Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

ROBUST POLE PLACEMENT VIA REFLECTION AXES POLYTOPES

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Abstract: A robust version of the output controller design for discrete-time systems is introduced. Instead of a single stable point a stable polytope is preselected in the closed loop characteristic polynomial coefficient space. A constructive procedure for generating stable polytopes is given starting from the unit hypercube of reflection coefficients of monic polynomials. This procedure is quite straightforward because for a special family of polynomials the linear cover of so-called reflection vectors is stable. The roots placement of reflection vectors is studied. A stability measure in a polytope is introduced in order to solve the problem by quadratic programming approach.

Keywords: robust control, pole placement, stability, discrete-time systems.

1. INTRODUCTION

The modal control or pole placement method is a common approach for designing closed-loop controllers in order to meet desired control specifications. If the model uncertainty is large some robust formulation of the problem is needed, such as multimodel (Ackermann, 1993; Magni, 2002) , polytopic model (Jetto, 2003; Rotstein *et al.*, 1991) or LMI approach (Scherer *et al.*, 1997).

In this paper a polytopic approach proposed in (Nurges, 2006b) is further developed. First, the methodology for generating stable polytopes via so-called reflection vectors (Nurges, 2005; Nurges, 2006b) is developed in more details. The notion of reflection axes of a polynomial is introduced as straight lines in the space of polynomial coefficients in directions of variation of single reflection coefficients. A necessary and sufficient stability condition is obtained as a polytope with vertices on reflection axes of a class of stable polynomials.

Second, in order to choose the best polytope for robust controller design the proposed family of polytopes is studied in respect of their poles placement.

Third, the robust output controller design task for polytopic plant model is solved by quadratic programming approach. It means, instead of a simplex of reflection vectors (Nurges, 2006b) we are using now a full polytope of reflection vectors.

The paper is organized as follows. In section 2 the problem of fixed order robust output control with a preselected polytope is stated and solved by quadratic programming approach. The third section is devoted to the the stable reflection axes polytopes building. In the fourth section some thumb rules are given for possible choices of reflection axes polytopes.

[1] Partially supported by the Estonian Science Foundation grant No. 6837

2. FIXED ORDER POLE ASSIGNMENT

Assume that a plant with parametric uncertainties is given. Our goal is to design an output controller of a fixed order so that the closed-loop poles are robustly assigned in a specific region.

For simplicity, let us first consider the problem of output controller design for a SISO plant with fixed parameters. Let the plant transfer function $G(z)$ of dynamic order m be given

$$G(z) = \frac{g(z)}{f(z)} = \\ = \frac{g_{m-1}z^{m-1} + \cdots + g_1 z + g_0}{z^m + f_{m-1}z^{m-1} + \cdots + f_1 z + f_0}$$

and we are looking for a controller $C(z)$ of dynamic order l with the transfer function

$$C(z) = \frac{q(z)}{p(z)} = \frac{q_l z^l + \cdots + q_1 z + q_0}{z^l + \cdots + p_1 z + p_0}.$$

It means that the closed loop characteristic polynomial

$$a(z) = f(z)p(z) + g(z)q(z)$$

is of degree $n = m + l$.

It is known in the literature (Keel and Bhattacharyya, 1999) that when $l = m - 1$ the above problem has a solution for arbitrary $a(z)$ whenever the plant has no common pole-zero pairs. In general for $l < m - 1$ exact attainment of the desired polynomial is impossible. Here we suggest the following approach.

Let us relax the requirement of attaining the desired polynomial $a(z)$ exactly and enlarge the target to a polytope $\mathcal{S}$ in polynomial coefficient space containing the point representing the desired closed-loop characteristic polynomial. Without any restrictions we can assume that $f_m = p_l = 1$ and deal in the following with monic polynomials $a(z)$.

Let us now introduce a stability measure ρ in accordance with the polytope $\mathcal{S}$

$$\rho = c^T c$$

where

$$Sc = a \tag{1}$$

and S is the nx$2n$ matrix of vertices of the target polytope $\mathcal{S}$. If all coefficients $c_i > 0$, $i = 1, ..., 2n$ and

$$\sum_{i=1}^{2n} c_i = 1$$

then the point a is placed inside the polytope $\mathcal{S}$.

It is easy to see that the minimum $\rho_{min} = \frac{1}{2n}$ is obtained by

$$c_1 = c_2 = ... = c_{2n} = \frac{1}{2n}.$$

Then the point a is placed in the center of the polytope $\mathcal{S}$. If some $c_j = 1$ then $\rho_{max} = 1$ is obtained. Then the point a coincides with the vertex s_j of the polytope $\mathcal{S}$. By minimizing the measure ρ ,

$$1/2n \ll \rho \ll 1$$

we guarantee, first, that the point a is placed in the polytope $\mathcal{S}$ and, second, as far as possible from all the vertices s_j , $j = 1, ..., 2n$.

Now we can formulate the following problem of controller design : find a controller $C(z)$ such that the stability measure ρ is minimal. In other words, we are looking for a controller which places the closed-loop characteristic polynomial $a(z)$ as close as possible to the center of the target polytope $\mathcal{S}$.

In matrix form we have

$$a = Gx \tag{2}$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} f_0 & 0 & ... & 0 & g_0 & 0 & ... & 0 \\ f_1 & f_0 & ... & 0 & g_1 & g_0 & ... & 0 \\ . & . & . & . & . & . & . & . \\ f_{m-1} & f_{m-2} & ... & f_{m-l-1} & g_{m-1} & g_{m-2} & ... & g_{m-l-1} \\ 1 & f_{m-1} & ... & f_{m-l} & 0 & g_{m-1} & ... & g_{m-l} \\ 0 & 1 & . & . & . & . & . & . \\ 0 & 0 & ... & f_{m-1} & 0 & 0 & ... & g_{m-1} \\ 0 & 0 & ... & 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

of dimensions $(m + l + 1)$ x $(2l + 2)$ and x is a $(2l + 2)$-vector of controller parameters $x = [p_0, ..., p_{l-1}, 1, q_0, ..., q_l]^T$.

The above controller design problem is due to relations (1) and (2) equivalent to the quadratic programming problem: find $\tilde{x}$ such that the minimum

$$J_1 = \min_{\tilde{x}} \tilde{x}^T \tilde{x} \tag{3}$$

is obtained subject to the linear restrictions

$$\begin{bmatrix} G & \vdots & -S \\ .. & . & .. \\ 0^T & \vdots & 1^T \end{bmatrix} \tilde{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$[0 \vdots I_{2n}]\tilde{x} \geq 0$$

where $\tilde{x}^T = [x^T \vdots c^T]$ is a $2(n + l + 1)$-vector.

Let us consider now the problem of fixed-order output controller design where the plant is subject to parameter uncertainty. We represent this by supposing that the given plant transfer function coefficients $f_0, ..., f_{m-1}$ and $g_0, ..., g_{m-1}$ are placed in a polytope $\mathcal{W}$ with vertices $d^j = [f_0^j, ..., f_{m-1}^j, g_0^j, ..., g_{m-1}^j]$, $j = 1, ..., M$

$$\mathcal{W} = conv\{d^j, j = 1, ..., M\}.$$

Because the relations (2) are linear in plant parameters we can claim that for an arbitrary fixed controller x the vector a of closed-loop characteristic polynomial coefficients is placed in a polytope $\mathcal{A}$ with vertices $a^1, ..., a^M$

$$\mathcal{A} = conv\{a^j, j = 1, ..., M\}$$

where

$$a^j = D^j x$$

and D^j is a $(m + l + 1)$x$(2l + 2)$ Sylvester matrix composed by the vertex plant d^j as in the case of exact model (2).

The problem of robust controller design can be formulated as follows : find a controller x such that all vertices a^j , $j = 1, ..., M$ are placed inside a stable target polytope $\mathcal{S}$.

This problem can be solved by quadratic programming task : find $\bar{x}$ which minimizes

$$J = \min_{\bar{x}} \bar{x}^T \bar{x} \tag{4}$$

by linear restrictions

$$\begin{bmatrix} \bar{G}^1 & -\bar{S} & 0 & 0 & ... & 0 \\ \bar{G}^2 & 0 & -\bar{S} & 0 & ... & 0 \\ .. & .. & .. & .. & .. & .. \\ \bar{G}^M & 0 & 0 & ... & 0 & -\bar{S} \end{bmatrix} \bar{x} = \begin{bmatrix} o \\ o \\ ... \\ o \end{bmatrix}, \tag{5}$$

$$[0 \vdots I_{2nM}]\bar{x} \geq 0 \tag{6}$$

where $\bar{G}^j = \begin{bmatrix} G^j \\ 0^T \end{bmatrix}$, $\bar{S} = \begin{bmatrix} S \\ 1^T \end{bmatrix}$, $o = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Here

$$\bar{x}^T = [x^T \vdots c_1^T ... c_M^T]$$

is a $2(nM + l)$-vector.

In the following section a method for the convex inner approximation of the stability region will be developed via so-called reflection coefficients of polynomials (see also (Nurges, 2005)). This method can be used to find a stable target polytope in order to solve the robust output pole assignment problem (4)-(6).

3. STABILITY REGION AND REFLECTION AXES

Polynomials are usually defined in terms of their coefficients or their roots. They can also be characterized by their reflection coefficients using Schur-Cohn type recursion (Diaz-Barrero *et al.*, 2004).

Let $a^{(n)}(z)$ be a monic polynomial of degree n with real coefficients $a_i \in \mathcal{R}$, $i = 0, ..., n$,

$$a^{(n)}(z) = z^n + ... + a_1 z + a_0.$$

The reciprocal polynomial $a^{(n)*}(z)$ of $a^{(n)}(z)$ is defined by

$$a^{(n)*}(z) = a_0 z^n + ... + a_{n-1} z + 1.$$

The reflection coefficients k_i , $i = 1, ..., n$ can be obtained from $a^{(n)}(z)$ by using backward Levinson's recursion (Diaz-Barrero *et al.*, 2004)

$$z a^{(i-1)}(z) = \frac{1}{1 - |k_i^2|}[a^{(i)}(z) - k_i a^{(i)*}(z)] \tag{7}$$

where $k_i = -a_0^{(i)}$ and $a_0^{(i)}$ denotes the last coefficient of an i-degree polynomial $a^{(i)}(z)$. From (7) the forward recursion can be obtained

$$a^{(i)}(z) = z a^{(i-1)}(z) + k_i a^{(i-1)*}(z). \tag{8}$$

The stability criterion via reflection coefficient is as follows (Diaz-Barrero *et al.*, 2004).

Lemma 1. A polynomial $a(z)$ has all its roots inside the unit disk if and only if its reflection coefficients are in the interval $(-1, 1)$, $-1 < k_i < 1, i = 1, ..., n$.

According to relations (8) the coefficient vector $a = (a_0, ..., a_{n-1})^T$ depends multilinearly on the reflection coefficients k_i, $i = 1, ..., n$. So a straight line parallel to a coordinate axis k_i will be transformed to a straight line $\mathcal{A}^i$ in the polynomial coefficient space.

Let us call the straight lines

$$\mathcal{A}^i = a(k| -\infty < k_i < \infty, k_j = const, \\ j \neq i, j = 1, ..., n)$$

reflection axes of the polynomial $a(z)$. Reflection axes of Schur polynomials will be useful for inner approximation of the stability region in the polynomial coefficient space.

The following assertions hold:

1. Through an arbitrary point $a \in \mathcal{R}^n$ n reflection axes $\mathcal{A}^i$, $i = 1, ..., n$ can be drawn.
2. Reflection axes $\mathcal{O}^i$ of the origin coincide with the coordinate axes a_i, $i = 1, ..., n$.

(3) Every reflection axis $\mathcal{A}^i$ of a stable point (Schur polynomial) cuts the stability boundary in two points $v^i(1) = a(k_i = 1)$ and $v^i(-1) = a(k_i = -1)$. These stability boundary points $v^i(\pm 1)$ are called reflection vectors of the Schur polynomial $a(z)$ (Nurges, 2005).
(4) Arbitrary line segments of a reflection axis $\mathcal{A}^i$ will be stable if only its endpoints $v_i^+ = a(\bar{k}^i) \in \mathcal{A}^i$ and $v_i^- = a(\underline{k}^i) \in \mathcal{A}^i$, $\underline{k}^i < \bar{k}^i$,$i = 1, ..., n$ are stable.

In the following the linear cover of reflection vectors $v^i(\pm 1)$, $i = 1, ..., n$ is called the reflection vectors (RV) polytope and the linear cover of reflection axes points v_i^+ and v_i^-, $i = 1, ..., n$ is called the reflection axes (RA) polytope.

Theorem 1(Nurges, 2006a). A reflection axes polytope $\mathcal{V}(a)$ of a stable polynomial $a(z)$ with reflection coefficients $k_2(a) = ... = k_{n-1}(a) = 0$ will be stable if and only if all the vertices $v_i^{+,-}(a)$, $i = 1, ..., n$ are stable and $-1 < k_1(a), k_n(a) < 1$.

4. POSSIBLE CHOICES OF POLYTOPES

In this section we study the stable reflection axes (RA) polytopes as possible candidates for a target polytope .

The following problems have to be solved:

1) choice of an initial polynomial $a(z)$ for generating a stable reflection vectors polytope $\mathcal{V}(a)$,
2) choice of vertices $v_i^+(a)$ and $v_i^-(a)$ on reflection axes.

According to Theorem 1 the initial polynomial $a(z)$ belongs to the family of polynomials with $k_1(a) \in (-1, 1)$, $k_n(a) \in (-1, 1)$ and $k_2(a) = ... = k_{n-1}(a) = 0$. Let us study the roots placement of such polynomials.

1. Let $k_1(a) \in (-1, 1)$ and $k_2(a) = ... = k_n(a) = 0$. Then $a(z) = z^n - k_1 z^{n-1}$ and $r_1 = k_1$, $r_2 = ... = r_n = 0$.
2. Let $k_n(a) \in (-1, 1)$ and $k_1(a) = ... = k_{n-1}(a) = 0$. Then $a(z) = z^n - k_n$ and the roots of $a(z)$ are placed symmetrically against the origin whereas $max|r_i| > k_n$.

Thus, a reasonable choice of an initial polynomial is following: $0 < k_1(a) < 1$, $|k_n(a)| << k_1(a)$, $k_2(a) = ... = k_{n-1}(a) = 0$.

In order to illustrate this statement let us calculate the root loci of innerpoints of some RV polytopes for $n = 3$. The robust root loci, obtained by taking 50 uniformly distributed random points within the RV polytopes around the initial points with $k_1(a) = 0.8; -0.8$, $k_2(a) = k_3(a) = 0$ and with $k_1(a) = k_2(a) = 0$, $k_3(a) = 0.8; -0.8$ are represented in Figures 1 and 2 respectively.

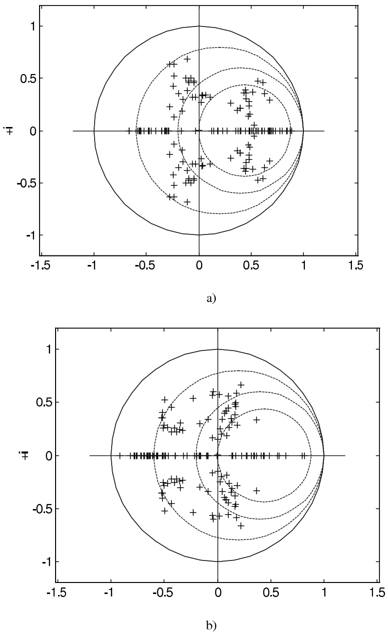

Fig. 1. Robust root locus of the polytope of reflection vectors $\mathcal{V}(a)$ for $k_1(a) = 0.8$ (a) and $k_1(a) = -0.8$ (b)

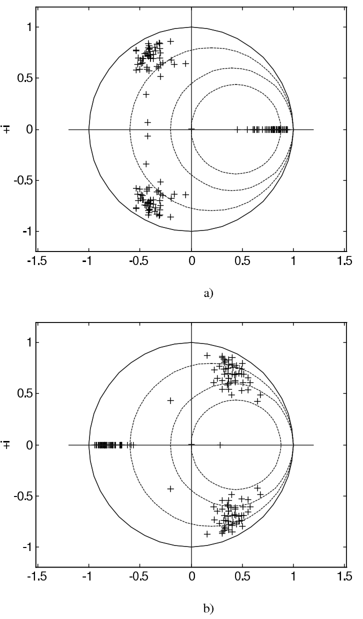

Fig. 2. Robust root locus of the polytope of reflection vectors $\mathcal{V}(a)$ for $k_n(a) = 0.8$ (a) and $k_n(a) = -0.8$ (b)

Now, let us study the problem of possible improvement of the root loci of reflection axes (RA) polytopes by moving the vertices of them nearer to the initial point a along the reflection axes $\mathcal{A}^i(a)$,$i = 1, ..., n$.

For performance purposes it is desirable to choose the vertices $v_i^{+,-}(a)$ of an RA polytope with $|k_i(v_i^{+,-}(a))| = \xi_i^{+,-} < 1$. The reasonable value of $\xi_i^{+,-}$ depends on the reflection axis number i and the sign of its direction.

Theorem 2. (Nurges, 2006b) Reflection vectors $v^i(1)$ and $v^i(-1)$, $i = 1, ..., n$ of a monic Schur polynomial $a(z)$ have i roots r_j, $j = 1, .., i$ on the stability boundary. The numbers of real and complex roots are determined by the sign and the parity of the reflection vector as follows:

1) the positive reflection vector $v^i(1)$ has
   - for i even $r_1 = 1$,
     $r_2 = -1$
     and $(i-2)/2$ pairs
     of complex roots on the unit circle,
   - for i odd $r_1 = 1$,
     and $(i-1)/2$ pairs
     of complex roots on the unit circle,
2) the negative reflection vector $v^i(-1)$ has
   - for i even $i/2$ pairs
     of complex roots on the unit circle,
   - for i odd $r_1 = -1$,
     and $(i-1)/2$ pairs
     of complex roots on the unit circle.

Taking into account the fact that the best stability boundary root in context of performance properties is $r = 1$ and a pair of complex stability boundary roots with a positive real part is a good one too, we can formulate the following thumb rules for choosing constants ξ_i :

(1) $\xi_1^+ > |\xi_2^-| > |\xi_1^-| > \xi_2^+ >> \xi_i^{+,-}, i > 2$,
(2) the greater the number i the smaller must be the value $|\xi_i^{+,-}|$,
(3) for i even $|\xi_i^+| \leq |\xi_i^-|$,
(4) for i odd $|\xi_i^+| \geq |\xi_i^-|$.

Let us consider now a very simple example to explain the main ideas of robust fixed order output controller design via the reflection vectors polytope preselection and quadratic programming task (4)-(6).

Example. Let us have the second order $m = 2$ uncertain plant with transfer function

$$G(z) = \frac{z + (0.6 \pm 0.1)}{z^2 - (0.8 \pm 0.2)z - 0.4}$$

and a proportional $l = 0$ output controller

$$C(z) = q.$$

Because $l < m - 1$ we can not choose an arbitrary closed-loop characteristic polynomial. Indeed,

$$a(z) = z^2 - [(0.8 \pm 0.2) + q]z + [(0.6 \pm 0.1)q - 0.4]$$

or in matrix form

$$a = Gx = \begin{bmatrix} f_0 & g_0 \\ f_1 & g_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q \end{bmatrix}.$$

Let us choose according to the above "thumb rules" the reflection coefficients $k_1^a = 0.2$ and $k_2^a = 0$. Then the generating polynomial $a(z) = z^2 - 0.2z$ has 4 reflection vectors (points $\mathbf{C}, \mathbf{D}, \mathbf{F}, \mathbf{A}$ in Fig.4 respectively)

$$\begin{aligned} v_1^+(a) &= [\ -1 \quad 0\ ], \\ v_1^-(a) &= [\ 1 \quad 0\ ], \\ v_2^+(a) &= [\ 0 \quad -1\ ], \\ v_2^-(a) &= [\ -0.4 \quad 1\ ]. \end{aligned}$$

and the 4 vertices of the target polytope $\mathcal{S} = conv(\mathbf{A}, \mathbf{C}, \mathbf{D}, \mathbf{F})$ are $\mathbf{C} = v_1^+(a)$, $\mathbf{A} = v_2^-(a)$ and $\mathbf{f} = v_1^-(a)$, $\mathbf{D} = v_2^+(a)$). So the matrix S of vertex polynomial coefficients is as follows

$$S = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & -0.4 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

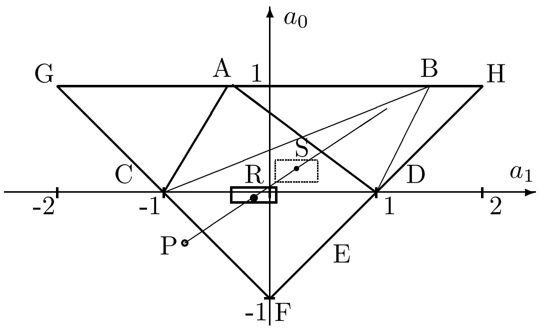

Fig. 3. Target polytopes for robust controller design.

Let us now solve the output controller design task for the nominal plant with $g_0 = 0.6$ and $f_1 = -0.8$ (point $\mathbf{P}$). Via quadratic programming with the optimization criterion J_1 we obtain the controller gain

$$q^1 = 0.6981$$

and the closed-loop characteristic polynomial (point $\mathbf{R}$)

$$a^1(z) = z^2 - 0.2595z - 0.0757.$$

For polytopic plant $g_0 = 0.6 \pm 0.1$ and $f_1 = -0.8 \pm 0.2$ with the optimization criterion J we obtain the controller gain

$$q^2 = 0.6776,$$

and the vertices of the rectangle of closed-loop characteristic polynomials (solid rectangle around $\mathbf{R}$)

$$\begin{aligned} a^{21}(z) &= z^2 + 0.0776z + 0.0744 \\ a^{22}(z) &= z^2 + 0.0776z - 0.0612 \\ a^{23}(z) &= z^2 - 0.3224z + 0.0744 \\ a^{24}(z) &= z^2 - 0.3224z - 0.0612. \end{aligned}$$

To illustrate the effect of the choice of reflection coefficients of the initial polynomial $a(z)$ we have solved the same task with $k_1^a = -0.8$. Then the target polytope $\mathcal{S}$ is the quadrangle **BCFD** where $\mathbf{B} = v_2^-(a) = [\,1\ 1.6\ 1\,]^T$. The optimal controller gain for the polytopic plant is

$$q^3 = 1.0117,$$

and the closed loop polytope (dotted rectangle around **S**)

$$\begin{aligned} a^{31}(z) &= z^2 + 0.4117z + 0.3082 \\ a^{32}(z) &= z^2 + 0.4117z + 0.1058 \\ a^{43}(z) &= z^2 + 0.0117z + 0.3082 \\ a^{34}(z) &= z^2 + 0.0117z + 0.1058. \end{aligned}$$

The root loci confirm that the initial polynomial $a_1(z)$ with $k_1 = 0.2$ is more suitable for generating a target simplex than $a_2(z)$ with $k_1 = -0.8$.

5. CONCLUSIONS

A novel procedure for robust output controller design is presented. Instead of a single target point a target polytope is preselected. The problem of robust controller design is then formulated and solved as an optimization task which guarantees the robust stability (polytopic closed loop characteristic polynomial is placed in a stable target polytope) and maximizes a stability margin (closed loop polytope is placed as far as possible from the vertices of this stable target polytope).

A constructive procedure for generating stable polytopes in polynomial coefficients space is given. In this paper instead of a simplex of reflection vectors (see (Nurges, 2006b)) we are using a full polytope of reflection vectors. This approach has some advantages: first, we can deal with greater uncertainty level (the volume of a full polytope of reflection vectors is considerably greater than the volume of a simplex), second, we have more flexibility by robust controller design . But this approach has a drawback too: the dimension of the quadratic programming task is considerably greater ($2(l+1)$ for a simplex (Nurges, 2006b) and $2(n+l+1)$ for a polytope of reflection vectors).

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