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Effect Output Noise in Iterative Learning Control

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Abstract: In this paper Iterative Learning Control(ILC) algorithm is analysed for a linear-time invariant SISO model with the effect of output noise and its properties derived. If the original plant is positive, it is shown that by using a fixed learning gain algorithm, the tracking error will converge and be predicted. Finally, through computational experiments, we confirm the correctness of the proposed properties.

1. INTRODUCTION

Iterative Learning Control (ILC) is an open-loop control technique to sequentially improve the control accuracy by performing a given task iteratively. Examples of such systems are robot manipulators that are required to repeat a given task with high precision, chemical batch processes, or, more generally, the class of repetitive systems. To achieve high-quality feedforward controls, researchers are focused on finding out the perfect control theory. Iterative Learning Control (ILC), an iterative update scheme which improves the quality of the feedforward signal from trial to trial, is applied to reduce the tracking error in repeated systems. As a means of controlling a class of uncertain dynamic systems, the method of Iterative Learning Control has been widely studied and used since its introduction.

The original ideal of using the previous information was introduced by Uchiyama (Uchiyama, 1978) in order to improve the performance of robot systems. However, the first step on to analyse the learning control and how the learning control scheme makes progress or improves the performance in repetition of the trials rigorous treatments of learning control was made simultaneously and independently by Arimoto et al. (1984), Casalino and Bartolini (1984), and Craig (1984) in around 1984. After that the ideal of ILC are developed in many papers, see [1-5].

This paper introduces the idea of ILC algorithm for discrete-time systems with the effect output noise and analyses the behaviour of this algorithm.

2. PROBLEM STATEMENT

As a starting point consider the following discrete-time, linear, time-invariant SISO system described by:

$$\left.\begin{array}{c} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array}\right\} \tag{1}$$

where initial state $x(0) = x_0$, $x(t) \in R^n$, $u(t) \in R$ and $y(t) \in R$ denote the state, input and output respectively. A, B and C in the state-space function (1) are the matrices with appropriate dimensions and it is assumed that CB is nonsingular and the system is controllable and observable. The time interval is finite from 0 to N, $t \in [0, N]$ (in order to simplify notation it is assumed that the sampling interval, t_s is unity). In addition, the reference signal is necessary and given a priority over that time with duration $[0, N]$.

The special feature of the ILC is that after the system has run over the time interval $[0, N]$, the system (1) is reset to its initial status. Then the system is required to repeat the same motion again. This repetition is the significant attribute of ILC. And it gives the system the possibilities to modify the next trial input $u(t)$ so that as the number of repetitions increases, the output $y(t)$ tracks the reference signal $r(t)$ more and more accurately. To be more precise, the main idea of ILC design is to find a recursive control law

$$u_{k+1} = F(e_{k+1}(t), e_k(t), u_k(t)) \tag{2}$$

So that

$$\lim_{k\to\infty} \parallel e_k \parallel = 0 \quad \text{and} \quad \lim_{k\to\infty} \parallel u_k - u^* \parallel = 0 \tag{3}$$

where u^* is the input that results in perfect tracking. In order to analyse the system more clearly, the system can be represented in the matrix equation (4) equivalently because it is defined over a finite time-interval.

$$y_k = G_e u_k \tag{4}$$

where

$$G_e = \begin{bmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & \dots & 0 \\ CAB & CB & \dots & 0 \\ . & . & . & \dots \\ . & . & . & \dots \\ CA^{N-1}B & CA^{N-2}B & \dots & 0 \end{bmatrix} \tag{5}$$

$$\left.\begin{array}{c} y_k = [y_k(0), y_k(1), y_k(2), \cdots, y_k(N)]^T \\ u_k = [u_k(0), u_k(1), u_k(2), \cdots, u_k(N)]^T \end{array}\right\} \tag{6}$$

and the dimension of G_e is $(N+1) \times (N+1)$.

Assume the reference signal $r(t)$ satisfies $r(0) = Cx_0$. Then for analysis, the matrix equation (4) is adjusted by the lifting technique as (in a similar manner to [6])

$$y_{k,l} = G_{e,l} u_{k,l} \tag{7}$$

note that

$$G_{e,l} = \begin{bmatrix} CB & 0 & \dots & 0 \\ CAB & CB & \dots & 0 \\ CA^2B & CAB & \dots & 0 \\ \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \dots \\ CA^{N-1}B & CA^{N-2}B & \dots & CB \end{bmatrix} \tag{8}$$

$$\left.\begin{aligned} y_{k,l} &= [y_k(1), y_k(2), y_k(3), \cdots, y_k(N)]^T \\ u_{k,l} &= [u_k(0), u_k(1), u_k(2), \cdots, u_k(N-1)]^T \end{aligned}\right\} \tag{9}$$

and the dimension of $G_{e,l}$ is $N \times N$.

Because it is assumed that CB is nonsingular, the matrix $G_{e,l}$ is invertible, there exists a u^* which satisfies $r = G_{e,l}u^*$ for an arbitrary reference $r(t)$. If $CB = 0$ and the first non-zero Markov parameter $CA^{m-1}B$ is where m is the relative degree of the plant, any matrix representation of the dynamical system can be generated by the same lifting technique.

3. THE ILC ALGORITHM WITH EFFECT OUTPUT NOISE

3.1 Derivation of the ILC Algorithm

This paper concentrates on the system with the effect output noise written in the form

$$y_k = G_e u_k + H_e n_k \tag{10}$$

where n_k is white noise which has the same variance in each trial and H_e is a filter. Now consider the following ILC control law suggested in Togai and Yamano (1985)

$$u_{k+1} = u_k + \beta e_k \tag{11}$$

where β is a learning gain introduced to add flexibility and influence performance. In addition, assume that the procedure is initiated with the choice of an arbitrary initial control time series u_0 leading to an initial error e_0.

More precisely, computation of the tracking error gives

$$\left.\begin{aligned} e_{k+1} &= r - y_{k+1} \\ e_{1,k+1} &= e_{k+1} + H_e n_{k+1} \end{aligned}\right\} \tag{12}$$

where $e_{1,k+1}$ is the noise free tracking error at the $(k{+}1)^{th}$ iteration. Using the control law (11) and the tracking error equation (12), the error evolution equations are given by

$$\left.\begin{aligned} e_{k+1} &= (I - \beta G_e)e_k + H_e(n_k - n_{k+1}) \\ e_{1,k+1} &= (I - \beta G_e)e_{1,k} + \beta G_e H_e n_k \end{aligned}\right\} \tag{13}$$

If the radius of $(I - \beta G_e)$ lies in the circle, then the radius error converges to zero. In the follows, the effect of noise is analysed.

3.2 Properties

Proposition 1: *The expectation of $\| e_{1,k+1} \|^2$ at the $(k{+}1)^{th}$ iteration satisfies*

$$E(\| e_{1,k+1} \|^2) = \| (I - \beta G_e)^{k+1} e_{1,0} \|^2 + \delta^2 A_k \tag{14}$$

where δ^2 is the variance of the white noise and A_k is an amplification factor.

$$A_k = \beta^2 \sum_{j=0}^{k} Tr((I - \beta G_e)^j G_e H_e H_e^T G_e^T ((I - \beta G_e)^j)^T) \tag{15}$$

Note $Tr(A)$ is the trace of an $n \times n$ square matrix A. It is defined to be

$$Tr(A) \equiv \sum_{i=1}^{n} a_{ii} \tag{16}$$

i.e. the sum of the diagonal elements.

Proof: Recall the error evolution equation

$$e_{1,k+1} = (I - \beta G_e)e_{1,k} + \beta G_e H_e n_k \tag{17}$$

Applying induction on (17), it results in

$$\begin{aligned} e_{1,k+1} = &(I - \beta G_e)^{k+1} e_{1,0} + \beta G_e (I - \beta G_e)^k H_e n_0 \\ &+ \beta G_e (I - \beta G_e)^{k-1} H_e n_1 + \cdots + \beta G_e H_e n_k \end{aligned} \tag{18}$$

The norm of $e_{1,k}$ has the form

$$\begin{aligned} \| e_{1,k+1} \|^2 = &\| (I - \beta G_e)^{k+1} e_{1,0} + \beta G_e (I - \beta G_e)^k H_e n_0 \\ &+ \beta G_e (I - \beta G_e)^{k-1} H_e n_1 + \cdots + \beta G_e H_e n_k \|^2 \end{aligned} \tag{19}$$

Before taking the next step, we should understand the following definitions

1) $n_i^T n_j = 0$ if $i \neq j$ where n is white noise.

2) $n_i n_i^T = \delta^2 I$ where I is the identity matrix.

3) $Tr(A) = Tr(A^T)$

Then we have the expectation of formula (20)

$$\begin{aligned} E(\| e_{1,k+1} \|^2) = &\| (I - \beta G_e)^{k+1} e_{1,0} \|^2 + \\ &\delta^2 \beta^2 \sum_{j=0}^{k} Tr((I - \beta G_e)^j G_e H_e H_e^T G_e^T ((I - \beta G_e)^j)^T) \end{aligned} \tag{20}$$

This completes the proof. The formula (14) gives the information about the limit of $E(\| e_{1,k+1} \|^2)$. The details are developed below.

Proposition 2: *The expectation of $\| e_{1,k+1} \|^2$ goes to $\delta^2 A_\infty$ when k goes to infinity if the learning gain β satisfies the inequality $| 1 - \beta CB | < 1$*

$$\lim_{k \to \infty} E(\| e_{1,k+1} \|^2) = \delta^2 A_\infty \tag{21}$$

where A_∞ is the amplification factor A_k when k goes to infinity.

Proof: Proposition 1 shows that $\| (I - \beta G_e)^{k+1} e_{1,0} \|^2$ in the formula (14) is the normal ILC algorithm. It converges to zero monotonically, i.e.

$$\lim_{k \to \infty} \| (I - \beta G_e)^{k+1} e_{1,0} \|^2 = 0 \tag{22}$$

if the gain β satisfies the inequality $\mid 1 - \beta CB \mid < 1$. Consequently $\lim_{k\to\infty} E(\| e_{1,k+1} \|^2) = \delta^2 A_\infty$ which proves the proposition.

Remark: Several similar results hold for the control signal sequence.

1. $E(\| u_{k+1} - u^* \|^2)$ at $(k+1)^{th}$ iteration is equal to $\| G_e^{-1}(I - \beta G_e)^{k+1} e_0 \|^2 + \delta^2 B_k$.

2. $\lim_{k\to\infty} E(\| u_{k+1} - u^* \|^2) = \delta^2 B_\infty$

where B_k is an amplification factor with the form

$$B_k = \beta^2 \sum_{j=0}^{k} Tr((I - \beta G_e)^j H_e H_e^T ((I - \beta G_e)^j)^T) \quad (23)$$

4. NUMERICAL EXAMPLE

To demonstrate the correctness of ILC algorithm, consider a plant having the transfer function

$$G(s) = \frac{s+1}{s^2 + 5s + 6}, \quad H(z) = 1 \quad (24)$$

Using the sample time 0.01 second and Zero-order hold method for discretisation, the corresponding discrete-time system becomes

$$G(z) = \frac{0.0098z - 0.0097}{z^2 - 1.951z + 0.9512} \quad (25)$$

Assume the reference signal $r = sin(t)$, $\beta = 0.5$, the white noise variance δ^2 is 0.0052, and Noise-to-signal ratio is 10%.

The results in Figure 1 show the important facts starts as Proposition 1 and 2.

a. The blue line and red line are $log_{10} \| e_{1,k+1} \|^2$ and $log_{10} E(\| e_{1,k+1} \|^2)$, respectively. The results confirm the theoretical proposition that $E(\| e_{1,k+1} \|^2)$ is the good prediction of $\| e_{1,k+1} \|^2$ for the given noise sequence.

b. In Proposition 2, $\delta^2 A_k$ (Green line) is the limit of $E(\| e_{1,k+1} \|^2)$ when k goes to infinity. In this special case it clearly shows after 100 iterations the red and green lines have reached the same level, i.e.

$$\lim_{k\to\infty} E(\| e_{1,k+1} \|^2) = \delta^2 A_\infty \approx 0.023 \quad (26)$$

c. The expression of $E(\| e_{1,k+1} \|^2)$ has two terms. Note that the first term (Black line)is the normal ILC tracking error evolution equation. In the normal ILC the tracking error goes to zero monotonically when k goes to infinity if $\mid 1 - \beta CB \mid < 1$. In Figure 1, the black line consists with this point clearly. The second term of the expression (14) is the prediction for the given noise sequence. Another fact in this figure is that the blue line represented $log_{10} \| e_{1,k+1} \|^2$ diverge from the black line after 50 iterations and converge with $\log_{10} \delta^2 A_k$.

The information form Figure 2 conform the correctness of Proposition 2. In this simulation, the red line represented $\log_{10} E(\| u_{k+1} - u^* \|^2)$ shows it is a good prediction of $\log_{10} \| u_{k+1} - u^* \|^2$. The limit of $E(\| u_{k+1} - u^* \|^2)$ is equal to $\delta^2 B_\infty$ which is about 90.5107.

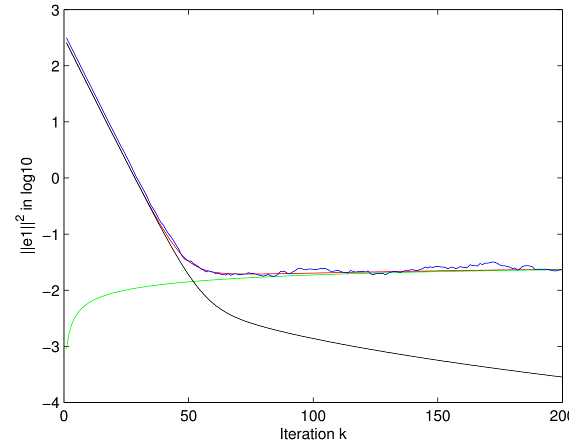

Figure 1: $\| e_{1,k} \|^2$ and $E(\| e_{1,k+1} \|^2)$
Blue: $\log_{10} \| e_{1,k} \|^2$
Red: $\log_{10} E(\| e_{1,k} \|^2)$
Green: $\log_{10} \delta^2 A_k$
Black: $\log_{10} \| (I - \beta G_e)^{k+1} e_{1,0} \|^2$

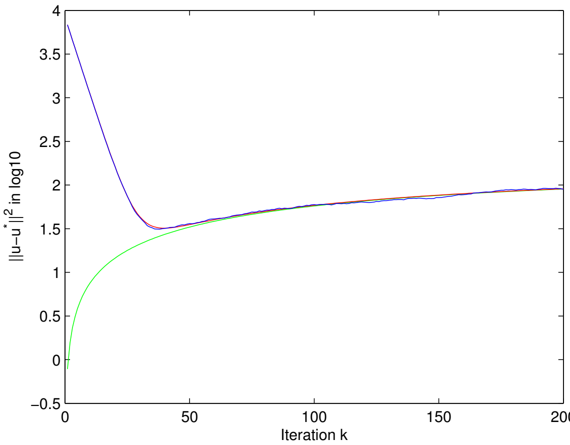

Figure 2: $\| u_{k+1} - u^* \|^2$ and $E(\| u_{k+1} - u^* \|^2)$
Blue: $\log_{10} \| u_{k+1} - u^* \|^2$
Red: $\log_{10} E(\| u_{k+1} - u^* \|^2)$
Green: $\log_{10} \delta^2 B_k$

5. CONCLUSION AND FUTURE WORK

In this paper the effect of noise at a single ILC algorithm is analysed. The ideal applied with the ILC control law $u_{k+1} = u_k + \beta e_k$ by requires the white noise at the output is proposed. The theoretical analysis of the algorithm shows that $\| e_{1,k} \|^2$ has a good prediction with the form $E(\| e_{1,k} \|^2)$ and this prediction has the limit when k goes to zero. Then we can predict the limit of $\| e_{1,k} \|^2$ by applied with the expectation.

Simulations were used to illustrate the theoretical findings in this paper. In the simulation examples both norm of the noise free tracking error and the expectation of $\| e_{1,k} \|^2$ were tested, and at least with these simulation examples the expectation seemed to result in better prediction. The simulation results also suggested the limit of the $E(\| e_{1,k} \|^2)$.

As a future work, POILC will be implemented as a new paradigm to solve the ILC problem when the original plant has the effect output noise. All the theories in this paper are based on the system positivity condition. In the future, the plant positivity condition will be removed and the high-order version of the algorithm will be analysed. Progress will be reported separately.

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