Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Frequency-Dependent Scaling Induced by Noncausal Linear Periodically Time-Varying Scaling for Discrete-Time Systems

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Abstract: A novel technique called noncausal linear periodically time-varying (LPTV) scaling was introduced recently, and it was shown that even static noncausal LPTV scaling has an ability of inducing frequency-dependent scaling if it is interpreted in the context of the conventional scaling approach. Motivated by this preceding study of ours, this paper studies to exploit this attractive property and demonstrates with numerical examples that it leads to quite effective robust stability analysis. The idea of noncausal LPTV scaling is then applied to the robust stability analysis of continuous-time systems via the Tustin transformation, and the effectiveness of such an approach is again demonstrated with a numerical example.

1. INTRODUCTION

In the study of sampled-data systems, the continuous-time lifting technique [Yamamoto (1994); Bamieh and Pearson (1992); Tadmor (1992); Toivonen (1992)] plays a significant role, and enables us to introduce the transfer operator and frequency response operator of sampled-data systems defined in an operator theoretic framework.

Based on such treatment, a general necessary and sufficient condition of a separator-type was given for robust stability of sampled-data systems [Hagiwara and Tsuruguchi (2004)]. This result, together with a generalized result [Hagiwara (2006)], suggested us to introduce a novel technique called causal/noncausal LPTV scaling for robust stability analysis of sampled-data systems [Hagiwara and Mori (2006); Hagiwara (2006)], including continuous-time systems as a special case. Furthermore, the effectiveness of such a technique, in particular *noncausal* LPTV scaling, has been demonstrated in Hagiwara (2006) and Hagiwara and Umeda (2007).

An interesting result about the use of noncausal LPTV scaling on continuous-time systems is that even static noncausal scaling has an ability of inducing some frequency-dependent scaling when it is interpreted in the context of conventional scaling approach. This paper aims at exploiting a parallel property and thus demonstrating the usefulness of noncausal LPTV scaling in the context of discrete-time systems. Regarding noncausal LPTV scaling of discrete-time systems, we have already established fundamental theoretical results and given a few supporting numerical examples in Hagiwara and Ohara (2007). With regard to this previous study, the significance of the present study lies in providing extended numerical analysis techniques in two respects. The first extension is to exploit noncausal LPTV scaling of the (D, G)-scaling type, while the numerical study in Hagiwara and Ohara (2007) was restricted to that of the D-scaling type. This leads to quite effective improvement of the robust stability analysis especially in the context of noncausal LPTV scaling. The second extension is to exploit an idea of what might be called "nested lifting." More precisely, we demonstrate the advantage of dealing with an N-periodic system as a special case of μN-periodic systems, where μ is a positive integer greater than 1. This corresponds to doubly applying the discrete-time lifting to the LTI system representation that is obtained by applying the lifting technique to the N-periodic system, where in the second application of lifting we regard the above intermediate LTI system as a special case of μ-periodic systems. The advantages of these extended approaches are demonstrated with numerical examples. The application of noncausal LPTV scaling of discrete-time systems to continuous-time systems via the aid of the Tustin transformation is also studied.

The contents of this paper are as follows. Section 2 states the problem studied in this paper and reviews some fundamental results. Section 3 reviews the definitions and properties of causal/noncausal LPTV scaling of discrete-time systems. Section 4 contains the arguments stated above, which constitutes the main contribution of this paper. Section 5 summarizes the arguments of the paper and gives some remarks on further research directions.

2. ROBUST STABILITY PROBLEM AND PRELIMINARIES

In this section, we state the problem we study in this paper, and review some fundamental results for the problem.

2.1 Robust Stability Problem

In this paper, we consider the discrete-time closed-loop system shown in Fig. 1 consisting of the nominal system G and the uncertainty Δ. We assume that G has q inputs and p outputs and is an internally stable, finite-dimensional (FD) linear periodically time-varying (LPTV) system with period N (i.e., an N-periodic system), where N is a positive integer. Also, we assume that Δ belongs to some given set $\boldsymbol{\Delta}$ satisfying the following assumption:

A1. Every $\Delta \in \boldsymbol{\Delta}$ is FD, N-periodic, and internally stable, and $\boldsymbol{\Delta}$ is a connected set such that $0 \in \boldsymbol{\Delta}$.

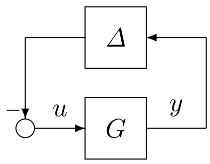

Fig. 1. Discrete-time system Σ_Δ with uncertainty Δ.

We also prepare the following alternative assumption, which is a special case of the above assumption.

A1'. Every $\Delta \in \boldsymbol{\Delta}$ is FD, LTI, and internally stable, and $\boldsymbol{\Delta}$ is a connected set such that $0 \in \boldsymbol{\Delta}$.

We introduce the state equations for G and Δ, respectively, given by

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k + D_k u_k \tag{1}$$

$$\xi_{k+1} = A_{\Delta k}\xi_k + B_{\Delta k} y_k, \quad -u_k = C_{\Delta k}\xi_k + D_{\Delta k} y_k \tag{2}$$

where A_k, B_k, C_k, D_k and $A_{\Delta k}$, $B_{\Delta k}$, $C_{\Delta k}$, $D_{\Delta k}$ are N-periodic matrices, i.e., $A_{k+N} = A_k$ for every integer k, and similarly for other matrices. In the following, we denote by Σ_Δ the closed-loop system shown in Fig. 1. Now, what we discuss in this paper is concerned with the robust stability analysis problem of the family $\boldsymbol{\Sigma}(\boldsymbol{\Delta}) := \{\Sigma_\Delta \mid \Delta \in \boldsymbol{\Delta}\}$.

2.2 Discrete-Time Lifting of N-Periodic Systems

It is well known [Bittanti and Colaneri (2000)] that the N-periodic system G can be associated with its LTI representation via the discrete-time lifting technique. That is, by defining $\widehat{x}_\nu := x_{\nu N}$, $\widehat{u}_\nu := [u_{\nu N}^T, u_{\nu N+1}^T, \cdots, u_{\nu N+N-1}^T]^T$ and $\widehat{y}_\nu := [y_{\nu N}^T, y_{\nu N+1}^T, \cdots, y_{\nu N+N-1}^T]^T$, the N-periodicity of G leads to an alternative representation of (1) given by

$$\widehat{G}: \ \widehat{x}_{\nu+1} = \widehat{A}\widehat{x}_\nu + \widehat{B}\widehat{u}_\nu, \quad \widehat{y}_\nu = \widehat{C}\widehat{x}_\nu + \widehat{D}\widehat{u}_\nu \tag{3}$$

where $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$ are appropriately defined constant matrices independent of ν. It is a fact that G is internally stable if and only if $\widehat{A}$ is Schur stable. In a similar fashion, Δ can be associated with the LTI representation

$$\widehat{\Delta}: \ \widehat{\xi}_{\nu+1} = \widehat{A}_\Delta\widehat{\xi}_\nu + \widehat{B}_\Delta\widehat{y}_\nu, \quad -\widehat{u}_\nu = \widehat{C}_\Delta\widehat{\xi}_\nu + \widehat{D}_\Delta\widehat{y}_\nu \tag{4}$$

with suitably defined $\widehat{A}_\Delta$, $\widehat{B}_\Delta$, $\widehat{C}_\Delta$ and $\widehat{D}_\Delta$. The feedback connection of the LTI systems $\widehat{G}$ and $\widehat{\Delta}$ is well-posed in the usual sense and internally stable if and only if Σ_Δ is well-posed and internally stable.

2.3 Stability Analysis via Separator

Let us introduce the discrete-time transfer matrices of $\widehat{G}$ and $\widehat{\Delta}$, respectively, given by

$$\widehat{G}(z) = \widehat{C}(zI - \widehat{A})^{-1}\widehat{B} + \widehat{D} \tag{5}$$

$$\widehat{\Delta}(z) = \widehat{C}_\Delta(zI - \widehat{A}_\Delta)^{-1}\widehat{B}_\Delta + \widehat{D}_\Delta \tag{6}$$

which we call the N-lifted transfer matrices of the N-periodic systems G and Δ, respectively. Then, we have the following result regarding the robust stability analysis of the family $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ [Hagiwara and Ohara (2007); Iwasaki and Hara (1998)], which plays a crucial role throughout the paper (see also Megretski and Rantzer (1997) for a closely related result).

Proposition 1. Suppose that $\boldsymbol{\Delta}$ satisfies the Assumption **A1** (or Assumption **A1'**) and Σ_Δ is well-posed for every $\Delta \in \boldsymbol{\Delta}$. Then, $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ is robustly stable if and only if there exists $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$, $z \in \partial\mathbf{D}$ such that

$$\begin{bmatrix} I & \widehat{G}(z)^* \end{bmatrix} \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \le 0 \quad (\forall z \in \partial\mathbf{D}) \tag{7}$$

$$\begin{bmatrix} -\widehat{\Delta}(z)^* & I \end{bmatrix} \widehat{\Theta}(z) \begin{bmatrix} -\widehat{\Delta}(z) \\ I \end{bmatrix} > 0 \ (\forall \Delta \in \boldsymbol{\Delta}, \ \forall z \in \partial\mathbf{D}) \tag{8}$$

where $\partial\mathbf{D}$ denotes the unit circle $\{z : |z| = 1\}$.

If both G and Δ are LTI, then, without applying the lifting technique, we can consider their usual transfer functions $G(\zeta)$ and $\Delta(\zeta)$ (the symbol for z-transformation is changed from z to ζ in the lifting-free treatment), and we have the following result.

Proposition 2. Suppose that G is LTI, $\boldsymbol{\Delta}$ satisfies the Assumption **A1'** and Σ_Δ is well-posed for every $\Delta \in \boldsymbol{\Delta}$. Then, $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ is robustly stable if and only if there exists $\Theta(\zeta) = \Theta(\zeta)^*$, $\zeta \in \partial\mathbf{D}$ such that

$$\begin{bmatrix} I & G(\zeta)^* \end{bmatrix} \Theta(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix} \le 0 \quad (\forall \zeta \in \partial\mathbf{D}) \tag{9}$$

$$[-\Delta(\zeta)^* \ \ I]\, \Theta(\zeta) \begin{bmatrix} -\Delta(\zeta) \\ I \end{bmatrix} > 0 \quad (\forall \Delta \in \boldsymbol{\Delta}, \ \forall \zeta \in \partial\mathbf{D}) \tag{10}$$

The matrices $\widehat{\Theta}(z)$ and $\Theta(\zeta)$ contained in the above propositions are called (dynamic) separators [Iwasaki and Hara (1998)]. We say that $\widehat{\Theta}(z)$ and $\Theta(\zeta)$ are static separators if they are in fact independent of z and ζ, respectively.

Before closing this section, we remark that we can view the N-periodic system G as a μN-periodic system, where μ is an arbitrary positive integer. If we view G in this way, we can also obtain another lifted representation and an associated transfer matrix. The latter will be called the μN-lifted transfer matrix of the N-periodic system G, and its use will be studied in Section 4.

3. CAUSAL/NONCAUSAL LPTV SCALING

Here, we simply review the definitions and properties of causal/noncausal LPTV scaling introduced and discussed in Hagiwara and Ohara (2007). Arguments about the rationale behind these definitions are omitted due to limited space; see Hagiwara and Ohara (2007) for details.

3.1 Definitions of Causal/Noncausal LPTV Scaling

Definition 1. We say that the separator $\widehat{\Theta}(z)$ induces causal LPTV scaling (or equivalently, $\widehat{\Theta}(z)$ is a causal LPTV separator) in an N-periodic feedback system if it can be represented as

$$\widehat{\Theta}(z) = \begin{bmatrix} \widehat{V}_1(z) & \widehat{V}_2(z) \end{bmatrix}^* \widehat{\Lambda} \begin{bmatrix} \widehat{V}_1(z) & \widehat{V}_2(z) \end{bmatrix} \tag{11}$$

where $\widehat{V}_1(z)$ and $\widehat{V}_2(z)$ are the N-lifted transfer matrices of a causal N-periodic system V_1 with q input and a causal N-periodic system V_2 with p input, respectively, and $\widehat{\Lambda} = \widehat{\Lambda}^*$ is a constant matrix of the form $\widehat{\Lambda} = \mathrm{diag}[\Lambda_1, \cdots, \Lambda_N]$ with the size of Λ_i being the same for all $i = 1, \cdots, N$.

Definition 1 applies also to the case with $N = 1$, i.e., an LTI feedback system, in which case (11) reduces to

$$\Theta(z) = V(z)^* \Lambda V(z), \ V(z) := [V_1(z), \ V_2(z)] \tag{12}$$

with the transfer matrix $V(z)$ of an LTI system V with $p+q$ inputs and a constant matrix $\Lambda = \Lambda^*$. This would be worth calling a causal LTI separator, which is nothing but the conventional separator in the analysis of LTI feedback systems, and is general enough in the sense that every matrix $\Theta = \Theta^*$ can be represented as

$$\Theta = V^* \Lambda V,\ \Lambda = \Lambda^* \tag{13}$$

Definition 2. We say that the separator $\widehat{\Theta}(z)$ induces noncausal LPTV scaling (or equivalently, $\widehat{\Theta}(z)$ is a noncausal LPTV separator) in an N-periodic feedback system if it can be represented as

$$\widehat{\Theta}(z) = \widetilde{\Gamma}^* \widehat{V}(z)^* \Gamma \widehat{V}(z) \widetilde{\Gamma} \tag{14}$$

where $\Gamma = \Gamma^*$ and $\widetilde{\Gamma}$ are constant matrices and $\widehat{V}(z)$ is the transfer matrix of a causal LTI system *defined on the lifted time axis* ν.

Definition 2 also applies to the case with $N = 1$, but it is obvious that (14) again reduces to the causal LTI separator (12) when $N = 1$. In other words, we can say that there exists no "(strictly) noncausal LTI separator," and hence it would be justified to refer to (12) simply as an LTI separator rather than a causal LTI separator.

3.2 Properties of Causal/Noncausal LPTV Scaling

Here we consider the case when G and Δ are both LTI, and review some properties of causal/noncausal LPTV scaling clarified in Hagiwara and Ohara (2007).

Theorem 1. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies the Assumption **A1'**. If there exists an LTI separator $\Theta(\zeta)$ satisfying (9) and (10), then there also exists a causal LPTV separator $\widehat{\Theta}(z)$ satisfying (7) and (8). In particular, if Θ is in fact a static LTI separator, there also exists a static causal LPTV separator $\widehat{\Theta}$ satisfying (7) and (8).

Roughly speaking, the above theorem (and its proof) says that if there exists a dynamic (resp. static) LTI separator that "resolves" the original lifting-free robust stability analysis problem, then the "lifted version" of that separator is also causal and dynamic (resp. static), and "resolves" the lifted restatement of the same problem. This in particular implies that we never lose anything in recasting the lifting-free problem into the lifted counterpart as far as the solvability issues of these problems are concerned. This suggests us to study possible advantages of treating the lifted counterpart instead of the original lifting-free problem; noncausal LPTV separators could possibly reduce the conservativeness of the robust stability analysis in the practical situation in which we cannot sweep over all possible separators $\Theta(\zeta)$ but have to restrict the class of $\Theta(\zeta)$ to a tractable one, e.g., a class of static separators.

The above observation is further supported from a frequency-domain viewpoint by the following theorem.

Theorem 2. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies the Assumption **A1'**. If there exists a separator $\widehat{\Theta}(z)$ satisfying (7) and (8), then there also exists a separator $\Theta(\zeta)$ satisfying (9) and (10). One such $\Theta(\zeta)$ is given by

$$\Theta(\zeta) := \mathrm{diag}[T_q(\zeta),\ T_p(\zeta)]^* \widehat{\Theta}(\zeta^N) \mathrm{diag}[T_q(\zeta),\ T_p(\zeta)] \tag{15}$$

where $T_m(\zeta) = [I_m, \zeta I_m, \cdots, \zeta^{N-1} I_m]^T$.

The above theorem, in particular (15), suggests that even a static noncausal LPTV separator $\widehat{\Theta}(z) = \widehat{\Theta}$ in the lifted treatment leads to $\Theta(\zeta)$ corresponding to the lifting-free treatment that is frequency-dependent (i.e., dependent on $\zeta \in \partial\mathbf{D}$ and thus a dynamic separator). To put it another way, the lifted treatment possibly has an ability to convert the problem of searching for frequency-dependent separators satisfying (9) and (10) in the lifting-free treatment into a simpler problem of finding a static separator satisfying (7) and (8). Compared with static Θ, it is obvious that ζ-dependent Θ has more freedom, and thus we could expect the above ability to reduce the conservativeness in the robust stability analysis.

The purpose of this paper is to exploit this advantage and demonstrate its effectiveness through numerical examples. Even though the above results are for the case with LTI G and Δ, we also show that this advantage can be exploited even when we are to deal with N-periodic G or Δ. Regarding the advantage, however, we remark here that it is known to be specific to *noncausal* (rather than causal) LPTV separators, as stated in the following theorem.

Theorem 3. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies the Assumption **A1'**. There exists a static causal LPTV separator $\widehat{\Theta}$ satisfying (7) and (8) if and only if there exists a static LTI separator Θ satisfying (9) and (10).

4. EXPLOITING FREQUENCY-DEPENDENT SCALING INDUCED BY STATIC NONCAUSAL LPTV SCALING

This section demonstrates the effectiveness of noncausal LPTV scaling with numerical examples. We confine ourselves to *static* noncausal LPTV scaling, but it leads to (almost) exact robust stability analysis. This can be attributed to the advantage of noncausal LPTV scaling suggested by Theorem 2, i.e., the fact that even static noncausal LPTV scaling induces frequency-dependent scaling in the context of the conventional treatment.

4.1 Noncausal LPTV Scaling of the (D, G)-Scaling Type

In our previous study [Hagiwara and Ohara (2007)], we confined ourselves to noncausal LPTV scaling of the D-scaling type. That is, we only considered the separators of the form

$$\widehat{\Theta} = \begin{bmatrix} -\gamma^2 \widehat{W}^T \widehat{W} & 0 \\ 0 & \widehat{W}^T \widehat{W} \end{bmatrix}, \quad \widehat{W}^T \widehat{W} > 0, \quad \gamma > 0 \tag{16}$$

In this subsection, we demonstrate the effectiveness of (D, G)-scaling especially in the context of noncausal LPTV scaling. Specifically, we confine ourselves to *static* noncausal LPTV separators for simplicity, but consider applying the so-called (D, G)-scaling by taking a more general separator of the form

$$\widehat{\Theta} = \begin{bmatrix} -\gamma^2 \widehat{W}^T \widehat{W} & \widehat{X} \\ \widehat{X}^T & \widehat{W}^T \widehat{W} \end{bmatrix},$$

$$\widehat{W}^T \widehat{W} > 0, \quad \widehat{X} + \widehat{X}^T = 0, \quad \gamma > 0 \tag{17}$$

in the conditions (7) and (8).

Example 1. We study a numerical example with the nominal system G that is a stable LTI system given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0.5 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

and $D = 0$. Here we assume that Δ is static, and is in fact given by $\Delta = \delta I$ with a time-invariant real scalar δ. We intend to analyze via noncausal LPTV scaling of the

Table 1. The results of static noncausal LPTV scaling for an LTI nominal system (Example 1).

N	1	2	3	4	5	6
$\gamma^{\text{noncausal}}_{\min,(D,G)}$	3.6215	1.0833	1.1008	1.0833	1.0833	1.0833
CPU time (sec)	4.63	5.53	10.44	13.00	18.05	40.53
$\gamma^{\text{noncausal}}_{\min,D}$	3.6215	1.7918	1.7146	1.6674	1.6533	1.6429
CPU time (sec)	4.64	4.99	6.27	10.34	12.21	18.73

(D, G)-scaling type (i.e., with (17)) a lower bound of the robust stability radius.

Since $\varDelta = \delta I$, it is easy to see that the condition (8) reduces to $|\delta| < 1/\gamma$, which is independent of $\widehat{W}$ and $\widehat{X}$. Also, the condition (7) can be rearranged as an LMI condition via the KYP lemma [Rantzer (1996)], and thus $\widehat{\Theta}$ of the form (17) with minimum γ (denoted by $\gamma^{\text{noncausal}}_{\min,(D,G)}$) can be computed. The results, together with the computation time with a PC with Pentinum 4, 3.6GHz, are shown in Table 1, where the reciprocal of each value gives a lower bound of the robust stability radius. This table shows that $\gamma^{\text{noncausal}}_{\min,(D,G)}$ is not necessarily monotonically decreasing with respect to N, but the results are consistent with Theorem 1 [†]. Table 1 also shows the results when $\widehat{X}$ is restricted to 0, or equivalently, when noncausal LPTV scaling of the D-scaling type is used as in our preceding study [Hagiwara and Ohara (2007)]; the reciprocal of $\gamma^{\text{noncausal}}_{\min,D}$ gives a lower bound of the robust stability radius obtained by applying noncausal LPTV scaling of the D-scaling type. We can thus see that even though the results remain much the same when $N = 1$, noncausal LPTV scaling of the (D, G)-scaling type leads to much better results when $N > 1$. Indeed, the results with $N = 2$ leads to the lower bound of the robust stability radius given by $1/\gamma^{\text{noncausal}}_{\min,(D,G)} = 1/1.0833 = 0.9231$, which is almost exact as shown in Fig. 2 with the thin solid line; the unshaded part corresponds to the stability region on the (δ_1, δ_2) plane that is obtained via fine gridding if δ were assumed to be 2-periodic and take δ_1 and δ_2 alternately—it can be seen from the figure that stability is retained under $-0.9231 < \delta < 1.5001$. The thick solid line on the figure corresponds to the case of D-scaling with $N = 5$, i.e., $1/1.6533 = 0.6049$.

To summarize, this example demonstrates the effectiveness of the frequency-dependent scaling induced by static noncausal LPTV scaling (in particular, of the (D, G)-scaling type) as suggested by Theorem 2. The same comments apply also to other examples to follow, and this is what is stressed by the title of the present paper.

4.2 μN-Lifted Treatment for N-Periodic Systems

We have mentioned about the μN-lifted transfer matrix of an N-periodic system at the end of Section 2, where μ is an arbitrary positive integer. This corresponds to the two-step procedure consisting of (i) first regarding the N-lifted LTI representation of the N-periodic system as if it were a given LTI system, and (ii) then considering the μ-lifted transfer matrix of the "given LTI system" viewed as a special case of μ-periodic systems. Regarding the latter step above, we have seen in the preceding section that it is quite promising and effective as far as the case with LTI nominal G is concerned, and thus it is quite reasonable to expect that considering the μN-lifted transfer matrix is also quite effective in the robust stability analysis with N-periodic G. Here we demonstrate that this is indeed the case.

[†] We can regard the case with $N = N_0$ as if it corresponds to the analysis with Proposition 2, while we can regard the case with $N = \mu N_0$ as if it corresponds to the analysis with Proposition 1, where N_0 and μ are arbitrary positive integers. Hence, Theorem 1 ensures that the result with $N = \mu N_0$ is no worse than that with $N = N_0$.

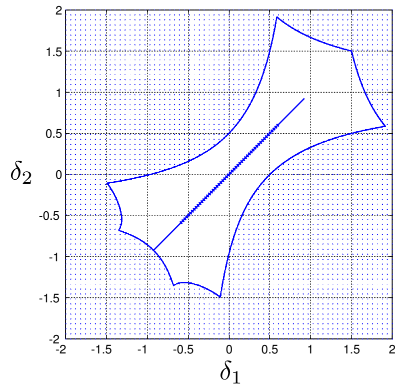

Fig. 2. Robust stability analysis for an LTI nominal system (Example 1).

Example 2. We study a numerical example with the nominal system G being a stable 2-periodic LPTV system given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.1 & 0.1 & 1 & -1 & 1.9 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.1 & -0.1 & 0.01 & -0.5 & 0.2 \end{bmatrix}$$

$$B_1 = B_2 = [\,0\ 0\ 0\ 0\ 1\,]^T, \quad C_1 = [\,0\ 0.6\ 0.6\ 1.5\ 0\,],$$
$$C_2 = [\,0\ 0.3\ 0.3\ 0.3\ 0\,], \ D_1 = D_2 = 0 \tag{19}$$

Here we assume that $\varDelta$ is static, and is in fact 2-periodic, taking the real scalar values δ_1 and δ_2 alternately. We intend to analyze via noncausal LPTV scaling a lower bound of the robust stability radius.

Here we introduce the 2μ-lifted transfer matrices of the 2-periodic systems G and $\varDelta$. Since $\varDelta$ is 2-periodic, the 2μ-lifted transfer matrix of $\varDelta$ is given by the matrix

$$\widehat{\varDelta} = I_\mu \otimes \text{diag}[\delta_1,\ \delta_2] \tag{20}$$

where I_μ denotes the $\mu \times \mu$ identity matrix and $\otimes$ denotes the Kronecker product. Now, let us denote the class of 2×2 diagonal matrices by $\boldsymbol{\varLambda}_2$, and define the class $\boldsymbol{\varLambda}_2^{\mu\times\mu}$ by the set of the matrices of the form

$$(L_{ij})_{i,j=1}^{\mu}, \quad L_{ij} \in \boldsymbol{\varLambda}_2 \tag{21}$$

We further define the class $\boldsymbol{W}_2^{\mu\times\mu}$ of symmetric matrices and the class $\boldsymbol{X}_2^{\mu\times\mu}$ of skew symmetric matrices by

$$\boldsymbol{W}_2^{\mu\times\mu} = \left\{ W \,|\, W = L + L^T,\ L \in \boldsymbol{\varLambda}_2^{\mu\times\mu} \right\},$$
$$\boldsymbol{X}_2^{\mu\times\mu} = \left\{ X \,|\, X = L - L^T,\ L \in \boldsymbol{\varLambda}_2^{\mu\times\mu} \right\}, \tag{22}$$

respectively. Then, we see that $\widehat{\Theta}$ given by (17) with $\widehat{W} \in \boldsymbol{W}_2^{\mu\times\mu}$ and $\widehat{X} \in \boldsymbol{X}_2^{\mu\times\mu}$ is a static noncausal LPTV separator of the (D, G)-scaling type. Also, it is not hard to see that, under such $\widehat{\Theta}$, the condition (8) reduces to the condition $\max(|\delta_1|, |\delta_2|) < 1/\gamma$, which is independent

Table 2. The results of static noncausal LPTV scaling for an LPTV nominal system (Example 2).

μ	1	2	3	4	5
$\gamma_{\min,\mu}^{\mathrm{noncausal}}$	3.4897	3.2195	3.1594	3.1445	3.1445
CPU time (sec)	5.32	6.83	8.61	8.58	11.56

of $\widehat{W}$ and $\widehat{X}$. Hence, as in the preceding subsection, we can compute $\widehat{\Theta}$ of the form (17) with the above-restricted forms of $\widehat{W}$ and $\widehat{X}$ and with minimum γ, where the minimum γ is denoted by $\gamma_{\min,\mu}^{\mathrm{noncausal}}$. The results are shown in Table 2, from which we can see that considering the μN-lifted transfer matrix for the N-periodic nominal G is successful in reducing the conservativeness in the robust stability analysis. The result with $\mu = 4$ is shown in Fig. 3 with the dash-two-dot square (the meaning of the figure is parallel to Fig. 2), and we can see that this result is indeed almost exact.

As a side remark, we state that if we consider the separator (17) with $\widehat{W}$ and $\widehat{X}$ both being the $2\mu \times 2\mu$ diagonal matrices, then it is a static *causal* LPTV separator of the (D, G)-scaling type [‡]. We can also minimize γ under this class of separators, which leads to $\gamma_{\min,\mu}^{\mathrm{causal}} = 3.4897$ for all $\mu = 1, \cdots, 5$ with our numerical computations (this corresponds to the dash square in Fig. 3, while the dash-dot square corresponds to the small-gain theorem). In other words, taking $\mu \geq 2$ does not lead to the improvement of the results, but this is in fact consistent with Theorem 3 if this theorem is interpreted after introducing the N-lifted transfer matrices of the N-periodic systems G and Δ (where $N = 2$ in the context here). Another independent interpretation of this lack of improvement with $\mu \geq 2$ is that considering the above *causal* LPTV separators corresponds to failing to rule out the possibility that the uncertainty Δ is 2μ-periodic [††], which naturally leads to conservativeness since our assumption here is that Δ is known to be 2-periodic.

4.3 Application of Discrete-Time Noncausal LPTV Scaling to Continuous-Time Systems

We have seen in the robust stability analysis of discrete-time systems that applying the discrete-time lifting and then introducing noncausal LPTV scaling are quite effective in reducing the conservativeness in the analysis. In this subsection, we intend to make use of the advantage even in the robust stability analysis of continuous-time systems with the aid of the Tustin transformation. This leads to an alternative algebraic approach to noncausal LPTV scaling of continuous-time systems, which was first studied in Hagiwara (2006) in an operator theoretic framework and strongly motivated the present study.

‡ As far as this example is concerned, such a separator in fact degenerates to a separator of the D-scaling type, since Δ is scalar. However, the above arguments can be extended to the case with Δ given by a matrix by appropriately modifying (22) in an obvious fashion, and the subsequent arguments remain the same in such a case, too.

†† Under the above *causal* LPTV separators, the condition (8) leads to $\max_{k=1,\cdots,2\mu} |\delta(k)| < 1/\gamma$ for the 2μ-periodic scalar $\Delta(k) = \delta(k)$, while under the above *noncausal* LPTV separators, the condition (8) does not lead to such a simple condition unless $\Delta(k) = \delta(k)$ is 2-periodic.

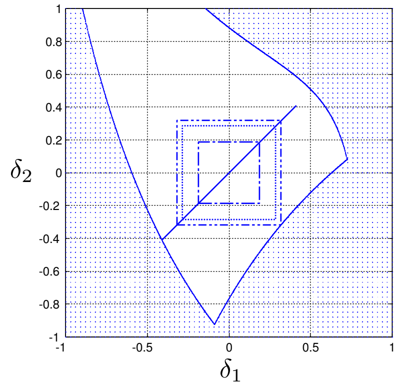

Fig. 3. Robust stability analysis for an LPTV nominal system (Example 2).

Example 3. We now consider the continuous-time counterpart of Fig. 1 composed of the continuous-time LTI system G_c given by

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -15 & -20 & -40 \end{bmatrix},\ B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ C_c = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad (23)$$

and $D_c = 0$, and the static uncertainty Δ_c. We consider the two situations about Δ_c: (i) $\Delta_c = \delta I$ with a time-invariant real scalar δ; (ii) $\Delta_c = \mathrm{diag}[\delta_1, \delta_2]$ with time-invariant real scalars δ_1 and δ_2. It is obvious that this continuous-time feedback system is stable if and only if the discrete-time feedback system in Fig. 1 with G and Δ given respectively by the Tustin transforms of G_c and Δ_c is stable. Thus, we can compute (a lower bound of) the robust stability radius for each of the above two situations by the discrete-time noncausal LPTV scaling. Here, note that the Tustin transform of Δ_c is nothing but itself since Δ_c is static. Regarding G_c, on the other hand, we take the fictitious sampling period $T = 2$, for simplicity, for the Tustin transformation and then consider the N-lifted transfer matrix of the resulting discrete-time LTI system.

In the first situation, we can introduce the separator $\widehat{\Theta}$ given by (17) so that (8) reduces to the condition $|\delta| < 1/\gamma$, and then γ is minimized with respect to (7), where the minimum γ is denoted by $\gamma_{\min,\mathrm{(i)}}^{\mathrm{noncausal}}$. The results are shown in Table 3 ($\gamma_{\min,\mathrm{(i)}}^{\mathrm{noncausal}}$) and their reciprocals give lower bounds of the robust stability radius, which clearly indicates the effectiveness of applying noncausal LPTV scaling. We also remark that the (D, G)-scaling in the conventional continuous-time system analysis leads to $\gamma_{\min} = 7.3618$, which obviously equals the result with $N = 1$ in Table 3.

In the second situation, on the other hand, we can see that we can use the separator (17) with $\widehat{W} \in \boldsymbol{W}_2^{\mu\times\mu}$ and $\widehat{X} \in \boldsymbol{X}_2^{\mu\times\mu}$ and we are led to the results shown in Table 3 ($\gamma_{\min,\mathrm{(ii)}}^{\mathrm{noncausal}}$). Again, their reciprocals give lower bounds of the robust stability radius, and we can confirm the effectiveness of noncausal LPTV scaling. In this case, too, the (D, G)-scaling in the continuous-time setting leads to the same result as that with $N = 1$ in Table 3.

The above results are shown in Fig. 4, where the unshaded part corresponds to the stability region on the (δ_1, δ_2)-plane, the solid line corresponds to the lower bound of the robust stability radius for the case (i) obtained with $N = 3$, and the dash square corresponds to that for the

Table 3. The results of static noncausal LPTV scaling for a continuous-time feedback system (Example 3).

N	1	2	3	4	5
$\gamma_{\min,(\mathrm{i})}^{\mathrm{noncausal}}$	7.3618	4.1242	2.8953	2.8953	2.8953
CPU time (sec)	6.53	9.47	9.82	15.02	20.52
$\gamma_{\min,(\mathrm{ii})}^{\mathrm{noncausal}}$	8.6904	4.8899	3.4678	3.4678	3.4678
CPU time (sec)	5.84	6.86	8.43	11.20	13.86

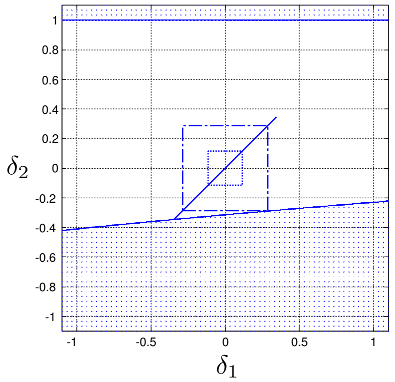

Fig. 4. Robust stability analysis for a continuous-time feedback system (Example 3).

case (ii) obtained with $N = 3$. We can see from this figure that we are successful in (almost) exact robust stability analysis even in the continuous-time setting. Note that the conventional (D, G)-scaling in the continuous-time setting for the case (ii) leads to only a quite conservative result as shown as the dash-dot square (which corresponds to the reciprocal of $\gamma_{\min,(\mathrm{ii})}^{\mathrm{noncausal}}$ for $N = 1$ in Table 3). We can observe similar conservativeness for the case (i) from Table 3 and Fig. 4.

5. CONCLUSION

Motivated by the study on noncausal linear periodically time-varying (LPTV) scaling for sampled-data/continuous-time systems [Hagiwara (2006); Hagiwara and Umeda (2007)], the discrete-time counterpart was introduced and its effectiveness in the robust stability analysis of discrete-time systems was studied in Hagiwara and Ohara (2007). In parallel to the case of sampled-data/continuous-time systems, it was shown that even static noncausal LPTV scaling has an ability of inducing frequency-dependent scaling if it is interpreted in the context of the conventional scaling treatment. Motivated by this observation, this paper studied to exploit this property and demonstrated with numerical examples that it leads to quite effective robust stability analysis. We also demonstrated that the idea of noncausal LPTV scaling can readily be applied to the robust stability analysis of continuous-time systems via the Tustin transformation, and gave a numerical example showing that such an approach is quite effective.

The above treatment of continuous-time systems, by the way, can be regarded as giving an alternative approach to noncausal LPTV scaling of continuous-time systems, which was first introduced in Hagiwara (2006); compared with the approach in Hagiwara (2006) that is operator-theoretic, the alternative approach here is algebraic and thus is quite simple. This was indeed one of the reasons why we were motivated to the present study in the discrete-time setting from the original study in Hagiwara (2006) in the sampled-data/continuous-time setting, and in that sense, we could say that we have arrived at a nearly equivalent "simplified method" for noncausal LPTV scaling of continuous-time systems. However, there are still some freedom in this simplified method; the fictitious sampling period for the Tustin transformation used in the simplified method is arbitrary, theoretically speaking, but it does affect the analysis results for fixed N, the parameter for discrete-time lifting. Hence, it would be an interesting topic to study how to choose this fictitious sampling period to arrive at an effective result. Also, it is important to clarify some mutual relationship between this noncausal LPTV scaling approach of continuous-time systems via the Tustin transformation and the purely continuous-time noncausal LPTV scaling introduced in Hagiwara (2006). These are left to our future studies.

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