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Exponential Dissipativity of Discrete-time Stochastic Systems and Robust Simultaneous Stabilization via Output Feedback ⋆

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Abstract: The paper studies robust simultaneous stabilization problem via output feedback for a set of nonlinear discrete-time uncertain systems. New version of dissipativity called exponential dissipativity is defined for discrete-time stochastic system with Markovian switching. It is proved that if this stochastic system with special choice of its parameters is exponentially dissipative then the original set of uncertain system is simultaneously stabilizable and the stabilizing control has robustness properties in the sense that it admits some feedback uncertainties. The linear robust simultaneous stabilization problem is considered as a particular case. In this case the results are obtained in the form of convergent algorithm for computing of output feedback gain, based on iterative solution of LMI's. This algorithm is applied to the problem of angular stabilization of longitudinal multi-regime aircraft motion.

1. INTRODUCTION

Stabilization via output feedback and simultaneous stabilization are "hard" problems in control theory, see Polyak and Shcherbakov (2005) and references therein. At the same time the control practice requires to solve these problems under parameters uncertainty and the "hardness" of the above problem increases. Moreover if we have obtained some stabilizing control the following question arises: whether this control admits some uncertainties such that the system saves the stability property?

There exist several ideas and approaches for the solution of the mentioned problems. This paper considers the stabilization problem for a set of nonlinear discrete-time uncertain systems via output feedback. To evaluate admissible uncertainties of stabilizing control we develop new stochastic version of Willems dissipativity, see Willems (1972) called here "exponential dissipativity". For obtaining the stabilizing control we use the comparison idea, see Bernstein (1987); Pakshin (2007) and introduce into consideration the stochastic system with Markovian switching such that if this system is exponentially stable in the mean square (ESMS), then the considered set of uncertain system is exponentially stable.

The dissipative systems theory has become an important tool in the investigations of stability and stabilization of nonlinear deterministic control systems, as evident from the works of numerous authors, see Hill and Moylan (1980); Byrnes et al. (1991); Byrnes and Lin (1994); Andrievskii and Fradkov (2006) and references therein.

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In recent years this theory has been extended to stochastic systems in various different ways by many authors, see Florchinger (1999); Thygesen (1999); Borkar and Mitter (2003); Aliyu (2004); Shaked and Berman (2005); Zhang and Chen (2006); Pakshin (2007) and references therein. Florchinger (1999); Thygesen (1999); Borkar and Mitter (2003); Shaked and Berman (2005); Zhang and Chen (2006) studied the controlled systems described by the Itô diffusion processes while Aliyu (2004) investigated the controlled systems with time delay and Markov jumps. Shaked and Berman (2005); Zhang and Chen (2006) develope H_∞ control theory, from the dissipation point of view, for a large class of stochastic, nonlinear, time-invariant systems with state and output feedback. Borkar and Mitter (2003) showed relevance of dissipativity ideas by examining the problem of ergodic control of partially observed diffusions. Pakshin (2007) studied dissipativity of Itô diffusion processes with Markovian switching.

The particular case of dissipativity called passivity is effectively used for deterministic systems, see Byrnes et al. (1991); Byrnes and Lin (1994); Polushin et al. (2000); Andrievskii and Fradkov (2006) and references therein. The main relevant result in this direction is following: if considered deterministic system is passive and zero-state detectable then any output feedback in the form of smooth first/third sector function stabilizes this system. This property is not valid for stochastic systems with control dependent noise, see Pakshin (2007) and this fact motivated to introduce in this paper the new notion of exponential dissipativity, study its properties, and apply it to robust simultaneous stabilization problem.

The paper is organized as follows. In Section 2 the notion of exponential dissipativity is defined and main general theorems are formulated. In section 3 these theorems are applied to robust simultaneous stabilization problem via output feedback. In linear case the convergent algorithm for obtaining the output feedback gain is proposed. This algorithm is based on iterative solution of LMI's. An example from the field of flight control ends the paper.

2. EXPONENTIAL DISSIPATIVITY

Consider nonlinear discrete-time system described by the following stochastic difference equations with Markovian switching:

$$x_{n+1} = a(x_n, r_n) + B(x_n, r_n)u_n + \sum_{l=1}^{s} \gamma_l [f_l(x_n, r_n) + G_l(x_n, r_n)u_n] v_{nl}, \qquad (1)$$

$$z_n = c(x_n, r_n), \quad n = 0, 1, \ldots \qquad (2)$$

where $x_n \in \mathbb{R}^m$ is the state vector, $u_n \in \mathbb{R}^k$ is the input vector, $z_n \in \mathbb{R}^q$ is the output vector, r_n is the homogenous Markov chain whose state space is the set of integers $\mathbb{N} = \{1, 2, ..., \nu\}$ and the matrix of transition probabilities $P = [P_{ij}]_1^\nu = [\text{Prob}\{r_{n+1} = j \mid r_n = i\}]_1^\nu$; $v_n = [v_{n1} v_{n2} \ldots v_{ns}]'$ is the Gaussian white noise defined on the complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with the natural filtration $\mathcal{F}_n$, $n = 0, 1 \ldots$, generated by v up to time n and with the identity covariance matrix; γ_l, $l = 1, \ldots, s$ are positive scalars; the initial state $[x_0 \ r_0]'$ is deterministic. Take the assumptions that the noise process v_n does not depend on the initial state and u_n is Markov process with respect to $\mathcal{F}_n$; $a(x, i), B(x, i), f_l(x, i), G_l(x, i)$ are smooth in x functions, such that $a(0, i) \equiv 0, B(0, i) \equiv 0, f_l(0, i) \equiv 0, G_l(0, i) \equiv 0$, $l = 1, \ldots, s$, $i \in \mathbb{N}$.

Denote $\mathcal{L}^2_{\mathcal{F}}([0, N], \mathbb{R}^k)$ the set of all $\mathcal{F}_n$-Markov input processes such that

$$\| u \|^2_{\mathcal{L}^2([0,N])} \triangleq \mathrm{E} \sum_{n=0}^{N} \| u_n \|^2 < \infty, \ N = 0, 1, \ldots,$$

where E is the expectation operator.

Consider a function $W : \mathbb{R}^m \times \mathbb{N} \times \mathbb{R}^q \to \mathbb{R}$ associated with the system (1), (2). This function is called the μ–supply rate on $[0, \infty)$ if it has the following property: for any $u \in \mathcal{L}^2_{\mathcal{F}}([0, N], \mathbb{R}^k)$ the system (1) with arbitrary deterministic initial conditions x_0, r_0 has the following property

$$\mathrm{E} \sum_{n=0}^{N} \mid W(u_n, r_n, z_n) \mid + \mid \mu(x_n, r_n) \mid < \infty, \ N = 0, 1 \ldots,$$

where $\mu(x, i)$ is continuous in x for all $i \in \mathbb{N}$ function such that $\mu(x, i) > 0$, $x \neq 0$, $\mu(0, i) = 0$.

Definition 1. System (1), (2) with μ–supply rate W is said to be exponentially dissipative on $[0, \infty)$, if there exists a nonnegative continuous function $V : \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ called the storage function, such that for all $N = 1, 2 \ldots$, $x_0 = x \in \mathbb{R}^m$, $r_0 = i \in \mathbb{N}$

$$\mathrm{E}_{x,i} V(x_N, r_N) - V(x, i) \leq \mathrm{E}_{x,i} \sum_{n=0}^{N-1} W(u_n, r_n, z_n) - \mu(x_n, r_n). \qquad (3)$$

The above inequality according to Willems (1972) can be called the exponential dissipation inequality. If the considered system is autonomous then the inequality (3) under known additional conditions expresses the condition of the ESMS, see Pakshin (1994). This fact explains the term "exponential dissipativity". It is possible to define the stochastic dissipativeness in more general form based on notions of the $\mathcal{F}_n$-stopping time and $\mathcal{F}_n$-super-martingale Thygesen (1999); Borkar and Mitter (2003), but such a generalization will not be considered in this paper.

Definition 2. The available storage $V_a(x, i)$ of the system (1), (2) with μ–supply rate $W(u, i, z)$ is the function defined for $N \geq 0$ by

$$V_a(x, i) = \sup_{u \in \mathcal{L}^2_{\mathcal{F}}([0,N-1],\mathbb{R}^k)} \ \sup_{N=1,2\ldots} \mathrm{E}_{x,i} \sum_{n=0}^{N-1} [-W(u_n, r_n, z_n) + \mu(x_n, r_n)]; \ V_a(x, i) = 0, \text{ if } N = 0.$$

As in the deterministic case the available storage plays important role in determining whether or not the system is dissipative. This is shown in the following theorem.

Theorem 3. The available storage $V_a(x, i)$ is finite for all $x \in \mathbb{R}^m, i \in \mathbb{N}$ if and only if the system (1), (2) is exponentially dissipative on $[0, \infty)$. Moreover, for any possible storage function V the following inequality holds $0 \leq V_a(x, i) \leq V(x, i) \ \ \forall x \in \mathbb{R}^m$, $i \in \mathbb{N}$ and V_a is itself a possible storage function.

Now consider the functions W and μ in the special quadratic forms

$$W(u, i, z) = z'Q(i)z + 2z'S(i)u + u'R(i)u,$$
$$\mu(x, i) = x'M(i)x, \ M(i) = M'(i) > 0, \ i \in \mathbb{N} \qquad (4)$$

Theorem 4. Suppose that the system (1), (2) is exponentially dissipative with storage function $V(x, i)$, satisfying the inequality

$$\lambda_1 \| x \|^2 \leq V(x, i) \leq \lambda_2 \| x \|^2, \ \lambda_1, \lambda_2 > 0, \qquad (5)$$

and the functions W and μ are given by (4). Let $\varphi : \mathbb{R}^q \to \mathbb{R}^k$ is continuous function such that $\forall z \neq 0$

$$z'Qz - 2z'S\varphi(z) + \varphi'(z)R\varphi(z) \leq 0, \ \varphi(0) = 0. \qquad (6)$$

Then the output feedback control

$$u = -\varphi(z) \qquad (7)$$

provides ESMS of the trivial solution $x_n \equiv 0$ of the system (1).

The proofs of both the theorems are discrete-time counterpart of the results by Pakshin (2007) and they are omitted because of limited space.

Theorem 4 can be interpreted in the following way. Suppose that some stabilizing output feedback control provides exponential dissipativity of the considered system. Then this systems will be stable for all additional feedbacks (7), satisfying (6). This property will be used in the sequel by studying the robustness of stabilizing control.

3. ROBUST SIMULTANEOUS STABILIZATION

3.1 Nonlinear systems

Let the set of deterministic nonlinear systems described by the following difference equations:

$$x_{n+1} = a_i(x_n) + B_i(x_n)u_n + \sum_{l=1}^{s} \sigma_{il}(n)(f_{il}(x_n) + G_{il}(x_n)u_n), \quad (8)$$

$$z_n = c_i(x_n),\ n = 0, 1, \dots,\ i \in \mathbb{N}, \quad (9)$$

where $\sigma_{il}(n),\ n = 0, 1, \dots,\ l = 1, \dots, s\ i = 1, \dots, \nu$, are uncertain parameters such that

$$|\ \sigma_{il}(n)\ | \le \delta_{il},\ n = 0, 1, \dots,\ l = 1, \dots, s,\ i \in \mathbb{N}, \quad (10)$$

other notations are the same as above. Consider the following *robust simultaneous stabilization problem*: determine the output feedback control law in the form

$$u = -\upsilon(z),\ \upsilon(0) = 0, \quad (11)$$

where $\upsilon : \mathbb{R}^q \to \mathbb{R}^k$ is continuous function such that the closed loop systems from the set (8), (9) are asymptotically stable for all $\sigma_{il}(n)$, satisfying (10). Denote

$$\bar{a}_i(x) = a_i(x) - B_i(x)\upsilon(z),\ \bar{f}_{il}(x) = f_i(x) - G_{il}(x)\upsilon(z),$$

and consider the following stochastic system with Markovian switching

$$x_{n+1} = (1 + \alpha(r_n))^{1/2}[\bar{a}(x_n, r_n) + B(x_n, r_n)\tilde{u}_n] + \sum_{l=1}^{s} \gamma_l(r_n)[\bar{f}_l(x_n, r_n) + G_l(x_n, r_n)\bar{u}_n]v_{nl}, \quad (12)$$

$$z_n = c(x_n, r_n),\ n = 0, 1, \dots, \quad (13)$$

where $\alpha(i) = \alpha_i > 0$ is some parameter and $\tilde{u}$ is some additional input (possible feedback uncertainty), $a(x,i) = a_i(x),\ B(x,i) = B_i(x),\ f_l(x,i) = f_{il}(x),\ G_l(x,i) = G_{il}(x),\ c(x,i) = c_i(x),\ \gamma_l(i) = \gamma_{il},\ l = 1, \dots, s,\ i \in \mathbb{N}$. The noise intensities and the uncertainty bounds are connected by the following inequalities

$$(\alpha_i - \sum_{l=1}^{s} \frac{\delta_{il}^2}{\Gamma_{il}}) > 0,$$

$$0 < \Gamma_{il} \le \gamma_{il}^2 - \delta_{il}(\sum_{j \ne l}^{s} \delta_{ij} + \delta_{il}) \quad i \in \mathbb{N}. \quad (14)$$

Theorem 5. Suppose that (11) is robust stabilizing control for the set of parameter-uncertain system (8), (9) that simultaneously renders the system (12), (13) exponentially dissipative with quadratic storage function

$$V(x) = x'Px. \quad (15)$$

and with quadratic μ–supply rate given by (4). Let $\varphi : \mathbb{R}^q \to \mathbb{R}^k$ be a continuous function satisfying (6). Then each output feedback control in the form

$$u = -(\upsilon(z) + \varphi(z)), \quad (16)$$

is robust stabilizing control for the set of system (8), (9).

The proof is based on results by Bernstein (1987) and Pakshin (2007).

So, the property of exponential dissipativity of stochastic comparison system (12), (13) allows to find the family of output feedback controllers (16) providing robust simultaneous stabilzation of the set of uncertain systems (8), (9). The function $\varphi(z)$ plays here the role of the "feedback uncertainty".

3.2 Linear systems

As an important particular case consider the set of linear discrete-time uncertain system described by the equations

$$x_{n+1} = A_i x_n + B_i u_n + \sum_{l=1}^{s} \sigma_{il}(n)(A_{il}x_n + B_{il}u_n), \quad (17)$$

$$z_n = Cx_n, \quad i \in \mathbb{N}.$$

The problem is to find the gain matrix F of the output feedback control

$$u_n = \bar{u}_n + \tilde{u}_n,\ \bar{u}_n = -Fz_n,\ \tilde{u}_n = -\varphi(z_n) \quad (18)$$

such that (18) provides exponential stability of all the closed loop systems from the set (17) under parameters uncertainties satisfying (10) and under feedback uncertainty $\varphi(z)$, satisfying (6). Denote $A_{ci} = A_i - B_i FC,\ A_{cil} = A_{il} - B_{il}FC$ and consider the system

$$x_{n+1} = A_{ci}x_n + B_i\tilde{u}_n + \sum_{l=1}^{s} \sigma_{il}(n)(A_{cil}x_n + B_{il}\tilde{u}_n), \quad (19)$$

$$z_n = Cx_n, \quad i \in \mathbb{N}.$$

Stochastic comparison system for (19) has the form

$$x_{n+1} = [1 + \alpha(r_n)]^{1/2}[A_c(r_n)x_n + B(r_n)\tilde{u}_n] + \sum_{l=1}^{s} \gamma_l(r_n)[A_{cl}(r_n)x_n + B_l(r_n)\tilde{u}_n]v_{nl},\ z_n = Cx_n, \quad (20)$$

where $A_c(i) = A_{ci},\ B(i) = B_i,\ A_{cl}(i) = A_{cil},\ B_l(i) = B_{il}$ The noise intensities $\gamma_l(i) = \gamma_{il}$ and uncertainty bounds δ_{il} satisfy the inequalities (14). Applying theorem 3 we obtain the following result.

Theorem 6. System (20) with quadratic μ–supply rate is exponentially dissipative if and only if the following matrix inequalities hold

$$H_i = H_i' > 0, \begin{bmatrix} \Lambda_{i11} & \Lambda_{i12} \\ \Lambda_{i12}' & \Lambda_{i22} \end{bmatrix} \le 0,\ i \in \mathbb{N}, \quad (21)$$

where

$$\Lambda_{i11} = A_{c\alpha i}' P_i A_{c\alpha i} - H_i + \sum_{l=1}^{s} \gamma_{il}^2 A_{cil}' P_i A_{cil} + M_i - C_i' Q_i C_i,$$

$$\Lambda_{i12} = A_{c\alpha i}' P_i B_{\alpha i} - C_i' S + \sum_{l=1}^{s} \gamma_{il}^2 A_{cil}' P_i B_{il}',$$

$$\Lambda_{i22} = B_{\alpha i}' P_i B_{\alpha i} - R_i + \sum_{l=1}^{s} \gamma_{il}^2 B_{il}' P_i B_{il}',$$

$$A_{c\alpha i} = (1+\alpha_i)^{1/2} A_{ci},\ B_{\alpha i} = (1+\alpha_i)^{1/2} B_i,\ P_i = \sum_{j=1}^{\nu} p_{ij} H_j.$$

The corresponding storage function has the form: $V(x,i) = x' H_i x,\ i \in \mathbb{N}$.

Letting $H_i = H,\ i \in \mathbb{N}$. we exclude dependency from the transition probabilities. Then we can formulate sufficient conditions of exponential dissipativity as follows.

Corollary 7. System with quadratic μ–supply rate is exponentially dissipative if the following matrix inequalities hold

$$H = H' > 0, \begin{bmatrix} \Gamma_{i11} & \Gamma_{i12} \\ \Gamma'_{i12} & \Gamma_{i22} \end{bmatrix} \leq 0, \ i \in \mathbb{N}, \tag{22}$$

where

$$\Gamma_{i11} = A'_{c\alpha i} H A_{c\alpha i} - H + \sum_{l=1}^{s} \gamma_{il}^2 A'_{cil} H A_{cil} + M_i - C'_i Q_i C_i,$$

$$\Gamma_{i12} = A'_{c\alpha i} H B_{\alpha i} - C'_i S + \sum_{l=1}^{s} \gamma_{il}^2 A'_{cil} H B'_{il},$$

$$\Gamma_{i22} = B'_{\alpha i} H B_{\alpha i} - R_i + \sum_{l=1}^{s} \gamma_{il}^2 B'_{il} H B'_{il}.$$

The corresponding storage function has the form: $V(x,i) = x'Hx, \ i \in \mathbb{N}$.

The problem will be solved if we obtain the matrices F, H, satisfying bilinear matrix inequalities (22). Unfortunately the solution of these inequalities with respect to pair F, H is connected with essential difficulties. We propose the following approach. First obtain the gain matrix F, which provides ESMS of the system (20) with $\tilde{u}_n \equiv 0$. Then the inequalities (22) are reduced to LMI's with respect to matrix H and this matrix can be easily found by the feasibility test, see Boyd et al. (1994). According to reasons above consider the stochastic system

$$x_{n+1} = [1 + \alpha(r_n)]^{1/2}[A(r_n)x_n + B(r_n)\bar{u}_n] + \sum_{l=1}^{s} \gamma_l(r_n)[A_l(r_n)x_n + B_l(r_n)\bar{u}_n]v_{nl}, \ z_n = Cx_n, \tag{23}$$

Suppose that the state feedback control

$$\bar{u}_n = -Kx_n \tag{24}$$

provides ESMS of the system (23). Then the gain matrix can be found as $K = \Upsilon X^{-1}$, see Boyd et al. (1994) for details, where the pair Υ, X is solution of the LMI's

$$X > 0, \ \begin{bmatrix} X & Z \\ Z' & D_i(X) \end{bmatrix} > 0, \tag{25}$$

$$Z = [(A_{\alpha i}X - B_{\alpha i}\Upsilon)' \ \gamma_1(A_{i1}X - B_{i1}\Upsilon)' \dots \gamma_N(A_{iN}X - B_{iN}\Upsilon)',], D_i(X) = \mathrm{diag}[M_{ai}^{-1} \ X \dots X]$$

for some matrix $M_{ai} = M'_{ai} > 0$. If the the equation

$$FC = K, \tag{26}$$

has exact solution with respect to matrix F, then this matrix is the gain matrix of output stabilizing control

$$\bar{u}_n = -Fz_n \tag{27}$$

and it can be easily found from the equation (26). Unfortunately it is possible only with a special structure of the matrix K. To find exact solution of (26) we try impose the structural constrains for the matrix K. Write singular value decomposition for the matrix C :

$$C = USV', \ U'U = I, \ V'V = I, \tag{28}$$

where U and V are orthogonal matrices, S is rectangular matrix which diagonal elements represent singular values of C, and other elements are zeroes. Let $V = [V_1 \ V_2]$, where $V_1 \in \mathbb{R}^{m \times q}, \ V_2 \in \mathbb{R}^{m \times (m-q)}$.

Define

$$F = KC^+, \tag{29}$$

where superscript + denotes Moore-Penrose inverse. Denoting $\hat{A}_{\alpha i} = V'A_{\alpha i}V$, $\hat{B}_{\alpha i} = V'B_{\alpha i}$, $\hat{K} = KV = [\hat{K}_1 \quad \hat{K}_2]$, where $\hat{K}_1 = KV_1, \ V_1 \in \mathbb{R}^{m \times q}, \ \hat{K}_2 = KV_2, \ V_2 \in \mathbb{R}^{m \times (m-q)}$ and taking into account (28) we have

$$A_{\alpha i} - B_{\alpha i}FC = V(\hat{A}_{\alpha i} - \hat{B}_{\alpha i}[\hat{K}_1 \quad \hat{K}_2] \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} V', \tag{30}$$

similarly

$$A_{ij} - B_{ij}FC = V\left(\hat{A}_{ij} - \hat{B}_{ij}[\hat{K}_1 \quad \hat{K}_2] \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}\right) V'. \tag{31}$$

Using these relations we can write

$$KC^+C = KV_1V'_1 = K(I - V_2V'_2) = K - KV_2V'_2. \tag{32}$$

The relations (30), (31) do not depend on a specific value of $\hat{K}_2$. At the same time if

$$\hat{K}_2 = KV_2 = 0, \tag{33}$$

then from (32) we obtain that equation (26) holds. So if K is the gain matrix of state feedback stabilizing control (24) satisfying (33), then (29) is the gain matrix of output feedback stabilizing control (27).

The gain matrix K nonlinearly depends on variables X and Υ from (25) and attempt to solve the LMIs (25) together with the constraint (33) is not effective. For this reason we take into consideration this constraint using Lagrange multipliers method.

Consider the the cost functional

$$J = \mathrm{E}[\sum_{n=0}^{\infty}(x'_nQ_a(r_n)x_n + u'_nR_a(r_n)u_n)], \tag{34}$$

where $Q_a(i) = Q_{ai} = Q'_{ai} \geq 0, \ R_a(i) = R_{ai} = R'_{ai} > 0, \ i = 1, \dots, \nu$. Let the function (15) satisfies the equations

$$L_iV(x) = -x'(Q_{ai} + C'F'R_{ai}FC)x, \ i = 1, \dots, \nu \tag{35}$$

where $L_iV(x) = \mathrm{E}[V(x_{n+1}] \mid x_n = x, \ r_n = i] - V(x)$ is stochastic first difference along trajectories of the system (23) with $\bar{u}$ given by (27).

Consider the following optimal stabilization problem. Find the control law in the form (24), which provides ESMS of the system (23) and minimizes the cost functional (34) along solutions of this system under constrains (33), (35). The constraints (35) is equal to the following system of the algebraic matrix equations, see Pakshin (1994, 1997)

$$(A_{\alpha i} - B_{\alpha i}K)'P(A_{\alpha i} - B_{\alpha i}K) - P + Q_{ai} + K'R_{ai}K + \sum_{j=1}^{N} \gamma_{ij}^2(A_{ij} - B_{ij}K)'P(A_{ij} - B_{ij}K) = 0, \ i \in \mathbb{N}. \tag{36}$$

The functional (34) can be rewritten in the form

$$J = \mathrm{trace}[PX_0], \tag{37}$$

where $X_0 = x_0x'_0$. So we have the minimization problem of the function (37) under constrains (36), (33). Solving this problem by Lagrange multipliers method and taking into account the results by Pakshin (1997) and Yu (2004) we obtain the following algorithm based on iterative solution of LMI's.

Step 1. Assign the matrices $M_{ai} > 0, \ Q_{ai} \geq 0, \ R_{ai} > 0, \ X_0 > 0$ and obtain the initial value of the gain matrix $K = K_0$ as a solution of LMI's (25).

Step 2. Solve the LMI's with respect to $Y_l = Y_l' > 0$ and with respect to $P_l = P_l' > 0$:

$$\sum_{i=1}^{\nu}(A_{\alpha i} - B_{\alpha i}K_l)Y_l(A_{\alpha i} - B_{\alpha i}K_l)' + \sum_{j=1}^{s}\gamma_{ij}^2(A_{ij} - B_{ij}K_l)Y_l(A_{ij} - B_{ij}K_l)' + X_0 - \nu Y_l < 0,$$

$$(A_{\alpha i} - B_{\alpha i}K_l)'P_l(A_{\alpha i} - B_{\alpha i}K_l) + \sum_{j=1}^{s}\gamma_{ij}^2(A_{ij} - B_{ij}K_l)'P_l(A_{ij} - B_{ij}K_l) + K_l'R_{ai}K_l + Q_{ai} - P_l < -I.$$

Step 3. Evaluate the gain increment

$$\Delta K_l = [\sum_{i=1}^{\nu} R_{ai} + B_{\alpha i}'P_lB_{\alpha i} + \sum_{j=1}^{s}\gamma_{ij}^2 B_{ij}'P_lB_{ij}]^{-1}[\sum_{i=1}^{\nu} B_{\alpha i}'P_lA_{\alpha i} + \sum_{j=1}^{s}\gamma_{ij}^2 B_{ij}'P_lA_{ij}][I - V_2(V_2'Y_l^{-1}V_2)^{-1}V_2'Y_l^{-1}] - K_l$$

Step 4. Update the gain $K_{l+1} = K_l + \beta_l\Delta K_l$, where $0 < \beta_l < 2$ and β_l is chosen so that the system (23) with $u_n = K_lx_n$ is ESMS. Set $l = l + 1$.

Step 5. If $\| K_lV_2 \| < \epsilon$, then stop the procedure and let $F = K_lC^+$, else go to step 2.

The following theorem gives a method of obtaining the parameter β_n, that provides both ESMS of the system (23) with $u_n = -K_{l+1}x_n$ for all the steps of the algorithm and convergence of this algorithm. Denote

$$M_{1i} = A_{\alpha i} - B_{\alpha i}K_l, \quad W = P_l,$$
$$M_{2i} = -B_{\alpha i}\Delta K_l,\ N_{ij} = A_{ij} - B_{ij}K_l,\ \tilde N_{ij} = -B_{ij}\Delta K_l,$$
$$Z_i = Q_{ai} + K_l'R_{ai}K_l + I, \quad i = 1,\ldots,\nu,\ j = 1,\ldots,N,$$
$$a_{il} = \| Z_i^{-1/2}(M_{2i}'WM_{2i} + \sum_{j=1}^{N}\gamma_{ij}^2\tilde N_{ij}'W\tilde N_{ij})Z_i^{-1/2} \|_2$$
$$b_{il} = 2 \| Z_i^{-1/2}(M_{1i}'WM_{2i} + \sum_{j=1}^{N}\gamma_{ij}^2 N_{ij}'W\tilde N_{ij})Z_i^{-1/2} \|_2 .$$

Theorem 8. (Convergence of the algorithm). Let the parameter $\beta_l, l = 1, 2, \ldots$ satisfies on each step the condition

$$\beta_l < \min_i \min\{\beta_{il}^+, 2\},$$

where β_{il}^+ is positive root of the quadratic equation

$$a_{il}\beta^2 + b_{il}\beta - 1 = 0.$$

Then the considered algorithm converges and the system (23) with $u_n = -K_lx_n$ is ESMS.

Corollary 9. The parameter β_{il} $(i = 1, 2, \ldots.\nu;\ l = 0, 1, \ldots)$ can be obtained on each step as solution of the LMI–optimization problem:

$$\beta_{il} \to \max,$$
$$0 < \beta_{il} < 2, \begin{bmatrix} \Phi_{i11} & \Phi_{i12} \\ \Phi_{i12}' & \Phi_{i22} \end{bmatrix} > 0, \tag{38}$$

where

$$\Phi_{i11} = P_l,$$
$$\Phi_{i12} = [(A_{\alpha i} - B_{\alpha i}K_l - B_{\alpha i}\beta_{il}\Delta K_l)'\ (\gamma_{i1}(A_{i1} - B_{i1}\ K_l - B_{i1}\beta_{il}\Delta K_l)' \ldots \gamma_{is}(A_{is} - B_{is}\ K_l - B_{is}\beta_{il}\Delta K_l)']P_l,\ \Phi_{i22} = \mathrm{diag}[P_l \ldots P_l].$$

4. EXAMPLE

In flight control practice it is very important to obtain an output feedback controller with a constant gain to stabilize the aircraft in all the possible flight modes. In this section we briefly demonstrate the application of the proposed method in the design of a control system for the linearized model of the angular longitudinal aircraft motion. This model is given by the following equations

$$\begin{aligned} \dot\vartheta &= \omega_z, \\ \dot\omega_z &= -a_{mz}^{\alpha}\vartheta - a_{mz}^{\omega z}\omega_z + a_{mz}^{\alpha}\Theta + a_{mz}^{\delta}\delta, \\ \dot\Theta &= -a_y^{\alpha}\vartheta + a_y^{\alpha}\Theta, \end{aligned} \tag{39}$$

where ϑ is the pitch angle, ω_z is the angular velocity, $\Theta = \vartheta - \alpha$, α is the angle of attack, δ is the elevator angle. In this case the state and control vectors of the system (1) are

$$x(t) = [\vartheta\ \omega_z\ \Theta]',\ \ u(t) = \delta(t),$$

Usually only ϑ and ω_z are available for direct measurement and we have

$$z(t) = [\vartheta\ \omega_z]'.$$

The considered aircraft has nine typical flight modes with uncertainty of each mode given by

$$a_{mz0}^{\alpha} - \Delta a_{mz}^{\alpha} \le a_{mz}^{\alpha} \le a_{mz0}^{\alpha} + \Delta a_{mz}^{\alpha},\ \Delta a_{mz}^{\alpha} = 0.05a_{mz0}^{\alpha},$$
$$a_y^{\alpha 0} - \Delta a_y^{\alpha} \le a_y^{\alpha} \le a_{y0}^{\alpha} + \Delta a_y^{\alpha},\ \Delta a_y^{\alpha} = 0.05a_{y0}^{\alpha},$$
$$a_{mz}^{\omega z0} - \Delta a_{mz}^{\omega z} \le a_{mz}^{\omega z} \le a_{mz0}^{\omega z} + \Delta a_{mz}^{\omega z}\ \Delta a_{mz}^{\omega z} = 0.05a_{mz0}^{\omega z},$$
$$a_{mz}^{\delta 0} - \Delta a_{mz}^{\delta} \le a_{mz}^{\delta} \le a_{mz0}^{\delta} + \Delta a_{mz}^{\delta},\ \Delta a_{mz}^{\delta} = 0.05a_{mz0}^{\delta}.$$

The numerical values of the parameters are the following Krasovskii (1973):

$$A_1^0 = \begin{bmatrix} 0 & 1 & 0 \\ -4.2 & -1.5 & 4.2 \\ 0.77 & 0 & -0.77 \end{bmatrix},\ B_1^0 = \begin{bmatrix} 0 \\ -7.4 \\ 0 \end{bmatrix},$$

$$A_2^0 = \begin{bmatrix} 0 & 1 & 0 \\ -7.1 & -1.9 & 7.1 \\ 1 & 0 & -1 \end{bmatrix},\ B_2^0 = \begin{bmatrix} 0 \\ -12.7 \\ 0 \end{bmatrix},$$

$$A_3^0 = \begin{bmatrix} 0 & 1 & 0 \\ -78 & -4.1 & 78 \\ 2.8 & 0 & -2.8 \end{bmatrix},\ B_3^0 = \begin{bmatrix} 0 \\ -57 \\ 0 \end{bmatrix},$$

$$A_4^0 = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -1.4 & 4 \\ 0.62 & 0 & -0.62 \end{bmatrix},\ B_4^0 = \begin{bmatrix} 0 \\ -7.5 \\ 0 \end{bmatrix},$$

$$A_5^0 = \begin{bmatrix} 0 & 1 & 0 \\ -116 & -2.36 & 116 \\ 2.3 & 0 & -2.3 \end{bmatrix},\ B_5^0 = \begin{bmatrix} 0 \\ -42 \\ 0 \end{bmatrix},$$

$$A_6^0 = \begin{bmatrix} 0 & 1 & 0 \\ -7.9 & -1.1 & 7.9 \\ 0.56 & 0 & -0.56 \end{bmatrix},\ B_6^0 = \begin{bmatrix} 0 \\ -13.8 \\ 0 \end{bmatrix},$$

$$A_7^0 = \begin{bmatrix} 0 & 1 & 0 \\ -55 & -0.66 & 55 \\ 0.84 & 0 & -0.84 \end{bmatrix},\ B_7^0 = \begin{bmatrix} 0 \\ -22.5 \\ 0 \end{bmatrix},$$

$$A_8^0 = \begin{bmatrix} 0 & 1 & 0 \\ -14.5 & -0.43 & 14.5 \\ 0.33 & 0 & -0.33 \end{bmatrix},\ B_8^0 = \begin{bmatrix} 0 \\ -8.6 \\ 0 \end{bmatrix},$$

$$A_9^0 = \begin{bmatrix} 0 & 1 & 0 \\ -18 & -0.31 & 18 \\ 0.34 & 0 & -0.34 \end{bmatrix},\ B_9^0 = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix},$$

$$C_i^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, i = 1, \ldots, 9.$$

Suppose that the control law is formed using on-board computer such that $u(t) = u(nT) = u_n,\ nT \le t < (n+1)T,\ \ n = 0, 1, ...,$ where T is the sample period. The problem is to stabilize the system (39) in all the modes for given uncertain parameters of each mode by means of constant static output feedback control $\bar{u}_n = -Fz_n$.

To find the gain matrix F of this control law we used the proposed algorithm. As a result of computing with the sample period $t = 0.015$ sec., we obtain the gain matrix $F = [-16.9\ \ -2.0]$. Figure 1 shows the typical step responses in all the modes with the obtained control law. It is checked using (21) that this control law saves the stabilizing properties under feedback uncertainty $\tilde{u} = \varphi(z)$ such that $-0.05Fz \le \varphi(z) \le 0.05Fz$.

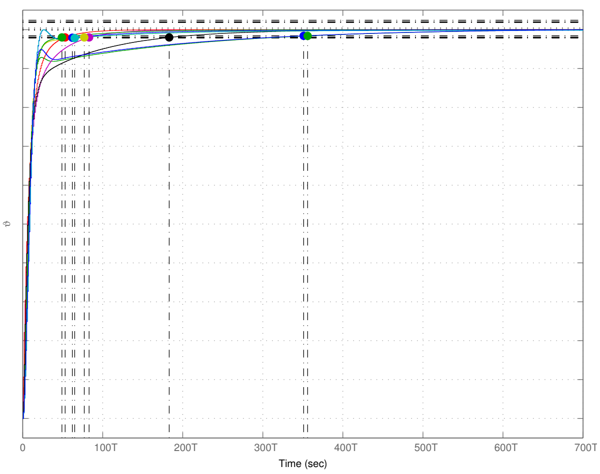

Fig. 1. The normalized step responses in different modes.

All LMI/LME programming was done within the framework of the YALMIP interface to the SeDuMi solver for MATLAB.

REFERENCES

B. R. Andrievskii and A. L. Fradkov. Method of passification in adaptive control, estimation and synchronization. *Automation and Remote Control*, volume 67, issue 9, pages 1699–1731, 2006.

M. D. S. Aliyu. Dissipative analysis and stability of nonlinear stochastic state-delayed systems. *Nonlinear Dynamics and Systems Theory*, volume 4, pages 243–256, 2004.

D. S. Bernstein. Robust static and dynamic output-feedback stabilization: Deterministic and stochastic perspectives. *IEEE Trans. Automat. Control*, volume 32, pages 1076–1084, 1987.

V. Borkar and S. Mitter. A note on stochastic dissipativeness. In *Directions in Mathematical Systems Theory and Optimization, Lecture Notes in Control Inform. 286*, pages 41–49. Springer, Berlin, 2003

C. I. Byrnes , A. Isidori and J. C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Automat. Control*, volume 36, pages 1228–1240, 1991.

C. I. Byrnes and Wei Lin. Losslessness, feedback equivalence and the global stabilization of discrete-time nonlinear systems. *IEEE Trans. Automatic Control*, volume 39, pages 83–98, January 1994.

S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. *Linear matrix inequalities in control and system theory.* SIAM, Philadelphia, 1994.

P. Florchinger. A passive system approach to feedback stabilization of nonlinear control stochastic systems. *SIAM J. Control Optimization*, volume 37, pages 1848–1864, 1999.

D. J. Hill and P. J. Moylan. Connection between finite-gain and asymptotic stability. *IEEE Trans. Automat. Control*, volume 25, pages 931–936, 1980.

A. A. Krasovskii *Systems of the authomatic flight control and their analytical design.* Nauka, Moscow, 1973.

P. V. Pakshin. Robust stability and stabilization of the family of jumping stochastic systems. *Nonlinear Analysis, Theory, Methods and Applications*, volume 30, pages 2855–2866, 1997.

P. V. Pakshin. Dissipativity of Ito diffusion processes with Markovian switching and problems of robust stabilization. *Automation and Remote Control*, volume 68, issue 9, 2007.

P. V. Pakshin. *Discrete Systems with Random Parameters and Structure.* Nauka, Moscow, 1994.

B. T. Polyak and P. S. Shcherbakov. Hard Problems in Linear Control Theory: Possible Approaches to Solution. *Automation and Remote Control*, volume 66, pages 681–718, 2005.

I. G. Polushin, A.L. Fradkov and D.J.Hill. Passivity and passification of nonlinear systems. *Automation and Remote Control*, volume 61, part 1, pages 355–388, 2000.

U. Shaked and N. Berman. H_∞ control for nonlinear stochastic systems: The output-feedback case. In *Preprints of the 16th IFAC World Congress*, Prague, Czech Republic, July 3-8, 2005, volume CD-ROM, pages 1–6.

U. H. Thygesen, On dissipation in stochastic systems. In *Proceedings of the American Control Conference*, San Diego, CA, pages 1430–1434, IEEE, 1999.

J. C. Willems. Dissipative dynamic systems. Part I: General theory. Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, volume 45, pages 321–393, 1972.

W. Zhang and B. S. Chen. State feedback H_∞ control for a class of nonlinear stochastic systems. *SIAM J. Control Optimization*, volume 44, pages 1973–1991, 2006.

Jen-te Yu. A convergent algorithm for computing stabilizing static output feedback gains *IEEE Trans. Automat. Control*, volume 49, pages 2271–2275, 2004.