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Delay-Dependent Exponential Stability of Linear Distributed Parameter Systems

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Abstract: Exponential stability analysis via Lyapunov-Krasovskii method is extended to linear time-delay systems in a Hilbert space. The operator acting on the delayed state is supposed to be bounded. The system delay is admitted to be unknown and time-varying with an *a priori* given upper bound on the delay. Sufficient delay-dependent conditions for exponential stability are derived in the form of Linear Operator Inequalities (LOIs), where the decision variables are operators in the Hilbert space. Being applied to a heat equation and to a wave equation, these conditions are represented in terms of standard Linear Matrix Inequalities (LMIs). The proposed method is expected to provide effective tools for robust control of distributed parameter systems with time-delay.

Keywords: distributed parameter systems, time-varying delay, stability, Lyapunov method, LMI.

1. INTRODUCTION

Time-delay often appears in many control systems and, in many cases, delay is a source of instability (Hale & Lunel, 1993). In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. Datko (1988), Logemann *et al.* (1996), Rebarber & Townley (1998), Nicaise & Pignotti (2006)). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

During the last decade, a considerable amount of attention has been paid to stability of Ordinary Differential Equations (ODEs) with uncertain constant or time-varying delays (see e.g. Kolmanovskii & Myshkis (1999), Niculescu (2001), Gu *et al.* (2003), Richard (2003)).

The stability analysis of Partial Differential Equations (PDEs) with delay is essentially more complicated. There are only a few works on Lyapunov-based technique for PDEs with delay. The second Lyapunov method was extended to abstract nonlinear time-delay systems in the Banach spaces in Wang (1994a). Stability conditions and exponential bounds were derived for some scalar heat and wave equations with constant delays and with Dirichlet boundary conditions in Wang (1994b), Wang(2006). Stability and instability conditions for wave delay equations were found in (Nicaise & Pignotti, 2006). Recently conditions for asymptotic stability of linear parabolic systems were derived in (Fridman & Orlov, 2007) via Lyapunov method. However, extension of the method of (Fridman & Orlov, 2007) to hyperbolic systems seems to be complicated.

In the present paper we study the exponential stability of general distributed parameter systems. A class of linear systems is considered, where a bounded operator acts on the delayed state. The system delay is admitted to be unknown and time-varying. Sufficient *delay-dependent* exponential stability conditions are derived in the form of LOIs, where the decision variables are operators in the Hilbert space. Being applied to a *heat* equation and to a *wave equation*, these conditions are represented in terms of standard finite-dimensional LMIs. We note that the delay-independent stability conditions were recently derived in (Orlov & Fridman, 2007), where the results were applied to the heat equation.

Notation and Preliminaries

The notation used throughout is fairly standard. The superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$ means that P is symmetric and positive definite, whereas $\lambda_{min}(P)$ and $\lambda_{max}(P)$ denote its minimum and maximum eigenvalues.

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot|$. The identity operator in $\mathcal{H}$ is denoted by I. Given a linear operator $P : \mathcal{H} \to \mathcal{H}$ with a dense domain $\mathcal{D}(P) \subset \mathcal{H}$, the notation P^* stands for the adjoint operator. Such an operator P is strictly positive definite, i.e., $P > 0$, iff it is self-adjoint in the sense that $P = P^*$ and there exists a constant $\beta > 0$ such that $\langle x, Px\rangle \geq \beta\langle x, x\rangle$ and for all$x \in \mathcal{D}(P)$, whereas $P \geq 0$ means that P is self-adjoint and nonnegative definite, i.e., $\langle x, Px\rangle \geq 0$ for all $x \in \mathcal{D}(P)$.

If an infinitesimal operator A generates a strongly continuous semigroup $T(t)$ on the Hilbert space $\mathcal{H}$ (see, e.g., Curtain & Zwart (1995) for details), the domain of the operator A forms another Hilbert space $\mathcal{D}(A)$ with the graph inner product $(\cdot,\cdot)_{\mathcal{D}(A)}$ defined as follows: $(x,y)_{\mathcal{D}(A)} = \langle x,y\rangle + \langle Ax,Ay\rangle,\ x,y \in \mathcal{D}(A)$. Moreover, the induced norm $\|T(t)\|$ of the semigroup $T(t)$ satisfies the inequality $\|T(t)\| \le \kappa e^{\sigma t}$ everywhere with some constant $\kappa > 0$ and growth bound σ.

The space of the continuous $\mathcal{H}$-valued functions $x : [a,b] \to \mathcal{H}$ with the induced norm $\|x\|_{C([a,b],\mathcal{H})} = \max_{s\in[a,b]} |x(s)|$ is denoted by $C([a,b],\mathcal{H})$. The space of the continuously differentiable $\mathcal{H}$-valued functions $x : [a,b] \to \mathcal{H}$ with the induced norm $\|x\|_{C^1([a,b],\mathcal{H})} = \max(\|x\|_{C([a,b],\mathcal{H})}, \|\dot{x}\|_{C([a,b],\mathcal{H})})$ is denoted by $C^1([a,b],\mathcal{H})$.

$L_2(a,b;\mathcal{H})$ is the Hilbert space of square integrable $\mathcal{H}$-valued functions on (a,b) with the corresponding norm; $L_2(a,b;\mathbf{R}) := L_2(a,b)$.

$W^{l,2}([a,b],R)$ is the Sobolev space of absolutely continuous scalar functions on $[a,b]$ with square integrable derivatives of the order $l \ge 1$

Given $x(\cdot) \in L_2([a,b],\mathcal{H})$, we denote $x^t = x(t+\theta) \in L_2([-h,0],\mathcal{H})$ for $t \in [a+h,b]$; to reduce the notational burden the dependence x^t on θ is subsequently suppressed.

Lemma 1. Wang (1994b) (Wirtinger's inequality). Let $u \in W^{1,2}([a,b],R)$ be a scalar function with $u(a) = u(b) = 0$. Then

$$\int_a^b u^2(\xi)d\xi \le \frac{(b-a)^2}{\pi^2}\int_a^b (u'(\xi))^2 d\xi. \tag{1}$$

Lemma 2. (Jensen's inequality). Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. For any linear bounded operator $R : \mathcal{H} \to \mathcal{H}, R > 0$, scalar $l > 0$ and $x \in L_2([a,b],\mathcal{H})$ the following holds:

$$l\int_0^l \langle x(s), Rx(s)\rangle ds \ge \langle \int_0^l x(s)ds, R\int_0^l x(s)ds\rangle. \tag{2}$$

We note that (2) follows from the Cauchy-Schwartz inequality

$$l\int_0^l \langle R^{\frac{1}{2}}x(s), R^{\frac{1}{2}}x(s)\rangle ds \ge \langle \int_0^l R^{\frac{1}{2}}x(s)ds, \int_0^l R^{\frac{1}{2}}x(s)ds\rangle.$$

2. PROBLEM FORMULATION

Consider a linear infinite-dimensional system

$$\dot{x}(t) = Ax(t) + A_1x(t-\tau(t)), \quad t \ge t_0 \tag{3}$$

evolving in a Hilbert space $\mathcal{H}$ where $x(t) \in \mathcal{H}$ is the instantaneous state of the system, the system delay $\tau(t)$ is a piecewise continuous function of class C^1 on each continuity subinterval and it satisfies

$$\inf_t \tau(t) > 0, \ \sup_t \tau(t) \le h \tag{4}$$

for some constant $h > 0$, A_1 is a linear bounded operator, A is an infinitesimal operator, generating a strongly continuous semigroup $T(t)$, and the domain $\mathcal{D}(A)$ of the operator A is dense in $\mathcal{H}$.

Throughout, solutions of such a system are defined in the Caratheodory sense, i.e., equation (3) is required to hold almost everywhere only. Let the initial conditions

$$x^{t_0} = \varphi(\theta),\ \theta \in [-h,0], \phi \in W \tag{5}$$

be given in the space

$$W = C([-h,0],\mathcal{D}(A)) \cap C^1([-h,0],\mathcal{H}). \tag{6}$$

Definition 1. A function $x(t) \in C([t_0-h,t_0+\eta],\mathcal{D}(A))$ is said to be a solution of the initial-value problem (3), (5) on $[t_0-h,t_0+\eta]$ if $x(t)$ is initialized with (5), it is absolutely continuous for $t \in [t_0,t_0+\eta]$, and it satisfies (3) for almost all $t \in [t_0,t_0+\eta]$.

The initial-value problem (3), (5) turns out to be well-posed on the semi-infinite time interval $[t_0,\infty)$ (Orlov & Fridman, 2007).

Our aim is to derive robust stability criteria for linear time-delay systems (3), (4), thus defined. The stability concept under study is based on the initial data norm

$$\|\phi\|_W = |A\phi(0)| + \|\phi\|_{C^1([-h,0],\mathcal{H})} \tag{7}$$

in space (6). Suppose $x(t,t_0,\phi)$ denotes a solution of (3), (5) at a time instant $t \ge t_0$.

Definition 2. System (3) is said to be exponentially stable with a decay rate $\delta > 0$ if there exists a constant $K \ge 1$ such that the following exponential estimate holds:

$$|x(t,t_0,\phi)|^2 \le Ke^{-2\delta(t-t_0)}\|\phi\|_W^2 \quad \forall t \ge t_0. \tag{8}$$

"Quasi delay-independent" exponential stability conditions, which become delay-independent for $\delta \to 0$, have been recently derived in (Orlov & Fridman, 2007). In the present paper we derive delay-dependent conditions.

3. DELAY-DEPENDENT EXPONENTIAL STABILITY

Consider Lyapunov-Krasovskii Functionals (LKFs), which depend on x and $\dot{x}$ (Kolmanovskii & Myshkis, 1999). Given a continuous functional

$$V : \mathbf{R} \times W \times C([-h,0],\mathcal{H}) \to \mathbf{R}, \tag{9}$$

its upper right-hand derivative along solutions $x^t(t_0,\phi),\ t \ge t_0$ of (3), (5) is defined as follows:

$$\dot{V}(t,\phi,\dot{\phi}) = \limsup_{s\to 0^+}\frac{1}{s}[V(t+s,x^{t+s}(t,\phi),\dot{x}^{t+s}(t,\phi)) - V(t,\phi,\dot{\phi})].$$

Lemma 3. (Orlov & Fridman, 2007) Let there exist positive numbers $\delta,\ \beta,\ \gamma$ and a continuous functional

$$V : \mathbf{R} \times W \times C([-h,0],\mathcal{H}) \to \mathbf{R}$$

such that the function $\bar{V}(t) = V(t,x^t,\dot{x}^t)$ is absolutely continuous for x^t, satisfying (3), and

$$\begin{aligned} &\beta|\phi(0)|^2 \le V(t,\phi,\dot{\phi}) \le \gamma\|\phi\|_W^2, \\ &\dot{V}(t,\phi,\dot{\phi}) + 2\delta V(t,\phi,\dot{\phi}) \le 0. \end{aligned} \tag{10}$$

Then (3) is exponentially stable with the decay rate δ and (8) holds with $K = \frac{\gamma}{\beta}$.

3.1 Delay-Dependent LOI in the Hilbert Space

In this section, the delay is assumed to be either slowly-varying with $\dot{\tau} \leq d < 1$, or fast-varying (with no restrictions on the delay-derivative). We consider a "simple" (as defined in Gu *et al.* 2003) LKF:

$$\begin{aligned} &V(t,x^t,\dot{x}^t) = \langle x(t), Px(t)\rangle \\ &+ \int_{t-h}^{t} e^{2\delta(s-t)}\langle x(s), Sx(s)\rangle ds \\ &+h \int_{-h}^{0}\int_{t+\theta}^{t} e^{2\delta(s-t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds d\theta \\ &+ \int_{t-\tau(t)}^{t} e^{2\delta(s-t)}\langle x(s), Qx(s)\rangle ds \end{aligned} \tag{11}$$

where $R, Q, S \in \mathcal{L}(\mathcal{H})$ and $R, Q, S \geq 0$. Moreover,

$$\begin{aligned} &\langle x, Px\rangle \leq \gamma_P[\langle x, x\rangle + \langle Ax, Ax\rangle], \\ &\langle x, Qx\rangle \leq \gamma_Q\langle x, x\rangle, \\ &\langle x, Rx\rangle \leq \gamma_R\langle x, x\rangle,\ \langle x, Sx\rangle \leq \gamma_S\langle x, x\rangle \end{aligned} \tag{12}$$

for all $x \in D(A)$ and for some positive constants $\gamma_P, \gamma_Q, \gamma_S, \gamma_R$.

Differentiating V, we find

$$\begin{aligned} &\dot{V}(t,x^t,\dot{x}^t) + 2\delta V(t,x^t,\dot{x}^t) \\ &\leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle \\ &-he^{-2\delta h}\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds + \langle x(t), (Q+S)x(t)\rangle \\ &-(1-\dot{\tau}(t))\langle x(t-\tau(t)), Qx(t-\tau(t)\rangle e^{-2\delta h} \\ &-\langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}. \end{aligned} \tag{13}$$

Following He *et al.* (2007), we employ the representation

$$\begin{aligned} -h\int_{t-h}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds &= -h\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \\ &-h\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \end{aligned} \tag{14}$$

and apply the Jensen's inequality (2)

$$\begin{aligned} &\int_{t-\tau(t)}^{t}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \\ &\geq \frac{1}{h}\langle \int_{t-\tau(t)}^{t}\dot{x}(s)ds, R\int_{t-\tau(t)}^{t}\dot{x}(s)ds\rangle, \\ &\int_{t-h}^{t-\tau(t)}\langle \dot{x}(s), R\dot{x}(s)\rangle ds \\ &\geq \frac{1}{h}\langle \int_{t-h}^{t-\tau(t)}\dot{x}(s)ds, R\int_{t-h}^{t-\tau(t)}\dot{x}(s)ds\rangle. \end{aligned} \tag{15}$$

Then, taking into account that $\dot{\tau} \leq d < 1$ and following Gouasbaut & Peaucelle (2006), we obtain

$$\begin{aligned} &\dot{V}(t,x^t,\dot{x}^t) + 2\delta V(t,x^t,\dot{x}^t) \\ &\leq 2\langle x(t), P\dot{x}(t)\rangle + 2\delta\langle x(t), Px(t)\rangle + h^2\langle \dot{x}(t), R\dot{x}(t)\rangle \\ &-\Big[\langle x(t)-x(t-\tau(t)), R(x(t)-x(t-\tau(t)))\rangle \\ &+\langle x(t-\tau(t))-x(t-h), R(x(t-\tau(t))-x(t-h))\rangle \\ &+(1-d)\langle x(t-\tau(t)), Qx(t-\tau(t)\rangle\Big]e^{-2\delta h} \\ &+\langle x(t), (Q+S)x(t)\rangle - \langle x(t-h), Sx(t-h)\rangle e^{-2\delta h}. \end{aligned} \tag{16}$$

We will derive stability conditions in two forms. The first one is derived by substituting the right-hand side of (3) for $\dot{x}(t)$. Setting $\eta(t) = col\{x(t), x(t-h), x(t-\tau(t))\}$, we find that

$$\dot{V}(t,x^t,\dot{x}^t) + 2\delta V(t,x^t,\dot{x}^t) \leq \langle \eta(t), \Phi_h\eta(t)\rangle \leq 0 \tag{17}$$

is satisfied if the following LOI

$$\begin{aligned} \Phi_h = &\begin{bmatrix} \Phi_{11} & 0 & PA_1 \\ 0 & 0 & 0 \\ A_1^*P & 0 & 0 \end{bmatrix} + h^2\begin{bmatrix} A^*RA & 0 & A^*RA_1 \\ 0 & 0 & 0 \\ A_1^*RA & 0 & A_1^*RA_1 \end{bmatrix} \\ &-e^{-2\delta h}\begin{bmatrix} R & 0 & -R \\ 0 & (S+R) & -R \\ -R & -R & 2R+(1-d)Q \end{bmatrix} \leq 0, \end{aligned} \tag{18}$$

holds provided that

$$\Phi_{11} = A^*P + PA + 2\delta P + Q + S. \tag{19}$$

The resulting inequality (18) is convex with respect to h: given $h_0 > 0$, it becomes feasible for all $\bar{h} \in [0, h_0]$ whenever it is feasible for h_0. The convexity follows from the fact that $\Phi_{\bar{h}} \leq \Phi_{h_0}$ since h^2 and $-e^{-2\delta h}$ multiply the non negative definite operators. Summarizing, the following result is obtained:

Theorem 1. Given $\delta > 0$, let there exist a positive definite operator P, bounded on $\mathcal{D}(A)$, and non negative definite operators $R, Q, S \in \mathcal{L}(\mathcal{H})$ such that the LOI (18) with notations (19) holds in the Hilbert space $\mathcal{D}(A)\times\mathcal{D}(A)\times\mathcal{D}(A)$. Then system (3) is exponentially stable for all differentiable delays $0 \leq \tau(t) \leq h$ with $\dot{\tau} \leq d < 1$. The inequality (8) is satisfied with this δ and $K = \max\{\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)\}/\beta$. Moreover, (3) is exponentially stable for all fast-varying delays $0 \leq \tau \leq h$ if the LOI (18) is feasible with $Q = 0$.

It may be difficult to verify the feasibility of (18), if the operator that multiplies h^2 (and depends on A) in Φ_h is unbounded. To avoid this, we will derive the second form of LOI by the descriptor method (Fridman, 2001), where the right-hand sides of the expressions

$$\begin{aligned} 0 &= 2\langle x(t), P_2^*[Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)]\rangle, \\ 0 &= 2\langle \dot{x}(t), P_3^*[Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)]\rangle \end{aligned} \tag{20}$$

with some $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ are added into the right-hand side of (16).

Setting $\eta_d(t) = col\{x(t), \dot{x}(t), x(t-h), x(t-\tau(t))\}$, we obtain that

$$\dot{V}(t,x^t,\dot{x}^t) + 2\langle x(t), P\dot{x}(t)\rangle \leq \langle \eta_d(t), \Phi_d\eta_d(t)\rangle \leq 0, \tag{21}$$

if the LOI

$$\begin{aligned} &\Phi_d = \\ &\begin{bmatrix} \Phi_{d11} & \Phi_{d12} & 0 & P_2^*A_1 + Re^{-2\delta h} \\ * & \Phi_{d22} & 0 & P_3^*A_1 \\ * & * & -(S+R)e^{-2\delta h} & Re^{-2\delta h} \\ * & * & * & -[2R+(1-d)Q]e^{-2\delta h} \end{bmatrix} \leq 0 \end{aligned} \tag{22}$$

holds, where

$$\begin{aligned}\Phi_{d11} &= A^*P_2 + P_2^*A + 2\delta P + Q + S - Re^{-2\delta h},\\ \Phi_{d12} &= P - P_2^* + A^*P_3,\ \Phi_{d22} = -P_3 - P_3^* + h^2 R.\end{aligned} \tag{23}$$

and $*$ denotes the symmetric terms of the matrix. Thus, the following result is obtained.

Theorem 2. Given $\delta > 0$, let there exist a positive definite operator P, bounded on $\mathcal{D}(A)$, non negative definite operators $R, Q, S \in \mathcal{L}(\mathcal{H})$, and indefinite operators $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ such that the LOI (22) with notations given in (23) holds in the Hilbert space $\mathcal{D}(A)\times\mathcal{D}(A)\times\mathcal{D}(A)\times\mathcal{D}(A)$. Then system (3) is exponentially stable with a decay rate δ for all differentiable delays (4) with $\dot\tau \le d < 1$. Moreover, (3) is exponentially stable for all fast-varying delays $0 \le \tau \le h$ if the LOI (22) is feasible with $Q = 0$.

3.2 Delay-Dependent Stability of the Heat Equation

Consider the heat equation

$$\begin{aligned}&u_t(\xi,t) = au_{\xi\xi}(\xi,t) - a_1 u(\xi, t-\tau(t)),\\ &t \ge t_0,\ 0 \le \xi \le \pi\end{aligned} \tag{24}$$

with the constant parameters $a > 0$ and a_1, with the time-varying delay $\tau(t)$, satisfying (4), and with the Dirichlet boundary condition

$$u(0,t) = u(\pi,t) = 0,\ t \ge t_0. \tag{25}$$

The boundary-value problem (24), (25) describes the propagation of heat in a homogeneous one-dimensional rod with a fixed temperature at the ends in the case of the delayed (possibly, due to actuation) heat exchange with the surroundings. Here a and a_1 stand for the heat conduction coefficient and for the coefficient of the heat exchange with the surroundings, respectively, $u(\xi,t)$ is the value of the temperature field of the plant at time moment t and location ξ along the rod. In the sequel, the state dependence on time t and spatial variable ξ is suppressed whenever possible.

The boundary-value problem (24), (25) can be rewritten as the differential equation (3) in the Hilbert space $\mathcal{H} = L_2(0,\pi)$ with the infinitesimal operator $A = a\frac{\partial^2}{\partial\xi^2}$ of double differentiation with the dense domain

$$\mathcal{D}(\frac{\partial^2}{\partial\xi^2}) = \{u \in W^{2,2}([0,\pi],\mathbf{R}) : u(0) = u(\pi) = 0\}, \tag{26}$$

and with the bounded operator $A_1 = -a_1$ of the multiplication by the constant $-a_1$. The infinitesimal operator A generates an exponentially stable semigroup (see, e.g., Curtain & Zwart (1995) for details).

We apply Theorem 2 and choose the LKF of the form

$$\begin{aligned}V(t,u^t,u_s^t) &= (p_1 - p_3 a)\int_0^\pi u^2(\xi,t)d\xi\\ &+p_3 a\int_0^\pi u_\xi^2(\xi,t)d\xi\\ &+\int_0^\pi \Big[r\int_{-h}^0\int_{t+\theta}^t e^{2\delta(s-t)}u_s^2(\xi,s)dsd\theta\\ &+s\int_{t-h}^t e^{2\delta(s-t)}u^2(\xi,s)ds\\ &+q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u^2(\xi,s)ds\Big]d\xi\end{aligned} \tag{27}$$

with some constants $p_1 > 0, p_3 > 0, s > 0, r > 0$ and $q \ge 0$. Then the operators in (11) take the form

$$P = -p_3(a\frac{\partial^2}{\partial\xi^2} + a) + p_1,\ R = r,\ Q = q,\ S = s. \tag{28}$$

In order to apply Theorem 2 we first note that $P > 0$ (see Fridman & Orlov, 2007). Now setting $P_2 = p_2$ and $P_3 = p_3$, where $p_2 > 0$ and $p_2 - \delta p_3 \ge 0$, we obtain that

$$\begin{aligned}&\langle\dot x, (P - P_2^* + A^*P_3)x\rangle = \langle\dot x, (p_1 - p_2 - ap_3)x\rangle;\\ &\langle x, A^*P_2x\rangle + \langle x, P_2^*Ax\rangle + 2\delta\langle x, Px\rangle\\ &= 2a(p_2 - \delta p_3)\int_0^\pi u_{\xi\xi}ud\xi + 2\delta(p_1 - ap_3)\int_0^\pi u^2 d\xi\\ &= -2a(p_2 - \delta p_3)\int_0^\pi u_\xi^2 d\xi + 2\delta(p_1 - ap_3)\int_0^\pi u^2 d\xi\\ &\le [-2ap_2 + 2\delta p_1]\int_0^\pi u^2 d\xi,\end{aligned}$$

where the latter inequality follows from the Wirtinger's inequality (1). Therefore, (22) holds if

$$\begin{aligned}&\begin{bmatrix}\phi_{11} & \phi_{12} & 0 & \phi_{14}\\ * & -2p_3 + h^2 r & 0 & -p_3 a_1\\ * & * & -(s+r)e^{-2\delta h} & re^{-2\delta h}\\ * & * & * & \phi_{44}\end{bmatrix} < 0,\\ &\phi_{11} = -2ap_2 + 2\delta p_1 + q + s - re^{-2\delta h},\\ &\phi_{12} = p_1 - p_2 - ap_3,\\ &\phi_{14} = -p_2 a_1 + re^{-2\delta h},\ \phi_{44} = -[2r + (1-d)q]e^{-2\delta h}.\end{aligned} \tag{29}$$

Summarizing the following result is obtained

Theorem 3. Given $\delta > 0$, let there exist scalars $p_1 > 0, p_2 > 0, p_3 > 0, s > 0, r > 0$ and $q \ge 0$ such that $p_2 - \delta p_3 \ge 0$ and the LMI (29) holds. Then the boundary-value problem (24), (25) is exponentially stable with a decay rate δ for all differentiable delays (4) with $\dot\tau \le d < 1$. Moreover, (24), (25) is exponentially stable with a decay rate δ for all fast varying delays (4) with no restrictions on $\dot\tau$ if LMIs $p_2 - \delta p_3 \ge 0$ and (29) are feasible with $q = 0$. If (29) holds for $\delta = 0$, then (24), (25) is exponentially stable with a sufficiently small decay rate.

Remark 1. The same LMI (29) guarantees the exponentially stability of the scalar ODE

$$\dot y(t) + ay(t) + a_1 y(t - \tau(t)) = 0, \tag{30}$$

System (30) corresponds to the first modal dynamics (with $k = 1$) in the modal representation of the Dirichlet boundary-value problem (24), (25)

$$\begin{aligned} &\dot{y}_k(t) + ak^2 y_k(t) + a_1 y_k(t - \tau(t)) = 0, \\ &k = 1, 2, \dots \end{aligned} \tag{31}$$

projected on the eigenfunctions of the operator $\frac{\partial^2}{\partial \xi^2}$ (this operator has eigenvalues $-k^2$, see e.g. Wu (1996)). The stability of (24), (25) implies the stability of (31). Thus the reduction of infinite-dimensional LOI of Theorem 2 to finite-dimensional LMI of Theorem 3 is tight, since the stability of (30) is necessary for the stability of (24), (25).

The above is consistent with the frequency domain analysis in the case of constant delays, where the characteristic equations of (24), (25) are given by

$$\lambda_k + ak^2 + a_1 e^{-\lambda_k \tau} = 0, \quad k = 1, 2, \dots \tag{32}$$

(see, e.g., Wu, 1996). The exponential stability of (24), (25) is shown in Huang & Vanderwalle (2004) to be determined by (32) with $k = 1$, i.e., by the stability of the ODE (30).

Example 1. Consider

$$\begin{aligned} &u_t(\xi, t) = 0.9 u_{\xi\xi}(\xi, t) - u(\xi, t - \tau(t)), \\ &t \geq 0, \ 0 \leq \xi \leq \pi \end{aligned} \tag{33}$$

with boundary condition (25) and $0 \leq \tau \leq h, \dot{\tau} \leq d < 1$. Applying Theorem 3 (with $\delta = 0$), we verify the feasibility of LMI (29) by using LMI toolbox of Matlab. Letting d to be 0.5 and unknown, respectively, we find the maximum values of h for which the system remains exponentially stable:

$$d = 0.5, \ h = 2.04; \ \textit{unknown } d, \ h = 1.34.$$

As shown before (see Remark 1), the latter results correspond also to the stability of ODE $\dot{y} = -0.9y(t) - y(t - \tau(t))$. These results for ODE coincide with the results of Example 7 in (He *et al.*, 2007).

3.3 Delay-Dependent Stability of the Wave Equation

Consider the wave equation

$$\begin{aligned} &u_{tt}(\xi, t) = a u_{\xi\xi} - \mu_0 u_t(\xi, t) - a_0 u(\xi, t) \\ &-a_1 u(\xi, t - \tau(t)), \quad t \geq 0, \ 0 \leq \xi \leq \pi \end{aligned} \tag{34}$$

with the Dirichlet boundary condition (25). The boundary-value problem (25), (34) describes the oscillations of a homogeneous string with fixed ends in the case of the delayed (possibly, due to actuation) stiffness restoration and dissipation. Here a stands for the elasticity coefficient, μ_0 stand for the dissipation coefficient, and a_0, a_1 stand for the restoring stiffness coefficients, the state vector $x = col\{u, u_t\}$ consists of the deflection $u(\xi, t)$ of the string and its velocity $u_t(\xi, t)$ at time moment t and location ξ along the string.

Let us introduce the operators

$$A = \begin{bmatrix} 0 & 1 \\ a\frac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ -a_1 & 0 \end{bmatrix} \tag{35}$$

where the domain $\mathcal{D}(\frac{\partial^2}{\partial \xi^2})$ of the double differentiation operator is still determined by (26). Then the boundary-value problem (25), (34) can be represented as the differential equation (3) in the Hilbert space $\mathcal{H} = L_2(0, \pi) \times L_2(0, \pi)$ with the infinitesimal operator A, possessing the domain $\mathcal{D}(A) = \mathcal{D}(\frac{\partial^2}{\partial \xi^2}) \times L_2(0, \pi)$ and generating an exponentially stable semigroup (see, e.g., Curtain & Zwart (1995) for details). We apply the conditions of Theorem 1. Since the delay appears only in u, we choose V as follows:

$$\begin{aligned} V &= ap_3 \int_0^\pi u_\xi^2(\xi, t) d\xi + \int_0^\pi [u(\xi, t) \ u_t(\xi, t)] P_0 \\ &\times \begin{bmatrix} u(\xi, t) \\ u_t(\xi, t) \end{bmatrix} d\xi + \int_0^\pi \Big[hr \int_{-h}^0 \int_{t+\theta}^t u_t^2(\xi, s) e^{2\delta(s-t)} ds d\theta \\ &+ s \int_{t-h}^t u^2(\xi, s) e^{2\delta(s-t)} ds + q \int_{t-\tau}^t u^2(\xi, s) e^{2\delta(s-t)} ds \Big] d\xi, \\ P_0 &= \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}, \ P_w = \begin{bmatrix} ap_3 + p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0. \end{aligned}$$

where $r > 0, \ s > 0, \ q \geq 0$. Then the operators P, Q, R in (18) are given by

$$\begin{aligned} &P = \begin{bmatrix} -ap_3 \dfrac{\partial^2}{\partial \xi^2} + p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} > 0, \ Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \\ &R = diag\{r, 0\} \geq 0, \ S = diag\{s, 0\} \geq 0. \end{aligned} \tag{36}$$

We have $h^2 A^* R A = diag\{0, h^2 r\}$, $h^2 A^* R A_1 = 0$, $h^2 A_1^* R A_1 = 0$. Now, integrating by parts and taking into account the inequality $p_3 > 0$ (extracted from $P_w > 0$) and Wirtinger's inequality we obtain that

$$\begin{aligned} &\langle x, Px \rangle = \int_0^\pi \left[-ap_3 u_{\xi\xi} u + u^T P_0 u \right] d\xi = a \int_0^\pi p_3 (u_\xi)^2 d\xi \\ &+ \langle x, P_0 x \rangle \geq \langle x, P_w x \rangle \geq \lambda_{min}(P_w) |x|^2 > 0 \end{aligned} \tag{37}$$

for all $x \in \mathcal{D}(A) \times L_2(0, \pi)$. It is thus shown that the operator P is strictly positive definite whereas the matrix operator Q is so by construction.

Finally, integrating by parts and applying Wirtinger's inequality (1) under condition $p_2 - p_3 \delta \geq 0$ yield

$$\begin{aligned} &-\langle x, P(A - \delta)x \rangle - \langle x, (A^* - \delta)Px \rangle = \int_0^\pi [u \ u_t] \\ &\times \Bigg\{ \begin{bmatrix} p_1 - ap_3 \dfrac{\partial^2}{\partial \xi^2} & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} \delta & 1 \\ a\dfrac{\partial^2}{\partial \xi^2} - a_0 & -\mu_0 + \delta \end{bmatrix} \\ &+ \begin{bmatrix} \delta & a\dfrac{\partial^2}{\partial \xi^2} - a_0 \\ 1 & -\mu_0 + \delta \end{bmatrix} \begin{bmatrix} p_1 - ap_3 \dfrac{\partial^2}{\partial \xi^2} & p_2 \\ p_2 & p_3 \end{bmatrix} \Bigg\} \begin{bmatrix} u \\ u_t \end{bmatrix} d\xi \\ &= -2a(p_2 - p_3 \delta) \int_0^\pi (u_\xi)^2 d\xi - \int_0^\pi [u \ u_t] \times \\ &\times \begin{bmatrix} -2p_2 a_0 + 2p_1 \delta & p_1 - (\mu_0 - 2\delta) p_2 - p_3 a_0 \\ p_1 - (\mu_0 - 2\delta) p_2 - p_3 a_0 & 2p_2 - 2(\mu_0 - \delta) p_3 \end{bmatrix} \\ &\times \begin{bmatrix} u \\ u_t \end{bmatrix} d\xi \leq \int_0^\pi [u \ u_t] (P_w C_\delta + C_\delta^T P_w) \begin{bmatrix} u \\ u_t \end{bmatrix} d\xi, \end{aligned} \tag{38}$$

where

$$C_\delta = \begin{bmatrix} \delta & 1 \\ -a - a_0 & -\mu_0 + \delta \end{bmatrix}. \tag{39}$$

From (38) it follows that (18) is feasible if the following LMIs are satisfied:

$$\begin{array}{l} p_2 \geq p_3\delta, \\ \begin{bmatrix} \phi_w & 0 & P_w \begin{bmatrix} 0 \\ -a_1 \end{bmatrix} + \begin{bmatrix} re^{-2\delta h} \\ 0 \end{bmatrix} \\ * & -(s+r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & -(2r + (1-d)q)e^{-2\delta h} \end{bmatrix} < 0, \end{array} \tag{40}$$

where $\phi_w = C_\delta^T P_w + P_w C_\delta + diag\{q + s - re^{-2\delta h}, h^2 r\}$. Summarizing the following result is obtained

Theorem 4. Given $\delta > 0$, let there exist a $2 \times 2-$matrix $P_w > 0$ and scalars $q \geq 0, r > 0, s > 0$ such that satisfy LMI (40). Then the wave time-delay equation (34) and with the Dirichlet boundary condition (25) is exponentially stable with a decay rate δ. Moreover, if LMIs (40) are feasible with $q = 0$, then the wave equation is exponentially stable with a decay rate δ for all fast varying delays $0 \leq \tau \leq h$. If the second LMI (40) holds for $\delta = 0$, then the wave equation is exponentially stable with a sufficiently small decay rate.

Remark 2. The same LMIs (40) appear to guarantee the ODE with delay $\dot{\bar{z}}(t) = C_0\bar{z}(t) + A_1\bar{z}(t-\tau(t)), \quad \bar{z}(t) \in \mathbf{R}^2$ or, equivalently, the scalar ODE

$$\ddot{y}(t) + \mu_0\dot{y}(t) + (a + a_0)y(t) + a_1 y(t - \tau(t)) = 0 \tag{41}$$

to be exponentially stable. As in the case of the heat equation, ODE (41) governs the first modal dynamics of the modal representation of the Dirichlet boundary-value problem (25), (34)

$$\begin{array}{l} \ddot{y}_k(t) + \mu_0\dot{y}_k(t) + (ak^2 + a_0)y_k(t) \\ +a_1 y_k(t - \tau(t)) = 0, \; k = 1, 2, \ldots \end{array} \tag{42}$$

on the eigenfunctions of the operator $\frac{\partial^2}{\partial\xi^2}$. Hence, the results of Theorem 4 are tight in the sense that the stability of ODE (41) is necessary for the stability of (25), (34).

Example 2. Consider the controlled wave equation

$$z_{tt}(\xi, t) = 0.1 z_{\xi\xi}(\xi, t) - 2z_t(\xi, t) + u, \tag{43}$$

with boundary condition (25), $t \geq t_0$, $0 \leq \xi \leq \pi, 0 \leq \tau \leq h, \dot{\tau} \leq d < 1$. Applying Theorem 4 to the open-loop system we find that (43) with $u = 0$ is exponentially stable with the decay rate $\delta = 0.05$. Considering next a delayed feedback $u = -z(\xi, t - \tau(t))$ and verifying conditions of Theorem 4, we find that the closed loop system is exponentially stable with a greater decay rate $\delta = 0.8$ for all $0 \leq \tau(t) \leq 0.31$.

4. CONCLUSIONS

A general framework is given for exponential stability analysis of linear time-delay systems in a Hilbert space with a bounded operator acting on the delayed state. Delay-dependent stability conditions are derived in terms of linear operator inequalities in the Hilbert space. In the case of a heat scalar equation and in the case of a wave scalar equation with the Dirichlet boundary conditions, these LOIs are reduced to finite-dimensional LMIs by applying new Lyapunov-Krasovskii functionals. The reduced-order LMIs coincide with the stability conditions for appropriate ODEs with delay, whereas the stability of the latter ODEs are necessary for the stability of the original boundary value problems.

LOIs are expected to provide effective tools for robust control of distributed parameter systems.

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