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Current Output Observer for Stochastic-Parameter Models and Application to Sensor Failure

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Abstract: In this paper, linear discrete-time systems with white stochastic parameters are considered. Most results on the optimal state estimation of linear discrete time systems with stochastic parameters rely strongly on the generalization of the one step prediction type Kalman filter to this type of systems. But it has been shown that the current output observer results in less estimation error as compared to the one step prediction Kalman Filter for the case of systems with deterministic parameters. In this work, the current output observer is generalized to stochastic-parameter systems and the estimation error performance improvement is mathematically shown. We have particularly directed our attention to the application to the sensor failure problem, which involves a stochastic model with non-Gaussian parameter distribution. Experimental results confirm our prediction and shows that the current output observer has a substantial benefit for the sensor failure problem over the one step prediction generalized Kalman filter solution.

1. INTRODUCTION

The focus of this paper is on state estimation of linear discrete-time systems and measurement models which contain white stochastic parameters. Because of the stochastic nature of the parameters, conventional Kalman filter (KF) (Kalman, 1960) does not work and therefore generalizations (Nahi, 1969; De Koning, 1984) have been proposed. These models are often called stochastic discrete-time systems with state multiplicative noise. They have applications in many areas such as target tracking in presence of failing sensors/missing data or delayed measurements (Nahi, 1969), communication network systems with missing or delayed measurements (Sinopoli et al, 2004), satellite attitude control (McLane, 1971), chemical reactor control (Rao, Ramakrishna, and Borwanker, 1974) population dynamics (Mohler and Kolodziej, 1980), macroeconomics (Aoki, 1976), robustness studies (Yaz, 1990; Wang and Yang, 2002) to name a few.

The typical method of estimation used to address this issue is by generalizing the Kalman filter (Kalman, 1960) to discrete-time systems with stochastic parameters. The first of these Kalman filter based generalization method was introduced in (Nahi, 1969) which considered the problem of uncertain observation by including a scalar stochastic variable in the measurement equation. Then the paper (De Koning, 1984) proposed the generalized Kalman filter (GKF) for the case where the elements of system and measurement matrices may be all stochastic white sequences. Several papers have used the same framework to address similar problems (Rajasekaran, Satyanarayana, and Srinath, 1971; Tugnait, 1981; Yaz, 1992; Yaz and Ray, 1996; Wu, Yaz, and Olejniczak, 1997; NaNacara and Yaz, 1997; Sinopoli, et al, 2004; Wang, Ho, and Liu, 2004; Hounkpevi and Yaz, 2007) to name a few.

In all these generalizations, only the one step prediction KF has been studied. But it is known that the current output observer provides smaller estimation error covariance than the one step prediction KF for systems with deterministic matrices. Therefore in this paper, we advance these studies to systems with stochastic parameters. We study the stochastic current output observer (SCOO). Explicit expression of the observer is given, and we focus particularly on the application to the random sensor failure problem.

In section 2, we present the general class of stochastic state space model under consideration. Then we present the existing generalized one step prediction solution. In section 4, formulae are provided for the new current output observer. Section 5 discusses the special case of the random sensor failure problem. Section 6 contains illustrative simulation examples. Section 7 is the conclusion section.

2. MODEL

Consider the stochastic dynamical system and the measurement model

$$\begin{cases} x_{k+1} = A_k x_k + v_k \\ y_k = C_k x_k + w_k \end{cases} \tag{1}$$

where $x_k \in R^n$, with x_0 having the expected value $E\{x_0\} = \bar{x}_0$ and covariance $E\left\{\tilde{x}_0 \tilde{x}_0^{\ T}\right\} = E\left\{\left(x_0 - \bar{x}_0\right)\left(x_0 - \bar{x}_0\right)^T\right\} = P_0$, v_k is zero mean white noise vector uncorrelated with x_0 and having covariance V, w_k is zero mean white noise that is uncorrelated with v_k and x_0 having covariance W, y_k is the measurement vector at time k, A_k and C_k are matrices of white noise (time-wise uncorrelated) sequences with known means ($\bar{A}_k$ and $\bar{C}_k$) and known covariances

($\overline{\tilde{A}_k \tilde{A}_k^T} = \overline{(A_k - \overline{A}_k)(A_k - \overline{A}_k)^T}$ and $\overline{\tilde{C}_k \tilde{C}_k^T} = \overline{(C_k - \overline{C}_k)(C_k - \overline{C}_k)^T}$) which are mutually uncorrelated and uncorrelated with other noises.

It is very important to point out here that this is different from the state space representation used in the regular KF where A_k and C_k are deterministic.

The problem that we consider in this paper is to derive and study the current output observer solution to the problem of linear minimum variance estimation of the state based on the knowledge of measurements and statistics of the system parameters. But we will first introduce the existing GKF solution.

3. THE GENERALIZED KALMAN FILTER SOLUTION

The solution that has been used to estimate the states of system (1) is the linear minimum variance GKF which was presented in the general form in (De Koning, 1984).

In this section, $\hat{x}_k$ is the one step prediction estimate of the state of the system at time k given measurements up through time $k-1$.

The linear minimum variance estimator is given by:

$$\hat{x}_{k+1} = \overline{A}_k \hat{x}_k + K_k \left(y_k - \overline{C}_k \hat{x}_k \right) \tag{2}$$

The estimation gain is found as:

$$K_k = \overline{A}_k P_k \overline{C}_k^T \left(\overline{C}_k P_k \overline{C}_k^T + \overline{\tilde{C}_k Q_k \tilde{C}_k^T} + W \right)^{-1} \tag{3}$$

and

$$Q_k \triangleq E\left\{ x_k x_k^T \right\} \tag{4}$$

$$Q_{k+1} = \overline{A_k Q_k A_k^T} + V = \overline{A}_k Q_k \overline{A}_k^T + \overline{\tilde{A}_k Q_k \tilde{A}_k^T} + V \tag{5}$$

$$Q_0 = P_0 + \overline{x}_0 \overline{x}_0^T \tag{6}$$

The estimation error covariance for this solution is given by the following recursive equation:

$$\begin{aligned} P_{k+1} &= \overline{A}_k P_k \overline{A}_k^T - \\ &\overline{A}_k P_k \overline{C}_k^T \left(\overline{C}_k P_k \overline{C}_k^T + \overline{\tilde{C}_k Q_k \tilde{C}_k^T} + W \right)^{-1} \overline{C}_k P_k \overline{A}_k^T + \\ &\overline{\tilde{A}_k Q_k \tilde{A}_k^T} + V \end{aligned} \tag{7}$$

4. STOCHASTIC CURRENT OUTPUT OBSERVER

Let us now present the SCOO solution to the problem of linear minimum variance estimation of the state based on the measurements $y_k, y_{k-1}, \cdots, y_0$.

<u>THEOREM 1</u>

The SCOO estimate of the state in system (1) is given by:

$$\hat{x}_k^c = \hat{x}_k + K_k^c \left(y_k - \overline{C}_k \hat{x}_k \right) \tag{8}$$

where $\hat{x}_k$ is obtained from (2) and the SCOO gain K_k^c is given as:

$$K_k^c = P_k \overline{C}_k^T \left(\overline{C}_k P_k \overline{C}_k^T + \overline{\tilde{C}_k Q_k \tilde{C}_k^T} + W \right)^{-1} \tag{9}$$

where Q_k is given by (5) and P_k is given by (7).

The estimation error covariance of SCOO is bounded above by that of the GKF as follows:

$$P_k^c \le P_k \tag{10}$$

<u>PROOF</u>

The system described by Equation (1) can be transformed into the following state and measurement equations by augmenting the state as follows:

$$\begin{cases} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} + \begin{bmatrix} v_k \\ 0 \end{bmatrix} \\ y_k = \begin{bmatrix} C_k & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} + w_k \end{cases} \tag{11}$$

where I and 0 are the identity matrix of dimension n and the matrix of zeros with suitable dimensions, respectively.

Model (11) can be put in the more compact form:

$$\begin{cases} X_{k+1} = \Phi_k X_k + V_k \\ y_k = M_k X_k + w_k \end{cases} \tag{12}$$

where $X_k = \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}$, $\Phi_k = \begin{bmatrix} A_k & 0 \\ I & 0 \end{bmatrix}$, $V_k = \begin{bmatrix} v_k \\ 0 \end{bmatrix}$ is a zero mean white noise vector of covariance $V_g = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}$, and $M_k = \begin{bmatrix} C_k & 0 \end{bmatrix}$.

The linear minimum variance estimator for this stochastic system is given by:

$$\hat{X}_{k+1} = \overline{\Phi}_k \hat{X}_k + G_k \left(y_k - \overline{M}_k \hat{X}_k \right) \tag{13}$$

The SCOO estimate is obtained by:

$$\hat{x}_k^c = \begin{bmatrix} 0 & I \end{bmatrix} \hat{X}_{k+1} \tag{14}$$

which yields (8).

The estimation gain for (13) is found as:

$$G_k = \bar{\Phi}_k R_k \bar{M}_k^{\ T} \left(\bar{M}_k R_k \bar{M}_k^{\ T} + \overline{\tilde{M}_k Q_k^g \tilde{M}_k^{\ T}} + W \right)^{-1} \tag{15}$$

where

$$Q_k^g = \begin{bmatrix} Q_k & \bar{A}_k Q_{k-1} \\ Q_{k-1} \bar{A}_k^T & Q_{k-1} \end{bmatrix} \tag{16}$$

and $K_k^c = \begin{bmatrix} 0 & I \end{bmatrix} G_k$ after simplification leads to (9).
The optimal value of the estimation error covariance is given by the following recursive equation:

$$\begin{aligned} R_{k+1} &= \bar{\Phi}_k R_k \bar{\Phi}_k^{\ T} - \\ &\bar{\Phi}_k R_k \bar{M}_k^{\ T} \left(\bar{M}_k R_k \bar{M}_k^{\ T} + \overline{\tilde{M}_k Q_k^g \tilde{M}_k^{\ T}} + W \right)^{-1} \bar{M}_k R_k \bar{M}_k^{\ T} + \\ &\overline{\tilde{\Phi}_k Q_k^g \tilde{\Phi}_k^{\ T}} + V_g \end{aligned} \tag{17}$$

and the estimation error covariance for the SCOO is obtained by:

$$\begin{aligned} P_k^c &= E\left\{ \left(x_k - \hat{x}_k^c \right) \left(x_k - \hat{x}_k^c \right)^T \right\} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} R_k \\ &= P_k - P_k \bar{C}^T \left(\bar{C} P_k \bar{C}^T + \overline{\tilde{C}_k Q_k \tilde{C}_k^{\ T}} + W \right)^{-1} \bar{C} P_k \\ &\leq P_k = E\left\{ \left(x_k - \hat{x}_k \right) \left(x_k - \hat{x}_k \right)^T \right\} \end{aligned} \tag{18}$$

Therefore, the SCOO solution may result in less mean square error (MSE) than the GKF solution. The above inequality does not imply that the error of the new proposed filter (SCOO) is uniformly smaller than the error of the GKF, however, the estimation error covariance of the SCOO is bounded above by that of the GKF.

Note that the above ordering is in the sense of Loewner (Horn and Johnson, 1991) ($A \geq B$ means $A - B \geq 0$ or a positive semi-definite matrix). Note also that the improvement in estimation error performance is achieved by minimal additional computational load given by (8) and (9) and there is no increase in the dimension of the generalized Riccati equation (7) used in SCOO.

5. SPECIAL CASE: APPLICATION TO THE SENSOR FAILURE PROBLEM

We will consider the special case of random sensor failure in this section. We will compare the SCOO method to the typical GKF method. The random sensor failure problem is defined by the following uncertain observation system.

5.1 Problem Statement

Consider the following stochastic dynamical system and the measurement model

$$\begin{cases} x_{k+1} = A_k x_k + v_k \\ y_k = \beta_k C_k x_k + w_k \end{cases} \tag{19}$$

where A_k and C_k are real matrices with deterministic coefficients, β_k are mutually independent scalar Bernoulli random variables whose outcomes are 1 or 0, having mean $\bar{\beta}$, and variance $\bar{\beta}\left(1-\bar{\beta}\right)$.

Model (19) describes the problem of missing measurement (when $\beta_k = 0$), a situation in which there may be a nonzero probability $\left(1-\bar{\beta}\right)$ that the measurement contains only noise (Nahi, 1969). This corresponds in a discrete time system, to the signal at some sampling time being completely absent from the observation. In this work we have only considered the case where the signal is either present or absent in the measurement. Intermediate cases where the signal is partially present can be treated by similar techniques.

5.2 The GKF Solution to the Problem

The GKF solution to this problem is a special case of the general solution presented in section 3 and is given as:

$$\hat{x}_{k+1} = A_k \hat{x}_k + K_k \left(y_k - \bar{\beta} C_k \hat{x}_k \right) \tag{20}$$

The estimation gain is found as:

$$K_k = \bar{\beta} A_k P_k C_k^{\ T} \begin{pmatrix} \left(\bar{\beta}\right)^2 C_k P_k C_k^{\ T} + \\ \bar{\beta}\left(1-\bar{\beta}\right) C_k Q_k C_k^{\ T} + W \end{pmatrix}^{-1} \tag{21}$$

where

$$Q_{k+1} = A_k Q_k A_k^{\ T} + V \tag{22}$$

with

$$Q_0 = P_0 + \bar{x}_0 \bar{x}_0^{\ T}$$

The estimation error covariance for this solution is given by the following recursive equation:

$$\begin{aligned} P_{k+1} &= A_k P_k A_k^{\ T} - \\ &\left(\bar{\beta}\right)^2 A_k P_k C_k^{\ T} \begin{pmatrix} \left(\bar{\beta}\right)^2 C_k P_k C_k^{\ T} + \\ \bar{\beta}\left(1-\bar{\beta}\right) C_k Q_k C_k^{\ T} + W \end{pmatrix}^{-1} C_k P_k A_k^{\ T} + V \end{aligned} \tag{23}$$

.

5.3 Solution using the SCOO

The solution to the sensor failure problem using the SCOO solution in section 4 is specified as:

Current output observer

$$\hat{x}_k^c = \hat{x}_k + K_k^c \left(y_k - \bar{\beta} C_k \hat{x}_k \right) \tag{24}$$

Current output gain

$$K_k^c = \bar{\beta} P_k C^T \left(\left(\bar{\beta}\right)^2 C P_k C^T + \bar{\beta}\left(1-\bar{\beta}\right) C Q_k C^T + W \right)^{-1} \qquad (25)$$

where: $\hat{x}_k$ is given by (20), K_k by (21), Q_k by (22) and P_k by (23).

6. SIMULATION EXAMPLES

In this section, we provide two simulation examples. In example 1, we consider the case where the stochastic parameters have Bernoulli distribution (random sensor failure model) and in example 2, the case where the stochastic parameters have Gaussian distribution. Example 1 provides the performance comparison of the SCOO with the GKF for a non-Gaussian distribution, and example 2 provides performance comparison between the two filters for Gaussian distribution.

6.1 Example 1: Non-Gaussian Case

In this example, we consider the problem of estimation for system with random sensor failure. We have applied the SCOO to the following system and compared the result with that obtained using the GKF method:

$$\begin{cases} x_{k+1} = 0.2 x_k + v_k \\ y_k = \beta_k x_k + w_k \end{cases}$$

with x_0 having $E\{x_0\} = \bar{x}_0 = 1$ and covariance $P_0 = 0.01$. The random variables v_k and w_k are zero mean white noise sequences with variance $V = 0.1$ and $W = 0.03$ respectively, β_k is a random sequences having Bernoulli distribution with mean $\bar{\beta} = 0.7$, and variance $\bar{\beta}\left(1-\bar{\beta}\right) = 0.21$. Using the SCOO method, we have estimated the state from the measurements and compared it to the GKF method. The experiment is computer simulated. The (ensemble) average state estimation error and the average (over time) MSE (AMSE) over 50 runs are computed and are given in Figure 1 and Table 1 (left portion) respectively. The AMSE of the SCOO is less than that of the GKF (0.0497 vs. 0.1037) as predicted by theory. The results show the effectiveness of the SCOO technique and illustrate what was stated in Theorem 1.

6.2 Example 2: Gaussian Case

We have applied the SCOO to the following system:

$$\begin{cases} x_{k+1} = 0.2 \alpha_k x_k + v_k \\ y_k = \gamma_k x_k + w_k \end{cases}$$

with x_0 having $E\{x_0\} = \bar{x}_0 = 1$ and covariance $P_0 = 0.01$. The random variables v_k and w_k are zero mean white noise sequences with variance $V = 0.1$ and $W = 0.03$ respectively, α_k and γ_k are mutually uncorrelated white sequences having Gaussian distribution with $N(1, 0.001)$. Using the SCOO method, we have estimated the state from the measurements and compared it to the GKF method. The experiment is computer simulated. The (ensemble) average state estimation error and the AMSE over 50 runs are computed and are given in Figure 2 and Table 1 (right portion) respectively. The AMSE of the SCOO is much smaller than that of GKF (0.0239 vs. 0.0985). These observations are in line with what we have already seen in example 1 and what is predicted by theory. This clearly justifies the advantage of the SCOO.

7. CONCLUSIONS

In this work, linear discrete-time systems with white stochastic parameters are considered. A current output observer is developed for state estimation of such systems. It is theoretically proven and shown by simulation studies that the current output solution can lead to smaller estimation error covariance for such systems. Examples have been provided for Gaussian as well as non-Gaussian stochastic parameters. Since the improvement in the estimation error performance is obtained with slightly higher computational load, these results will potentially benefit estimation for systems with random sensor failures.

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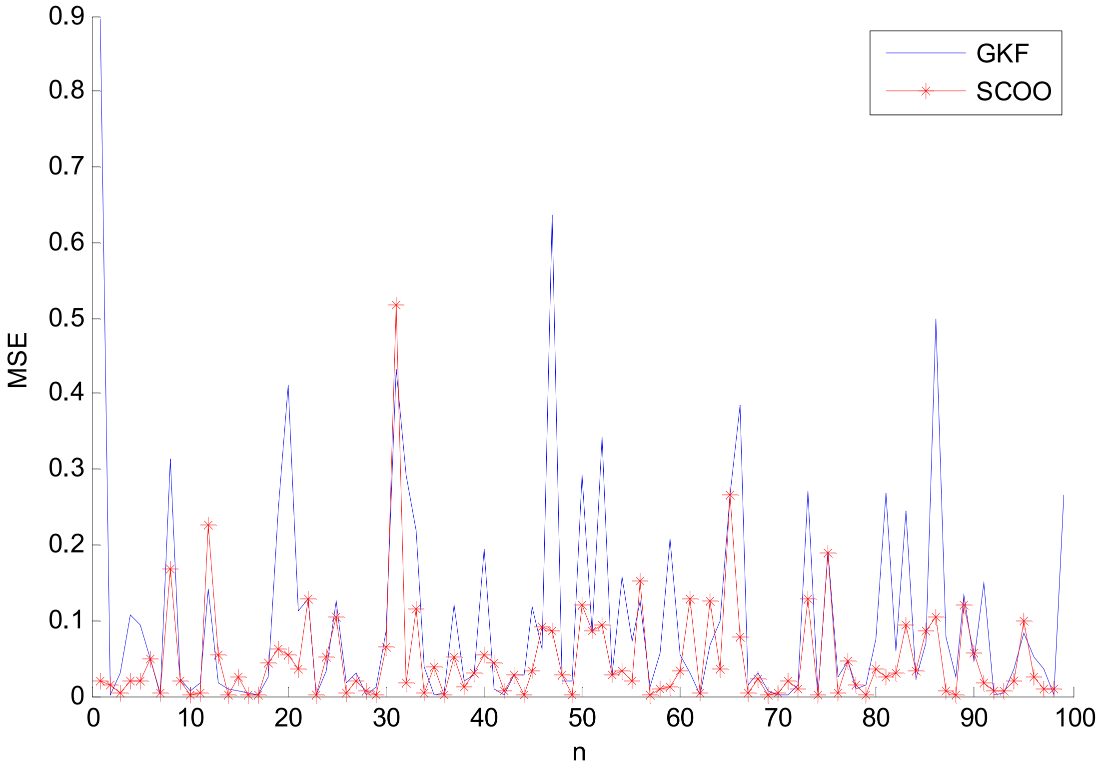

Figure 1: MSE Comparison between the GKF and the SCOO - Non-Gaussian parameters

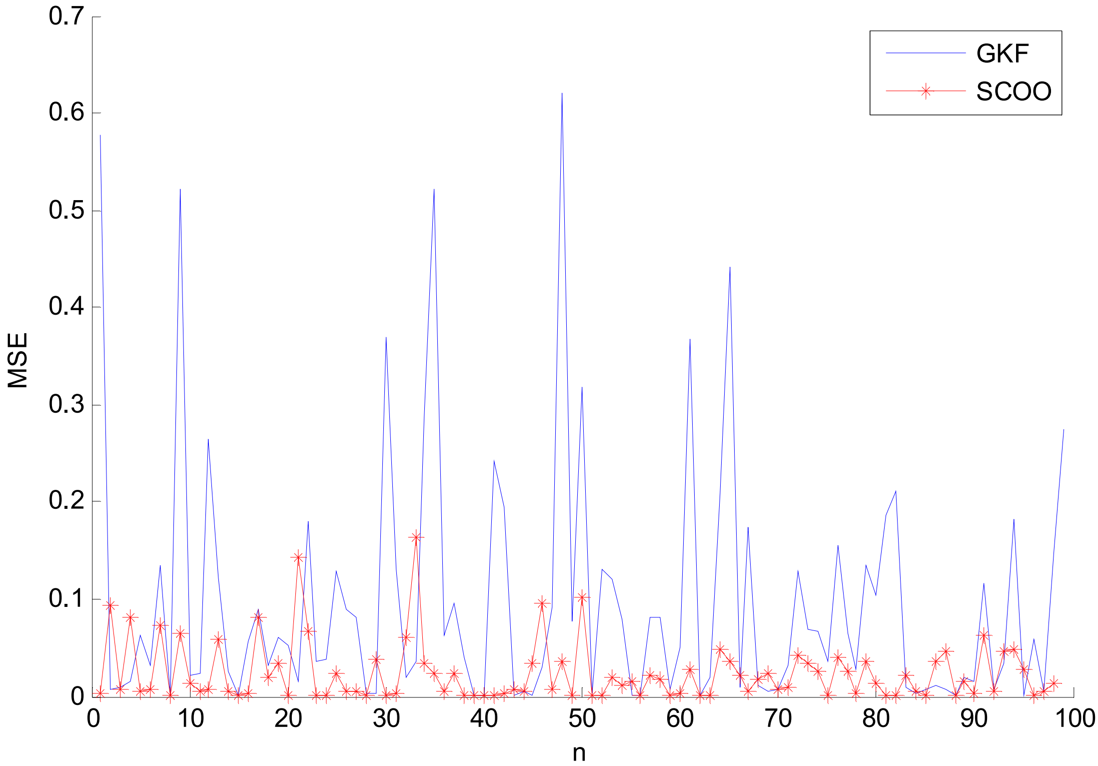

Figure 2: MSE Comparison between the GKF and the SCOO – Gaussian parameters

Example 1		Example 2	
Method	AMSE	Method	AMSE
GKF	0.1037	GKF	0.0985
SCOO	0.0497	SCOO	0.0239

Table 1: The comparison of the AMSE between the different methods used. Left: example 1 results, right: example 2 results