drive controller design is presented, and it is shown that this simple technique results in a controller that compares favorably with those designed using more sophisticated approaches.

Keywords: Systems with saturation; Stochastic linearization; Root locus; Constrained control.

1. INTRODUCTION

Consider the SISO tracking system of Figure 1. Here, $P(s)$ is the plant, $KC(s)$ is the controller, $K > 0$, and $F_\Omega(s)$ is a coloring filter with $3dB$ bandwidth Ω, which generates the reference r from standard white noise w; the signals y and u are, respectively, the system output and the input to the saturation element defined as:

$$sat_\alpha(u) = \begin{cases} \alpha, & u > +\alpha \\ u, & -\alpha \le u \le \alpha \\ -\alpha, & u < -\alpha. \end{cases} \tag{1}$$

To study this system, we use the method of stochastic linearization (see Roberts and Spanos, 1990), whereby the saturation element is replaced by a stochastically linearized gain, $N(K)$, defined as

$$N(K) = \text{erf}\left(\frac{\alpha}{\sqrt{2}\left\|\frac{F_\Omega(s)KC(s)}{1+N(K)KC(s)P(s)}\right\|_2}\right), \tag{2}$$

where

$$\text{erf}\,(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt, \tag{3}$$

and the 2-norm of a transfer function is understood as

$$\|H\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|^2\,d\omega}. \tag{4}$$

The value of $N(K)$ can be calculated from (2) using the standard bisection algorithm. The resulting quasilinear system, shown in Figure 2, is known to provide a sufficiently accurate approximation of the original nonlinear system (see Gökçek et al., 2000; Eun et al., 2005, and Section 2 for more details).

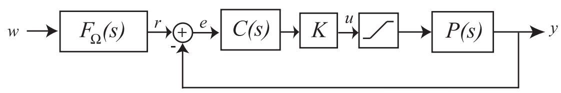

Fig. 1. Closed loop system with saturating actuator

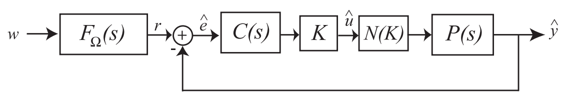

Fig. 2. Equivalent quasilinear system

The locus traced by the closed loop poles of the quasilinear system of Figure 2 is referred to as the saturated root locus (*S-root locus*). It is the object of study in this paper.

Specifically, the following two phenomena are investigated: *S-termination* and *S-truncation*. S-termination occurs when, as $K \to \infty$, the S-root locus terminates prematurely, i.e., before the open loop zeros are reached. S-truncation occurs when the output signal is truncated by the saturating actuator. We investigate how both S-termination and S-truncation depend on all the transfer functions in Figure 1, as well as on the level of saturation α.

Although there are a large number of publications on systems with saturating actuators (see, for instance, recent monographs by Hu and Lin, 2001; Kapila, 2001; Saberi et al., 2001), the root locus approach has not been investigated. Some intuitive recommendations for dealing with root locus under actuator saturation can be found in standard textbooks (see, for instance, Franklin et al., 1998).

The outline of this paper is as follows: Sections 2 and 3 introduce the formal definition of the S-root locus and present methods of S-root locus construction respectively.

Section 4 outlines an S-root locus design methodology. Section 5 discusses the issue of amplitude truncation, and Section 6 presents a method for calibrating the S-root locus. An application to a hard disk drive control problem is described in Section 7 and, finally, in Section 8 the conclusions are formulated. Due to space limitations, many details and all proofs are omitted and can be found in Ching et al. (2007b).

2. DEFINITIONS

2.1 Definitions

Denote the equivalent gain of the quasilinear system as

$$K_{eq}(K) \triangleq KN(K). \tag{5}$$

As follows from (2), $K_{eq}(K)$ can be obtained from the equation

$$K_{eq}(K) = K\text{erf}\left(\frac{\alpha}{\sqrt{2}K\left\|\frac{F_\Omega(s)C(s)}{1+K_{eq}(K)P(s)C(s)}\right\|_2}\right). \tag{6}$$

Definition 1. (S-Closed Loop Poles). The saturated closed loop poles (*S-poles*) of the nonlinear system of Figure 1 are the poles of the quasilinear system of Figure 2, i.e., the poles of the transfer function from r to $\hat{y}$:

$$T(s) = \frac{K_{eq}(K)C(s)P(s)}{1+K_{eq}(K)C(s)P(s)}. \tag{7}$$

Typical of the method of stochastic linearization, the standard deviation of the tracking error, $\sigma_{\hat{e}}$, in the quasilinear system is close (generally within 10%) to the standard deviation, σ_e, of the original system with a saturating actuator (see Gökçek et al., 2000). For example, if

$$P(s) = \frac{1}{s(s+1)}, C(s) = 1, \Omega = 1rad/s, \alpha = 0.05,$$
$$K = 5, F_\Omega(s) = \sqrt{\frac{3}{\Omega}}\left(\frac{\Omega^3}{s^3+2\Omega s^2+2\Omega^2 s+\Omega^3}\right), \tag{8}$$

it follows from (6) that $K_{eq}(K) = 0.0399$ and $\sigma_{\hat{e}} = 0.999$, while simulation of the corresponding nonlinear system yields $\sigma_e = 0.986$ (i.e., an error of 2%). Additional examples can be found in Eun et al. (2005). Since $\sigma_{\hat{e}}$ is defined by the poles of $T(s)$, we conclude that the saturated closed loop poles characterize the tracking performance of the original system.

Definition 2. (S-Root Locus). The S-root locus is the path traced by the saturated closed loop poles when $K \in [0, \infty)$.

Since $K_{eq}(K)$ enters (7) as a usual gain, and since

$$0 \le K_{eq}(K) \le K,$$

the S-root locus is a proper or improper subset of the unsaturated root locus. In the next section, a method for contructing the S-root locus is presented.

3. S-ROOT LOCUS

As in the unsaturated case, we are interested in the points of origin and termination of the S-root locus. The points of origin clearly remain the same as in the unsaturated case. The points of termination, however, may not be the same because, as it turns out, $K_{eq}(K)$ may not tend to infinity as K increases. To discriminate between these two cases, we need the equation

$$\beta = \left\|\frac{F_\Omega(s)C(s)}{1+\left(\frac{\alpha\sqrt{2/\pi}}{\beta}\right)P(s)C(s)}\right\|_2. \tag{9}$$

Lemma 1. If $K_{eq}(K)$ is unique for all K, then

i) $K_{eq}(K)$ is continuous and strictly monotonically increasing.
ii) Equation (9) admits *at most* one positive solution.

Theorem 1. Assume that $T_\gamma(s)$, given by

$$T_\gamma(s) = \frac{F_\Omega(s)C(s)}{1+\gamma P(s)C(s)}, \tag{10}$$

is stable for all $\gamma > 0$ and (6) has a unique solution $K_{eq}(K)$ for all $K > 0$. Then,

$$\text{(i)} \lim_{K\to\infty} K_{eq}(K) = \frac{\alpha\sqrt{2/\pi}}{\beta} < \infty \tag{11}$$

if and only if (9) admits a unique solution $\beta > 0$;

$$\text{(ii)} \lim_{K\to\infty} K_{eq}(K) = \infty \tag{12}$$

if and only if $\beta = 0$ is the only real solution of (9).

Theorem 2. Assume that $T_\gamma(s)$ of (10) is stable only for $\gamma \in [0, \Gamma), \Gamma < \infty$, with $\lim_{\gamma\to\Gamma} \|T_\gamma(s)\|_2 = \infty$, and (6) admits a unique solution $K_{eq}(K)$ for all $K > 0$. Then,

$$\lim_{K\to\infty} K_{eq}(K) = \frac{\alpha\sqrt{2/\pi}}{\beta} < \Gamma, \tag{13}$$

where $\beta > 0$ is the unique positive solution of (9).

Note that (13) implies that, under the conditions of Theorem 2, the S-root locus can never enter the right half plane.

As it follows from Theorems 1 and 2, in the limit of $K \to \infty$, the quasilinear system of Figure 2 has a closed loop transfer function given by

$$T_{term}(s) = \frac{\kappa C(s)P(s)}{1+\kappa C(s)P(s)}, \tag{14}$$

where

$$\kappa = \lim_{K\to\infty} K_{eq}(K). \tag{15}$$

Definition 3. (S-Termination Points). The S-termination points of the S-root locus are the poles of $T_{term}(s)$.

Thus, as $K \to \infty$, the S-poles travel monotically along the S-root locus from the open loop poles to the S-termination points. If $\kappa = \infty$, then the S-termination points coincide with the open loop zeros; otherwise, the S-root locus terminates prematurely.

Example 1. Consider the system of Figure 2 with

$$P(s) = \frac{s+15}{s(s+2.5)}, \alpha = 0.1, \tag{16}$$

and $C(s), \Omega$, and $F_\Omega(s)$ as defined in (8).

It is straightforward to verify that, for this system, $T_\gamma(s)$ is stable for all $\gamma > 0$ and (6) admits a unique solution for $K > 0$, i.e., the conditions of Theorem 1 are satisfied. Since (9) admits a positive solution $\beta = 0.709$, (11) and (15) result in $\kappa = 0.1125$.

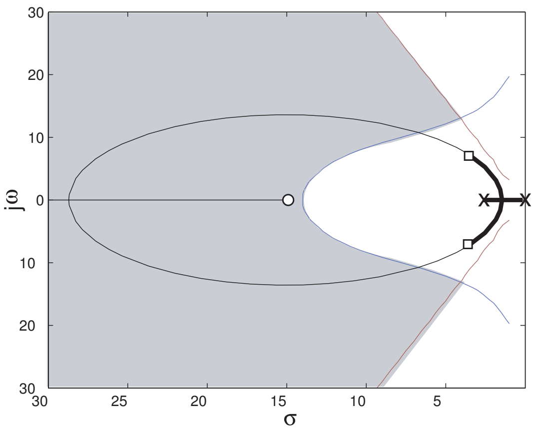

Fig. 3. Unsaturated (thin line) and S-root locus (thick line) for Example 1 with $\alpha = 0.1$

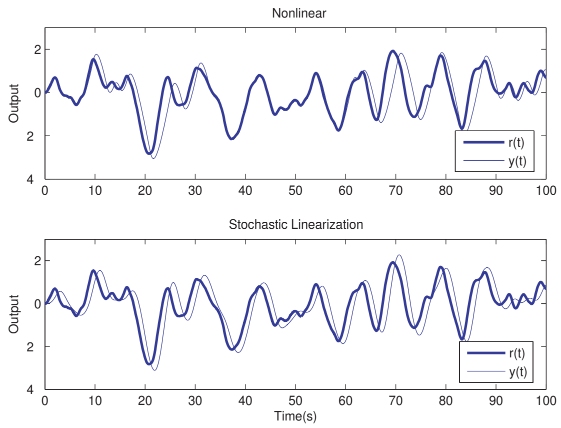

Fig. 4. Tracking quality with S-poles located near the S-termination points ($K = 150$) for Example 1 with $\alpha = 0.1$

Figure 3 shows both the unsaturated and S-root locus for Example 1. Here, and in all subsequent figures, the S-termination points are indicated by squares. The shaded area of Figure 3 represents the admissible domain for a high quality of tracking derived in Ching et al. (2007a). Clearly, the S-root locus never enters the admissible domain, and the achievable tracking quality is limited by the S-termination points. Figure 4 shows the output of the nonlinear and quasilinear systems when $K = 150$ (i.e., when the saturated closed loop poles are located close to the S-termination points). Clearly, the tracking quality of the stochastically linearized system is poor due to dynamic lag. As predicted, the same is true for the original nonlinear system.

To improve the tracking performance, we increase α to 0.2. As illustrated in Figure 5, this causes the S-root locus to enter the admissible domain, and hence, choosing K large enough, results in a high quality of tracking. This is verified in Figure 6, where we see that the quality of tracking is good for both the stochastically linearized and original nonlinear systems.

Example 2. Consider the system of Figure 2 with

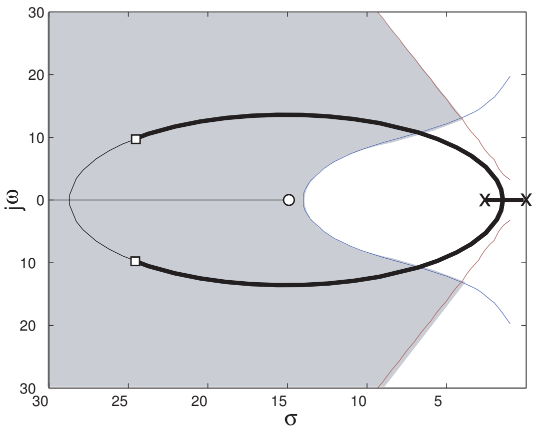

Fig. 5. Unsaturated and S-root locus for Example 1 with $\alpha = 0.2$

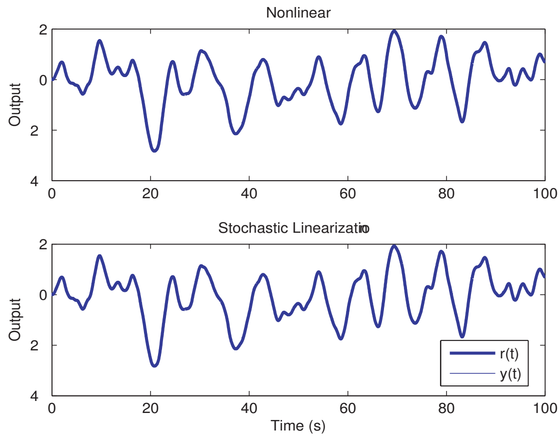

Fig. 6. Tracking quality with poles located near the S-termination points ($K = 150$) for Example 1 with $\alpha = 0.2$

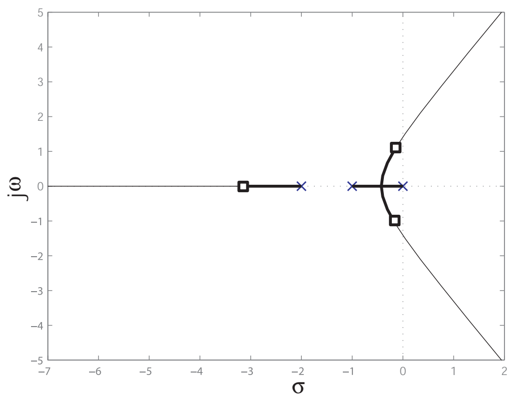

Fig. 7. Unsaturated and S-root locus for Example 2

$$P(s) = \frac{1}{s(s+1)(s+2)}, \alpha = 1, \tag{17}$$

and $C(s)$, Ω, and $F_\Omega(s)$ as defined in (8). It is easily verified that $T_\gamma(s)$ is stable only for $\gamma \in [0, 5.96)$ and (6) has a unique solution for $K > 0$, i.e., the conditions of Theorem

2 are satisfied. Noting that $\beta = 1.1$ is the solution to (9), it follows from (13) and (15) that $\kappa = 0.722$. The resulting S-root locus, illustrated in Figure 7, never enters the right half plane.

4. DESIGN METHODOLOGY

The results of Section 3 are obtained under the assumption that (6) has a unique solution $K_{eq}(K)$ for all $K > 0$. If the solution is not unique, the behaviors of the original and stochastically linearized systems become much more complex and are characterized by jumping phenomena among the sets of S-poles defined by different solutions $K_{eq}^i(K)$, $i = 1, 2, .., r$, where r is the number of solutions of (6). This behavior is analyzed in detail in Ching et al. (2007b). However, since it is complex and, to a cetain extent, unpredictable, we propose a design methodology that avoids multiple solutions of (6).

Specifically, we propose to select, if possible, a controller $C(s)$, which results in a unique $K_{eq}(K)$ for all $K > 0$, and then select K so that the closed loop S-poles are at the desired locations. A question arises as to when such a $C(s)$ does exist. The answer is as follows:

Theorem 3. If $P(s)$ is stable and minimum phase and $F_\Omega(s)/P(s)$ is strictly proper, there exists $C(s)$ such that the solution of (6) is unique for all $K > 0$.

In the remainder of the paper, we assume that $K_{eq}(K)$ is unique for all K.

5. S-ROOT LOCUS AND AMPLITUDE TRUNCATION

In the previous sections we have used the S-root locus to characterize the dynamics of systems with saturating actuators. However, the performance may be poor not only due to the location of S-poles, but also due to output truncation by the saturation. Accordingly, in this section we introduce and compute S-truncation points, which characterize the region of the S-root locus where truncation does not occur. To accomplish this we use the notion of trackable domain introduced in Eun et al. (2005).

The trackable domain, TD, is defined by the magnitude of the largest step input that can be tracked in the presence of saturation. For the system of Figure 1, it is defined as (see Eun et al., 2005):

$$TD = \left| \frac{1}{KC(0)} + P(0) \right| \alpha, \tag{18}$$

where $C(0)$ and $P(0)$ are the DC gains of the controller and plant, respectively. Clearly, the trackable domain is infinite when $P(s)$ has at least one pole at the origin; otherwise, it is finite, assuming that $C(0) \neq 0$. Although (18) is based on step signals, it has been shown in Eun et al. (2005) that TD can also be used to characterize the tracking quality of random signals. In particular, Eun et al. (2005) introduced a tracking quality indicator to account for the finite trackable domain vis-à-vis the size of the signal to be tracked. This indicator was defined as

$$I_0 = \frac{\sigma_r}{TD}, \tag{19}$$

where σ_r is the standard deviation of the reference signal. Specifically, when $I_0 < 0.4$, tracking practically without

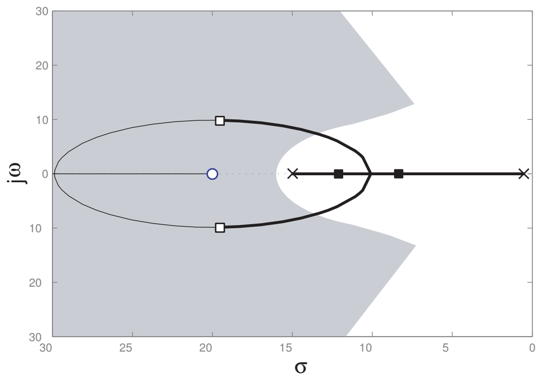

Fig. 8. S-root locus for Example 3 with $\alpha = 0.8$

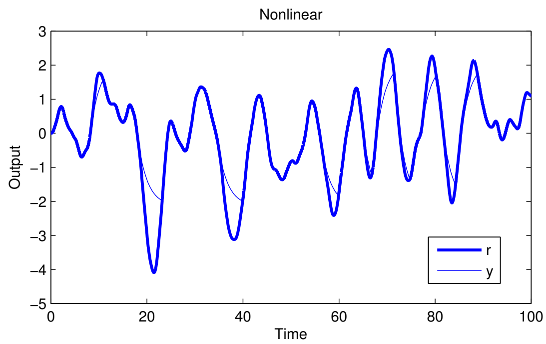

Fig. 9. Tracking quality when $K = 20$ for Example 3 with $\alpha = 0.8$

output truncation is possible, whereas for $I_0 > 0.4$ it is not. Keeping in mind that TD is monotonically decreasing in K and utilizing this threshold, we introduce the following definition.

Definition 4. (S-truncation points). The S - truncation points are defined as the poles of

$$T_{tr}(s) = \frac{K_{tr}C(s)P(s)}{1 + K_{tr}C(s)P(s)}, \tag{20}$$

where $K_{tr} = K_{eq}(K_{I_0})$ is the truncation gain, and

$$K_{I_0} = \min\{K > 0 : I_0 > 0.4\}. \tag{21}$$

Note that when $P(s)$ has a pole at the origin, $I_0 = 0$, and the S-root locus has no S-truncation points.

To illustrate the utility of S-truncation points, consider the following example.

Example 3. Consider the system of Figure 1 defined by

$$P(s) = \frac{s + 20}{(s + 15)(s + 0.5)}, C(s) = 1, \alpha = 0.8, \Omega = 1,$$

and $F_\Omega(s)$ as in (8). As obtained from Theorem 1, the limiting gain of this system is $\kappa = 20.015$ (the S-termination points are obtained through (14)). Using (21), we determine that $K_{I_0} = 2.2$, and hence the truncation gain can then be evaluated as $K_{tr} = K_{eq}(2.2) = 2.15$. The resulting S-root locus is given in Figure 8, where the S-truncation points are denoted by solid squares. Although the S-root locus enters the shaded region for high quality dynamic tracking, the position of the truncation points limit the achievable performance. Figure 9 illustrates the output response of the system when the control gain is

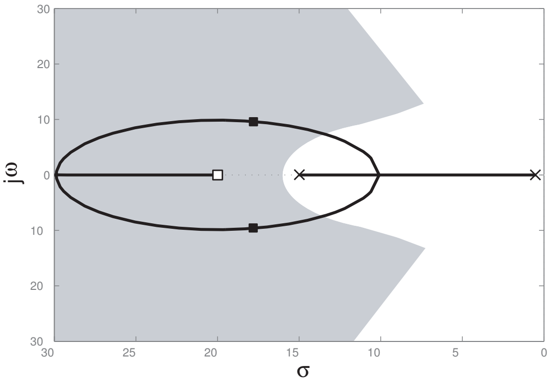

Fig. 10. S-root locus for Example 3 with $\alpha = 1.5$

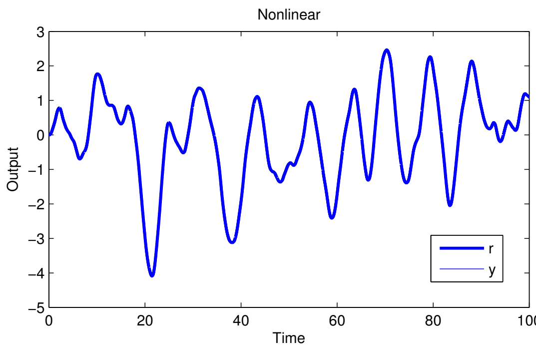

Fig. 11. Tracking quality for Example 3 with $\alpha = 1.5$

$K = 20 > K_{I_0}$. The saturated closed loop poles are located at $-16 \pm j9.5$ (i.e., in the admissible domain), but beyond the truncation points. This leads to good dynamic tracking, but with a clipped response. Clearly, this can be remedied by increasing α, as illustrated in Figure 10, which shows the S-root locus when $\alpha = 1.5$. Figure 11 shows the corresponding plot for the same location of S-poles as before. Obviously, the clipping practically does not occur (the output overlays the reference).

In conclusion, note that if S-truncation points exist, then they occur prior to the S-termination points (since the latter correspond to $K = \infty$).

6. CALIBRATION OF THE S-ROOT LOCUS

Let s^* be an arbitrary point on the S-root locus, i.e.,

$$1 + K_{eq}(K)\, C(s^*)\, P(s^*) = 0, \tag{22}$$

where

$$0 \le K_{eq}(K) \le \kappa. \tag{23}$$

To calibrate the S-root locus implies to find the particular K such that (22) is satisfied (i.e., the S-poles are located at s^*). This is accomplished by the following Theorem.

Theorem 4. For arbitrary s^* on the S-root locus, there exists a unique $K^* > 0$ such that $K = K^*$ satisfies (22). Moreover, K^* is the unique solution for

$$K_{eq} = K^* erf\left(\frac{\alpha}{\sqrt{2}K^* \left\| \frac{F_\Omega(s)C(s)}{1+K_{eq}P(s)C(s)} \right\|_2}\right), \tag{24}$$

where

$$K_{eq} = \frac{1}{|C(s^*)\, P(s^*)|}. \tag{25}$$

Note that (24) can be solved by a standard bisection algorithm. Figure 12 illustrates the differences in calibration between an unsaturated and S-root locus (using the system of Example 1).

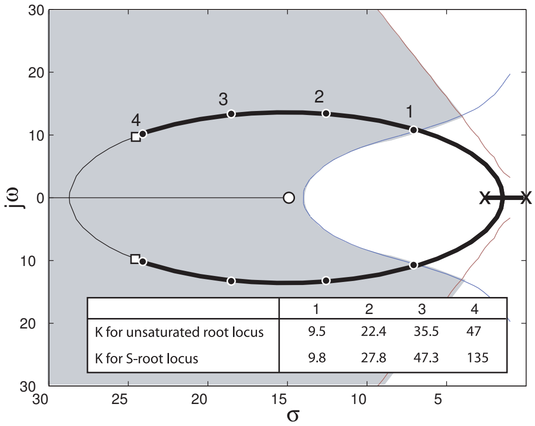

Fig. 12. Difference in calibration between unsaturated and S-root locus

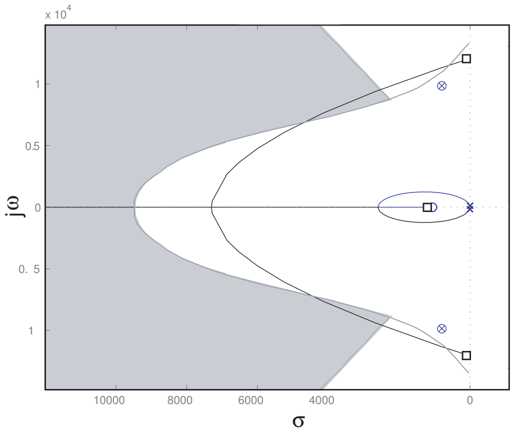

Fig. 13. S-Root Locus for Hard Disk Drive

7. APPLICATION: HARD DISK DRIVE

We consider the hard disk drive servo problem of Ching et al. (2007a), where the control objective is to maintain the disk head above a circular track that exhibits random irregularities that can be modelled as a bandlimited noise. The model for the plant was given as

$$P_D(s) = \frac{4.382 \times 10^{10} s + 4.382 \times 10^{15}}{s^2\left(s^2 + 1.596 \times 10^3 s + 9.763 \times 10^7\right)}, \tag{26}$$

while the controller was selected as

$$C(s) = \frac{K\,(s + 1058)\left(s^2 + 1596s + 9.763 \times 10^7\right)}{\left(s^2 + 3.719 \times 10^4 s + 5.804 \times 10^8\right)^2}. \tag{27}$$

From Goh et al. (2001), we impose the additional assumption that the input to the plant is constrained by a

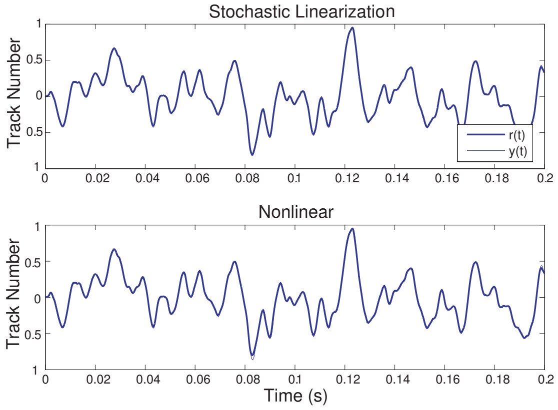

Fig. 14. Tracking performance, Hard Disk Drive example

saturation with $\alpha = 0.006$. The bandwidth of the reference is $\Omega = 692 rad/s$, and $F_\Omega(s)$ is as defined in (8). For this system, the conditions of Theorem 2 hold, and it results in

$$\kappa = 1.0509 \times 10^6.$$

The corresponding S-root locus is shown in Figure 13, where the admissible domain is indicated by the shaded region. Note that since $P_D(s)$ contains poles at the origin, the S-root locus does not have S-truncation points. To meet the performance specification, we select K so that

$$K_{eq}(K) = 5.7214 \times 10^5, \tag{28}$$

which results in a pair of dominant S-poles located at $(-2.83 \pm j9.06) \times 10^3$, i.e., in the admissible domain. Using Theorem 4 with (28) yields:

$$K = 6.208 \times 10^5.$$

Figure 14 illustrates the tracking quality for both the stochastically linearized and original nonlinear systems. As predicted, the nonlinear system exhibits a tracking quality similar to that of the stochastically linearized system, and achieves a standard deviation $\sigma_e = 0.047\sigma_r$. It is noteworthy that this performance matches that obtained in Eun et al. (2005), where a nonlinear antiwindup controller was utilized.

8. CONCLUSIONS

This paper has presented a new methodology - the *S-root locus* - to design controllers for tracking random reference signals in the presence of saturation. The technical approach utilizes stochastic linearization, whereby the saturation nonlinearity is replaced by a static gain that depends on the variance of the signal at its input. The poles of the resulting quasilinear system are the so-called S-poles and can be used to predict the quality of tracking. Specifically, good tracking requires that the S-poles be located in desirable regions of the complex plane. The path traced by the S-poles as the controller gain changes is the so-called S-root locus and is always a subset of the unsaturated root locus, i.e., the root locus obtained by removing the saturation. Hence, the S-root locus methodology introduced in this paper is similar to the classical root locus methodology in that both require certain poles to be located in desirable regions of the complex plane.

There are, however, significant differences between the S-root locus and the classical root-locus methodologies. Specifically:

a) *S-termination:* As the control gain tends to infinity, the equivalent gain may tend to a finite value (see Theorem 1). This phenomenon is referred to as S-termination and results in the S-root locus terminating at points prior to the open loop zeros.
b) *S-truncation:* The phenomenon of S-truncation refers to quality degradation due to output truncation. This occurs when the magnitude of the signal to be tracked lies outside of the trackable domain specified, in particular, by the level of saturation.

The formulation of the S-root locus design methodology and the analytic description of the above phenomena constitute the main original contributions of this paper.

The results presented herein have the potential to be broadly applicable because saturation is a ubiquitous limitation of control systems. Moreover, as suggested by Section 7 and our experience, these results have the potential to yield linear controllers that match the performance of much more elaborate controllers in the literature. However, only extensive use of this new methodology will confirm its potential.

Future work will treat the problems of filtering and control in the presence of sensor nonlinearities such as dead zones, quantization and saturation.

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