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Kalman Filter Decomposition in the Time Domain Using Observability Index

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Abstract:

Considering the Kalman–Bucy Filter (KBF) from an engineering point of view it is always important to know in advance, before KBF implementation, which variables are practically "good" and which are "bad" observable and how long it will take to estimate all of them in the presence of measurement noise to some appropriate (not necessarily theoretically optimal) level. This article presents an approach to measuring the observability by a special index that has the physical meaning of signal to nose ratio. This approach leads to the decomposition of the KBF in the time domain into two filters applied consecutively in time: the filter estimating the transitional process caused by the uncertainty in initial conditions and the filter estimating the system steady state. In turn; this results in mitigation of the computational requirements and in a simplification of the filter implementation by the engineers.

1. INTRODUCTION

The Kalman–Bucy Filter has become a very popular mathematical tool for solving diverse applied problems. The filter's property is that it can provide optimal estimates for all observable variables of the dynamical system that meets KBF theory assumptions. Since the publishing in 1960 by R.E. Kalman [3] and in 1961 by R.E Kalman and R.S Bucy [4] of two famous articles on a new approach to Linear Filtering and Prediction problems, the set of mathematical equations considered in these articles for obtaining an optimal estimate of a linear dynamical system state vector with minimum of the mean of the squared error has been widely adopted in many scientific and technical applications. This set of the equations (continuous or recursive) has obtained the name of Kalman Filter (KF) or Kalman-Bucy Filter (KBF). Many publications have been dedicated to KBF sub-optimization to make it less computationally demanding and more robust (for example [6, 7]) as well as for explanation and popularization of the KBF principles [1, 3, 11]. However, from an engineering point of view for many designers the filter still remains to be a mathematical magic "black box". There are two polar points of view on KBF: "KBF can solve any engineering problem with excellent accuracy" or "KBF is very sensitive to the system model, computationally demanding and can hardly be practically used". Acceptance one of these philosophies leads to the formal programming of the original KBF equations or rejection of the KBF and using traditional methods for automatic control and communication with analog or digital filters, correspondingly. This article is based on previous publications of the author [6– 10] and has the intent to show that a comprehensive analysis of system observability for the considered applied problem performed prior to KBF implementation might resolve the antagonism between the philosophies mentioned above. It is proposed to introduce a quantitative observability measure (observability index-signal/noise ratio) that would evaluate the time required for estimation of a certain state vector component in the presence of measurement noise to some level of unbiased estimation. As such, it allows one to restrict the estimated vector components by the number of the last one that can be confidently obtained within allowed observation period. It allows one also to divide the observation period into two stages: transitional process estimation and steady state estimation. At the first stage, the estimated dynamical system can be considered as " purely deterministic" and at the second as "purely stochastic". For a time invariant (stationary) system, the KBF coefficients can be expressed by simple set of analytical functions of time instead of the recursive computation of the KBF matrix Riccati equation for the covariance matrix. At the steady state stage only constant gains can be kept. For many cases, the time variable and stationary (constant) gains can be applied consecutively: time variable at the transitional stage to insert the roughly estimated initial conditions and constant at the steady state to get the final fine unbiased estimation.

2. OBSERVABILITY MEASURE

2.1 Observability Criteria

The meaning of the term observability (this concept was introduced by R. Kalman) can be considered from both deterministic and stochastic points of view. In the deterministic case, the observability is the possibility to determine the initial state of the linear dynamical system with some measurements of system state vector. In the stochastic case this is the possibility to decrease initial uncertainty (covariance matrix) about the system state vector using the state vector measurements accompanied by noise. In both cases, this is a fundamental system characteristic that only indicates the existence of a potential for the estimation of the

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system state vector rather than providing any quantitative information about the estimation quality. In both cases, the criteria of observability are almost identical and don't take into account any random disturbances applied to dynamical system. Only the system free motion is considered. Let us consider a linear dynamical system that is given by the following matrix differential equation in the state-vector differential equation form [1]

$$\begin{aligned} \dot{x} &= F(t)x + G(t)w(t), \\ z &= H(t)x + v(t), \end{aligned} \qquad (1)$$

where: x is an n-vector of the system state, w is an m-vector of external disturbances, z is a p-vector of measurements, v is a p-vector measurement noise, $F(t)$ is an $(n \times n)$ system dynamics matrix, $G(t)$ is an $(n \times m)$ disturbances matrix, $H(t)$ is a $(p \times n)$ measurements matrix.

Let us the following information about (1) is given:

F, G, H are known matrices of time (in the stationary case, constants),

$$\begin{aligned} &E[x(t_0)] = 0, E[x(t_0)x^T(t_0)] = M_0, \\ &E[w(t)] = 0, E[w(t)w^T(\tau)] = Q(t)\delta(t-\tau), \\ &E[v(t)] = 0, E[v(t)v^T(\tau)] = R(t)\delta(t-\tau), \\ &E[w(t)v^T(\tau)] = E[v(t)w^T(\tau)] = E[w(t)x^T(t_0)] = 0, \end{aligned} \qquad (2)$$

where: M_0 is the initial state covariance matrix, $R(t)$ is the covariance matrix of measurement noise, $Q(t)$ is the covariance matrix of disturbance noise, $\delta(t-\tau)$ is the Dirac delta function. Hence, $w(t)$ and $v(t)$ are Gauss white noise processes. In the stationary case, the matrices Q and R are constants and have the meaning of spectral densities of the white noises $w(t)$ and $v(t)$, correspondingly. The following criteria can be applied for system (1) observability analysis [1, 3]: A. Stochastic case

$$I = \int_{t_0}^{t} \Phi^T(\tau, t_0) H(\tau) R^{-1}(\tau) H(\tau) \Phi(\tau, t_0) d\tau > 0\,, \quad (3)$$

where: $R > 0$. B. Deterministic case

$$I = \int_{t_0}^{t} \Phi^T(\tau, t_0) H(\tau) H(\tau) \Phi(\tau, t_0) d\tau > 0\,, \quad (4)$$

where: $\Phi(t, t_0)$ is the system transition matrix corresponding to the matrix F in (1). In other words, if the integrals (3) (in the stochastic case) and (4) (in the deterministic case) are positive definite, then system (1) is completely observable and all n components of vector x can be estimated with the measurements Z The R-matrix in (3) is assumed to be positive definite and affects only the scale (4) w.r.t (3), hence both approaches to the observability analysis are in complete agreement. The solution to (1) for free motion can be expressed by the following formula [1]

$$x(t) = \Phi(t, t_0) x(t_0)\,. \quad (5)$$

If this solution is known, then the direct approach to analyzing observability [6] can be used. It is based on the following definition of observability [3], [10]: if from the condition

$$Z = H(t)\Phi(t, t_0)x(t_0) \equiv 0 \quad (6)$$

follows that

$$x(t_0) = 0\,, \quad (7)$$

then the system (1) is completely observable.

In the stationary case (when the matrixes F and H are constants), the observability property can be analysed with the determination of the rank of the observability matrix N [1, 3, 10]

$$N^T = \left[H^T F^T H^T (F^2)^T H^T \ldots\ldots (F^{(n-1)})^T H^T \right]. \quad (8)$$

If $rankN = n$ then all n components of the x-vector will be observable.
One can see that observability defined by this way is an inherent system property that depends on matrices F and H properties only. The abovementioned criteria allow one to determine if the system (1) state vector x is observable in principle, however, at least in the stochastic case, these criteria don't answer all questions connected to the considered problem. Indeed, it is clear that the system initial state can be set by its measured output variable (signal) and its linearly independent derivatives, but to determine a high order derivative in presence of noise is very difficult and not always practically possible. Another concern can be expressed about a priori estimate error covariance and random disturbances, applied to the system. It raises the question if applying the KBF to an observable system will always decrease the initial covariance about system state vector independently on the ratios between the matrices M_0, Q and R. We will try to discuss these issues introducing an additional parameter – an observability index that would present some quantitative observability measure in addition to basic criteria (4), (5).

2.2 Substantially Deterministic and Stochastic Systems

Depending on the ratios between values of initial conditions, the elements of matrix Q and the time of observation, the system (1) can be considered as almost deterministic or almost stochastic.

Let us consider differential equation for the covariance matrix for equation (1). It can be written as follows [1]

$$\dot{M} = FM + MF^T + GQG^T , \quad (9)$$

where: M is the covariance matrix of the vector x in (1).

The solution (9) can be presented in the following form

$$M(t) = \Phi(t,t_0)M_0\Phi^T(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)Q(\tau)G^T(\tau)\Phi^T(t,\tau)d\tau. \quad (10)$$

It has two different components: free motion covariance caused by the initial conditions vector x_0 with initial covariance M_0 and forced motion covariance caused by the disturbance noise $w(t)$ with covariance matrix Q or as follows

$$M(t) = M_{M_0}(t) + M_Q(t), \quad (11)$$

where: $M_{M_0} = \Phi(t,t_0)M_0\Phi^T(t,t_0)$ and

$$M_Q(t) = \int_{t_0}^{t} \Phi(t,\tau)G(\tau)Q(\tau)G^T(\tau)\Phi^T(t,\tau)d\tau$$

Depending on the ratios between M_{M_0} and M_Q during the observation period, the system (1) can be considered as substantially stochastic or substantially deterministic. When the following inequality takes place

$$diagM_{M_0} >> diagM_Q , \quad (12)$$

($diagM = M_{ii}, i = 1,..n$ means the diagonal elements of the matrix) the deterministic motion is dominant and the system can be considered as substantially deterministic and during the time of observation when the opposite inequality takes place

$$diagM_Q >> diagM_{M_0} , \quad (13)$$

the system can be considered as substantially stochastic. It is clear that for an asymptotically stable system, the deterministic component M_{M_0} will decay and after the time of decaying of the transitional process, the system becomes substantially stochastic. In this case, for a time invariant system, the steady state covariance matrix $M(t \to \infty) = M^*$ can be found from the following algebraic matrix equation

$$FM^* + M^*F^T + GQG^T = 0 . \quad (14)$$

When $diagM_0 >> diagM^*$, then during the time of the transitional process, system (1) can be considered as substantially deterministic and after this time as substantially stochastic.

2.3 Estimation of stochastic system with KBF as an equivalent deterministic system

Let us consider the estimation of system (1) with a KBF, assuming that system meets the observability criterion (3) and the all n components of the vector x are observable. In this case, the KBF should provide a stable optimal estimate x. The KBF equations for the estimation of (1) in continuous form can be written as follows [1]

$$\begin{aligned} &\dot{\hat{x}} = F(t)\hat{x} + K(z - H(t)\hat{x}), \hat{x}_0 = 0, \\ &K = PH(t)^T R(t)^{-1}, \\ &\dot{P} = F(t)P + PF(t)^T - PH(t)^T R(t)^{-1} H(t)P + \\ &+G(t)Q(t)G(t)^T, P_0 = M_0 \end{aligned} \quad (15)$$

where: $K = K(t)$ is the KBF matrix gain, $P = P(t)$ is the KBF estimate errors covariance matrix that should be found from the solution of the third matrix equation (Riccati type) in (15).

It can be shown [7], that the KBF (15) can be presented as two filters working in parallel: one for steady state estimation of original system (1), where all transitional processes have been completed and it has become substantially stochastic, estimating the steady state motion of (1), and another one for a substantially deterministic system with modified matrix F and modified initial conditions, estimating the transitional process in a modified deterministic system. Mathematically it results in the presented KBF covariance matrix and gain as follows: $K = K^* + \tilde{K}$ and $P = P^* + \tilde{P}$, where P^* and K^* are related to steady state estimation in the stochastic system (original KBF (15) with $t \to \infty$), $\tilde{P}$ and $\tilde{K}$ related to transitional process estimation in the deterministic system, correspondingly. Let us assume that the matrixes P^* and K^* for the steady state KBF have been found by some way and consider the deterministic system. This modified deterministic system is as follows

$$\begin{aligned} &\dot{x} = F^*(t)x, \\ &z = H(t)x + v(t), \end{aligned} \quad (16)$$

where: $F^*(t) = F(t) - K^*(t)H(t), K^*(t) = P^*(t)H^T(t)R^{-1}(t).$

The covariance matrix $P^* = P(t \to \infty)$ is the steady state covariance matrix that is defined by solving the Riccati equation in (15) for $t \to \infty$. For the case of time invariant system it can be found from the following algebraic matrix equation

$$FP^* + P^*F^T - P^*H^T R^{-1} HP^* + GQG^T = 0. \quad (17)$$

Then for system (16), the modified KBF becomes as follows

$$\begin{aligned} &\dot{\hat{x}} = F^* \hat{x} + \tilde{K}(z - H\hat{x}), \hat{x}_0 = 0, \\ &\tilde{K} = \tilde{P} H^T R^{-1}, \\ &\dot{\tilde{P}} = F^* \tilde{P} + \tilde{P} F^* - \tilde{P} H^T R^{-1} H \tilde{P}, \tilde{P}_0 = P_0 - P_0^*. \end{aligned} \quad (18)$$

Note: the index "~" will represent the transient components for the matrixes P and K.

For the deterministic system (16), the covariance equation in (18) can be solved with the following formula [2]

$$\tilde{P}(t) = \Phi^*(t,t_0)[\tilde{P}_0^{-1} + \int_{t_0}^{t} \Phi^{*T}(t,\tau) H^T(\tau) R^{-1}(\tau) H(\tau) \Phi^*(t,\tau)]^{-1} \Phi^{*T}(t,t_0), \quad (19)$$

where: $\Phi^*(t,t_0)$ is the system (16) transition matrix, corresponding to the matrix F^*.

Examining (10), (11) and (19), one can conclude that (19) establishes the connection between the covariance of the errors of estimating the initial and current states of the deterministic system (16). The covariance matrix of the errors in estimation of the initial state of (16) is determined by the following formula

$$D(t) = [\tilde{P}_0^{-1} + \int_{t_0}^{t} \Phi^{*T}(t,\tau) H^T(\tau) R^{-1}(\tau) H(\tau) \Phi^*(t,\tau) d\tau]^{-1}, \quad (20)$$

where: $D(t) = E[\tilde{x}_0 \tilde{x}_0^T], \tilde{x}_0 = \hat{x}_0(t) - x_0$ and

$\hat{x}_0(t)$ can be estimated with the following observer

$$\hat{x}_0 = D(t) \int_{t_0}^{t} \Phi^T(t,\tau) H^T(\tau) R^{-1}(\tau) z(\tau) d\tau \quad (21)$$

that can be derived by applying the Least Square Method [1]

to the following system

$$z = H(t)\Phi^*(t,t_0)x_0 + v(t) \quad (22)$$

Hence, we have converted the original problem of the estimation of the current state of the stochastic system (1) into the problem of the estimation of the initial state of an equivalent deterministic system (16). For this case, all of the above mentioned observability criteria were developed and now we can try to modify them to measure the observability quantitatively.

2.4 Observability Index

The first attempt to introduce the observability (estimability) index was undertaken by author in [7, 8], where mainly polynomial signals (KBF for polynomial signals can be found also in [11]) were considered. A more general discussion is presented in this article. Looking at (20), one can conclude that when the integral in (20) will dominate over the inverse initial covariance matrix $\tilde{P}_0^{-1}$, then the estimate errors covariance will be almost independent of the initial uncertainty about the system initial state. In this case, the unbiased estimation process [1] will start from this time instant. This condition can be written as follows

$$diag \tilde{P}_0^{-1} << diag \int_{t_0}^{t_*} \Phi^{*T}(t,\tau) H^T(\tau) R^{-1}(\tau) H(\tau) \Phi^*(t,\tau) d\tau \quad (23)$$

where: t_* is the time domain, where (23) is satisfied.

In many practical cases, the matrices $\tilde{P}_0$ and R are diagonal.

Then (23) can be written for each separate i-th component of the vector x as follows

$$1 << \frac{\tilde{P}_{0_{ii}}}{\sigma_{v_i}^2} \int_{t_0}^{t_{*i}} diag[\Phi^{*T}(t,\tau) H^T(\tau) H(\tau) \Phi^*(t,\tau)] d\tau, \quad (24)$$

where: $\sigma_{v_i}^2$ are the diagonal elements of the matrix R.

Let us introduce the observability index for i-th component of x by the following formula

$$\chi_i(t) = \frac{\tilde{p}_{0ii} \int_{t_0}^{t} diag[\Phi^{*T}(t,\tau) H^T(\tau) H(\tau) \Phi^*(t,\tau)] d\tau}{\sigma_{v_{ii}}^2}. \quad (25)$$

In many practical cases this formula allows to compute the index χ even analytically and get some very important general conclusions about considered system observability features. Then, the condition of unbiased estimation (24) can be rewritten as follows

$$\chi_i(t_{*i}) >> 1, i = 1,2,...n. \quad (26)$$

The time instant t_{*i} when the i-th component of vector x starts unbiased estimation process can be found from the inequality (26). In (25), the numerator can be interpreted as the work of signal ($s(t) = H(t)\Phi^*(t,t_0)x_0$), performed for the time t, and the denominator is measurement noise spectral density (in the considered frequency range). As such, (26) has the physical meaning of ratio deterministic signal energy to noise power spectral density ($\chi_i(t) = \frac{\int_{t_0}^{t} s_i^2(\tau) d\tau}{\sigma_{v_{ii}}^2}$). Hence, the unbiased estimation process for each of an observable component of x starts then, when corresponding to this component the signal work to measurement noise power spectral density (observability index) exceeds the value of one.

3. KBF DECOMPOSITION IN THE TIME DOMAIN

3.1General Approach

In many practical cases, condition (12) is satisfied (at least at the beginning of the observation period). In these cases, the following condition

$$diagP_0 >> diagP^* \qquad (27)$$

is usually satisfied as well. If (27) is satisfied, then at least for a time invariant system, the optimal KBF filter with connected in parallel K^* and $\tilde{K}$ can be replaced with a suboptimal scheme, applying these coefficients consecutively in time as follows

$$\begin{aligned} &\dot{\hat{x}} = F\hat{x} + K(z - H\hat{x}), \\ &K = \begin{cases} \tilde{K}, t_0 \le t \le t_*, \\ K^*, t > t_*, \\ 0, \quad if \quad z \equiv 0, \end{cases} \end{aligned} \qquad (28)$$

where: t_* is the time required for unbiased estimation of all n components of vector x. This time is found from the condition (27).

The gains $\tilde{K}$ and K^* are found with the following formulas

$$\begin{aligned} &\tilde{K} = \tilde{P}H^T R^{-1}, \\ &\dot{\tilde{P}} = F\tilde{P} + \tilde{P}F^T - \tilde{P}H^T R^{-1} H\tilde{P}, \tilde{P}_0 = P_0, \\ &K^* = P^* H^T R^{-1}, \\ &FP^* + P^* F^T - P^* H^T R^{-1} HP^* + GQG^T. \end{aligned} \qquad (29)$$

In other words, $\tilde{K}$ is computed for system (1), considered as a substantially deterministic ($w \equiv 0$), and K^* - considering (1) as a substantially stochastic system, where only steady state motion, caused by the random disturbance w takes place. For many applied problems (see, for example [7, 8]) the coefficients $\tilde{K}(t)$ and K^* can even be found analytically and programmed as functions of time ($\tilde{K}$) and constants (K^*), correspondingly. In some cases, when any reliable information about the Q matrix is absent. It can be set with the guaranteed approach, as an equivalent white noise providing a P^* ellipsoid that will be similar to F.Chernousko's Q guaranteed ellipsoid [9]. The suboptimal filter (28) was firstly considered in [7] and called there as the "filter with bounded growing memory" (FBGM). Being almost identical to the KBF it is more computationally economical and more robust for many engineering applications. Applying (28) in conjunction with the use of the observability index χ allows one to avoid any of the unexpected effects of filtering process divergence as well as performing superfluous computations in contrast to the formal implementation of the KBF in its original form.

4. CONCLUSION

The proposed observability index allows one to measure the observability property quantitatively and has the physical meaning of the ratio of deterministic signal energy to noise power spectral density. When the index becomes greater than one, then the corresponding estimated state vector component becomes free from any bias caused by the initial uncertainty about the system state vector. Applying the observability index to the system analysis allows one to implement the KBF with transitional and steady state suboptimal gains consecutively in time as a FBGM that has important advantages for practical filter implementation.

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Appendix A. EXAMPLE OF THE FIRST ORDER DIFFERENTIAL EQUATION STATIONARY SYSTEM

Let us consider the scalar case of system (1) with following equations

$$\dot{x} = -\frac{1}{T}x + \frac{1}{T}w, \quad \text{(A.1)}$$
$$z = x + v,$$

where: $T = 100$ s is system time constant, $x_0 = 10$ m,

$$F = -\frac{1}{T}, M_0 = 100m^2, G = \frac{1}{T}, H = 1,$$
$$Q = \sigma_w^2 = 200m^2 s, R = \sigma_v^2 = 1m^2 s.$$

The covariance matrix equation for (A.1) is as follows

$$\dot{M} = -\frac{2}{T}M + \frac{1}{T^2}\sigma_w^2. \quad \text{(A.2)}$$

The solution of (A.2) can be written as follows

$$M(t) = M_0 e^{-\frac{2}{T}t} + \frac{\sigma_w^2}{2T}(1 - e^{-\frac{2}{T}t}). \quad \text{(A.3)}$$

Steady state covariance can be found from (A.3) if one puts $t \to \infty$. This covariance is as follows $M^* = \frac{\sigma_w^2}{2T} = 1m^2$ As it can be seen from (A.3), after $t > \frac{3}{2}T(150s)$ the system can be considered as substantially stochastic ($M \approx M^*$).

Considering the KBF for system (A.1), one can find that the steady state covariance equation is as follows

$$P^{*2} + \frac{2}{T}P^* - \frac{\sigma_v^2 \sigma_w^w}{T^2} = 0. \quad \text{(A.4)}$$

The algebraic equation (A.4) has the following solution

$$P^* = \frac{\sigma_v^2}{T}(\sqrt{1 + \frac{\sigma_w^2}{\sigma_v^2}} - 1) = 0.1317m^2 (\sigma^* = 0.363m) \quad \text{(A.5)}$$

The modified deterministic system (16) will be as follows

$$\dot{x} = -\frac{1}{T^*}x, \quad \text{(A. 6)}$$

where: $T^* = T(1 + \frac{\sigma_w^2}{\sigma_v^2})^{-\frac{1}{2}} = 7.07s$.

The transition matrix for (A.6) is as follows

$$\Phi(\tau) = e^{-\frac{1}{T^*}\tau} \quad \text{(A.7)}$$

Using formula (25), one can find that the observability index will be expressed by the following formula

$$\chi(t) = \frac{(P_0 - P_*)T^*}{2\sigma_v^2}(1 - e^{-\frac{2}{T^*}t}). \quad \text{(A.8)}$$

When $t >> \frac{T^*}{2}(3.535s)$, then $\chi(t) \to \frac{(P_0 - P^*)T^*}{2\sigma_v^2}$ $t = t_*$. Hence, unbiased estimation starts somewhere after 3.5 s. However, until 150 s of the observation period, the system is almost deterministic. Therefore, the FBGM (28), (29) can be applied with switching gains from

$$\tilde{K} = \frac{2P_0 e^{-\frac{2}{T}t}}{2\sigma_v^2 + P_0 T(1 - e^{-\frac{2}{T}t})}, \text{ to}$$

$$K^* = \frac{1}{T}(\sqrt{1 + \frac{\sigma_w^2}{\sigma_v^2}} - 1) \text{ at switching time instant } t_*$$

Plots of the simulation of the KBF and the FBGM for the considered example are presented in Fig. 1-3

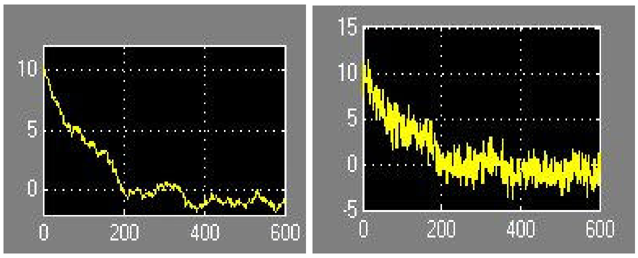

Figure 1. Model (1); $x(t)$ -left, $z(t)$ -right [m], [s]

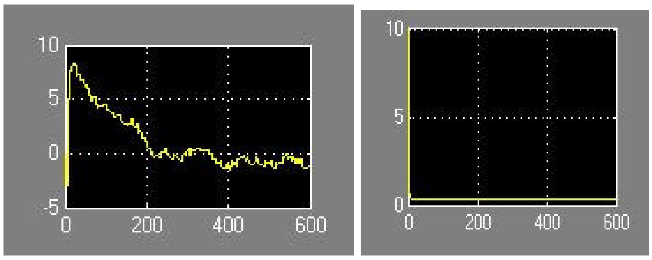

Figure 2. KBF; $\hat{x}(t)$ -left, $\sigma(t) = \sqrt{P(t)}$ -right [m], [s]

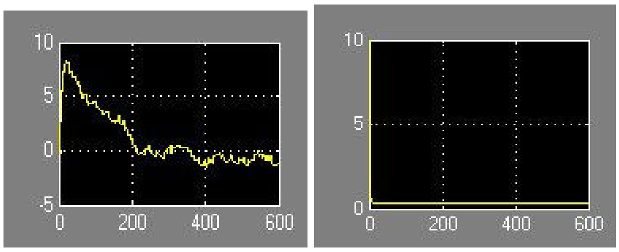

Figure 3. FBGM; $\hat{x}(t)$ -left, $\sigma(t) = \sqrt{P(t)}$ -right [m], [s]

Switching time; $t_* = 10s$

This example demonstrates that both filters (KBF and FBGM) provide practically very close estimation accuracy. Unbiased estimation starts very fast; after about 10 sec since the starting filtering. This agrees well with the time evaluated with the observability index.