Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A Fix-Up for the EKF Parameter Estimator

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Abstract: We have reduced recursive parameter estimation to Kalman filtering, with a few added fixes. By incorporating projections in the parameter gain updates and parameter variance estimates, the recursive maximum likelihood method asymptotically becomes a reformulation and fix-up of the extended Kalman filter used as a parameter estimator (EKFPE), except that an additional n x n linear symmetric matrix must also be updated for each parameter estimate. Estimates for both the process and measurement noise variances, as well as for structural parameters, have been proven globally convergent to a local maximum of the likelihood function. This obviates the usual guesswork in finding noise variances when fitting data using the EKFPE, and assures the existence of the innovations representation for the recursive maximum likelihood method. Slightly non-linear and also slightly unstable linear, as well as drastically time-varying stable linear, system parameters can be estimated even in severe noise environments On average, the rate of convergence of parameter estimates appears to be faster than other methods if no projection limit is hit.

1. INTRODUCTION

By adjoining an np-dimensional, constant-in-time parameter vector $\boldsymbol{\theta}$ to the n-dimensional state vector $\mathbf{x}(t)$, and linearizing the resulting dynamic equations, the Kalman filter can be extended to become an estimator (EKFPE) of the structural parameters (i.e. appearing in only the **A**, **B**, or **C** matrices of the state equation) in a stochastic linear system. EKFPE has been used to estimate parameters for stochastic processes since shortly after the introduction of the Kalman filter in 1960. However, EKFPE (also unscented EKFPE) can not estimate the process noise variance (Wiberg *et al.,* 2000). It is important to incorporate process noise in dynamic models because if only white measurement noise is considered, the residuals may not be white and the resulting parameter estimates may be biased (Ljung, 1987). So, guesses must be made for values of the process and measurement noise variances in EKFPE to fit data records, and iterative searches performed over a range of variance values to achieve acceptable data fits. In the case when all the data is collected offline and stored, the later introduction of maximum likelihood methods for parameter estimation, such as the system ID toolbox in MATLAB, now give a convenient way to avoid guesswork instead of using the EKFPE.

Recursive parameter estimation algorithms are much trickier to implement than offline algorithms. Recursive algorithms are needed for failure detection, adaptive control and other uses. There are many effective recursive parameter algorithms for time-invariant stable linear dynamic systems, including recursive prediction error (Ljung, 1979), subspace (Van Overschee and DeMoor, 1996), particle filtering (Arulampalam et al., 2002), instrumental variables (Sinha and Rao, 1991) and errors-in-variables (Chen, 2007). The fixed up EKFPE (FEKFPE) introduced here appears to converge faster than all of them if no projection limits are hit, and none of them incorporate the parameter gain magnitude limits that are necessary to guarantee convergence.

The state-space representation of EKFPE has led to many difficulties, e.g., parameters not being identifiable. Here we point out another (hitherto unmentioned) difficulty of EKFPE and of some recursive maximum likelihood methods, namely, the non-existence of the innovations representation and consequent non-existence of parameter update gains unless certain projection operations are performed. Incorporation of these projection operations guarantees convergence (as defined later) of another version of the EKFPE also introduced here which we denote FEKFPE2.

Two versions of the fixed up EKFPE are presented. The first version, denoted FEKFPE, is simpler, but proven convergent in Appendix B as yet only for state x dimension equal to one for a continuous-time limit. The second version, denoted FEKFPE2, is derived in Appendix A and is more complicated, but is globally convergent in discrete time for arbitrary state dimension. For the dozen or so examples we have tried, there is little difference between the two versions and we conjecture FEKFPE is convergent for any n-dimensional state vector, $\mathbf{x}$. Therefore emphasis is given to FEKFPE in this paper.

A description of the derivation of FEKFPE2 is given in Appendix A. FEKFPE2 is 3-OM (Wiberg *et al.,* 2000) reformulated to eliminate Kronecker products, which permits efficient use of MATLAB to avoid long computation time. Also a projection operation in 3-OM becomes a limit on the magnitude of the gain of the parameter update plus a requirement to keep the estimate of any parameter error variance positive. These two results are the major contributions of this paper. They show that 3-OM and the recursive maximum likelihood method (RML) of MATLAB are asymptotically equivalent to the EKFPE when fixed up. Furthermore, the importance of limiting the magnitude of the parameter update gains is introduced here.

To explain how FEKFPE comes from FEKFPE2, we need to define some terms. Call structural parameters $\boldsymbol{\theta}_j$ for j = 1, 2,…*np,* as distinguished from parameters appearing in the process noise variance, which are denoted $\boldsymbol{\rho}_i$ for i = 1, 2,…*nq.* For any variable, define the difference between true values and their estimates by the superscript tilde, e.g. the difference between **x** and the estimate for **x**, denoted $\hat{\mathbf{x}}$, is $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$. FEKFPE2 consists of updates of estimates, estimate variances, and estimates of third order conditional moments of $\tilde{\mathbf{x}}, \tilde{\mathbf{x}}, \tilde{\boldsymbol{\theta}}_j$, and $\tilde{\mathbf{x}}, \tilde{\mathbf{x}}, \tilde{\boldsymbol{\rho}}_i$, which are n x n matrices for each j and i. FEKFPE simply eliminates the former matrices from FEKFPE2, but keeps the latter. The elimination of updating the n x n matrices associated with all the structural parameters, and only retaining these updates for those parameters in the process noise variance, is a major simplification. Consequently, recursive parameter estimation becomes as simple as the EKFPE with a slight fix-up.

By inheritance from 3-OM, FEKFPE2 is globally convergent and has fast transient response for asymptotically stable, controllable, and observable linear systems, time-varying or not, whose parameters are identifiable. It is applicable to all nonlinear systems in which the EKF also applies. It is recursive and estimates structural parameters and both measurement and process noise variances. One disadvantage is that now it must be coded by the user for each case, which makes FEKFPE preferable to FEKFPE2. Another disadvantage is that FEKFPE and FEKFPE2 are in state space form, with its inherent dangers. A website is being prepared with code that includes UDU (Bierman, 1977) for use with high-dimensional states. Here, space considerations limit examples to one nonlinear and one unstable case, with others on the website.

2. LINEAR STOCHASTIC MODELS CONSIDERED

The stochastic state n-vector $\mathbf{x}(t)$ takes values at discrete-time t intervals normalized to unity, so t = 0,1,2,… With input m-vector **u**, and output l-vector **y**, the linear state equation is

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(t) + \mathbf{B}(\boldsymbol{\theta})\mathbf{u}(t) + \mathbf{v}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t). \end{aligned} \tag{1}$$

The process white noise is the Gaussian n-vector sequence $\mathbf{v}(t)$, zero-mean, variance $\mathbf{Q}(\boldsymbol{\rho})$ and uncorrelated with initial condition $\mathbf{x}(0)$ and $\mathbf{w}(t)$. The measurement white noise $\mathbf{w}(t$)is a Gaussian l-vector sequence that is zero-mean with variance **R**. **R** is found from the residuals, and not estimated directly.

The problem is to estimate structural parameters $\boldsymbol{\theta}$ and process noise parameters $\boldsymbol{\rho}$ by measuring input and output data **u(t)** and **y(t).** The **A**, **B**, **C, Q** and **R** matrices are time varying, with the time variable suppressed. This is the usual Kalman filtering set-up. Note no parameters $\boldsymbol{\theta}$ enter in the **C** matrix, because any observable system can be arranged into the form (1) by redefinition of state variables. The n x n matrix **A**, n x m matrix **B**, and n x n matrix **Q** are parameterized

$$\mathbf{A}(\boldsymbol{\theta}) = \mathbf{A}_0 + \mathbf{A}_1\boldsymbol{\theta}_1 + ... + \mathbf{A}_{np}\boldsymbol{\theta}_{np}$$

$$\mathbf{B}(\boldsymbol{\theta}) = \mathbf{B}_0 + \mathbf{B}_1\boldsymbol{\theta}_1 + ... + \mathbf{B}_{np}\boldsymbol{\theta}_{np} \tag{2}$$

$$\mathbf{Q}(\boldsymbol{\rho}) = \mathbf{Q}_0 + \mathbf{Q}_1\boldsymbol{\rho}_1 + ... + \mathbf{Q}_{nq}\boldsymbol{\rho}_{nq}. \tag{3}$$

In (2), $\boldsymbol{\theta}$ is constrained to be in a region *DS* of parameter space where ($\mathbf{A}(\boldsymbol{\theta})$, $\mathbf{B}(\boldsymbol{\theta})$*,* **C**) are asymptotically stable, observable, and controllable and $\boldsymbol{\theta}$ is identifiable. Generally, identifiability depends on **u(t), x(0)** and is difficult to relate to the likelihood function. In (3), for i = 1, 2,...*nq,* the n x n matrix $\mathbf{Q}_i$ is symmetric and $\boldsymbol{\rho}$ is constrained to be in the region of parameter space *DN* where $\mathbf{Q}(\boldsymbol{\rho})$ is nonnegative definite and $\boldsymbol{\rho}$ is identifiable. A nonlinear parameter, not in the form of (2) or (3) can be redefined as a set of different linear parameters and estimated by slightly modifying FEKFPE as in Wiberg et al. (2000) to retain the correlation between parameters. For example, the equation $(1/b)\mathbf{x}(t+1) + (\sin b)\mathbf{x}(t) = \mathbf{u}(t)$ can be rewritten as $\mathbf{x}(t+1) = a\mathbf{x}(t) + b\mathbf{u}(t)$ with a and b correlated as $a = -b\sin b$. The linear model can also be extended to unstable and nonlinear models for which the EKF works, as in the Examples section.

3. EKF PARAMETER ESTIMATOR

The extended Kalman Filter (EKF) estimates the $\boldsymbol{\theta}$ parameters by adjoining $\boldsymbol{\theta}$ to **x** as $\xi = col\,(\mathbf{x}, \boldsymbol{\theta})$, so ξ is an $(n+np)$-vector. Since $\boldsymbol{\theta}$ is constant, $\boldsymbol{\theta}(t+1) = \boldsymbol{\theta}(t)$. Together with (1), this gives an equation for ξ, for which the EKF can be found from (Anderson and Moore, 1979) to be, after partition and for j = 1,2,…*np,* the measurement update:

$$\begin{aligned} \hat{\mathbf{x}}(t\,|\,t) &= \hat{\mathbf{x}}(t\,|\,t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t) \\ \hat{\boldsymbol{\theta}}_j(t\,|\,t) &= \hat{\boldsymbol{\theta}}_j(t\,|\,t-1) + \mathbf{k}_j^{\mathrm{T}}(t\,|\,t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t) \\ \mathbf{P}(t\,|\,t) &= [\mathbf{I} - \mathbf{K}(t)\mathbf{C}]\mathbf{P}(t\,|\,t-1) \\ \mathbf{k}_j(t\,|\,t) &= [\mathbf{I} - \mathbf{K}(t)\mathbf{C}]\mathbf{k}_j(t\,|\,t-1) \\ \boldsymbol{\sigma}_j(t\,|\,t) &= \boldsymbol{\sigma}_j(t\,|\,t-1) - \\ &\quad - \mathbf{k}_j^{\mathrm{T}}(t\,|\,t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\mathbf{C}\mathbf{k}_j(t\,|\,t-1), \end{aligned}$$

and the time update:

$$\hat{\mathbf{x}}(t+1\,|\,t) = \hat{\mathbf{A}}(t)\hat{\mathbf{x}}(t\,|\,t) + \hat{\mathbf{B}}(t)\mathbf{u}(t) \tag{4}$$

$$\hat{\boldsymbol{\theta}}_j(t+1\,|\,t) = \hat{\boldsymbol{\theta}}_j(t\,|\,t)$$

$$\mathbf{P}(t+1\,|\,t) = \hat{\mathbf{A}}(t)\mathbf{P}(t\,|\,t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + \sum_{j=1}^{np}[\boldsymbol{\psi}_j(t)\mathbf{k}_j^{\mathrm{T}}(t\,|\,t)\hat{\mathbf{A}}^{T}(t) +$$

$$+ \hat{\mathbf{A}}(t)\mathbf{k}_j(t\,|\,t)\boldsymbol{\psi}_j^{\mathrm{T}}(t) + \boldsymbol{\sigma}_j(t\,|\,t)\boldsymbol{\psi}_j(t)\boldsymbol{\psi}_j^{\mathrm{T}}(t)] + \mathbf{Q}(\hat{\boldsymbol{\rho}}(t\,|\,t))$$

$$\mathbf{k}_j(t+1\,|\,t) = \hat{\mathbf{A}}(t)\mathbf{k}_j(t\,|\,t) + \boldsymbol{\psi}_j(t)\boldsymbol{\sigma}_j(t\,|\,t)$$

$$\boldsymbol{\sigma}_j(t+1\,|\,t) = \boldsymbol{\sigma}_j(t\,|\,t).$$

The above equations are the extended Kalman filter parameter estimator (EKFPE). In the above, the hat is the estimate of any variable, e.g., $\hat{\mathbf{x}}$ is the estimate of **x**, and the T superscript indicates the matrix transpose. Also, let E be the expectation operator conditioned on past $\mathbf{y}(t)$. Then **P**, $\mathbf{k}_j$, and $\boldsymbol{\sigma}_j$ are estimates of E{ $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^{\mathrm{T}}$ }, E{ $\tilde{\mathbf{x}}\tilde{\boldsymbol{\theta}}_j$ } and E{ $\tilde{\boldsymbol{\theta}}_j^2$ }, respectively. Finally, in (4) above, we define

$$\begin{aligned}
&\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t \mid t-1) \\
&\mathbf{V}(t) = \mathbf{C}\mathbf{P}(t \mid t-1)\mathbf{C}^T + \mathbf{R} \\
&\mathbf{K}(t) = \mathbf{P}(t \mid t-1)\mathbf{C}^T\mathbf{V}^{-1}(t) \\
&\hat{\mathbf{A}}(t) = \mathbf{A}(\hat{\boldsymbol{\theta}}(t \mid t)) \text{ and } \hat{\mathbf{B}}(t) = \mathbf{B}(\hat{\boldsymbol{\theta}}(t \mid t)) \\
&\boldsymbol{\psi}_j(t) = \mathbf{A}_j\hat{\mathbf{x}}(t \mid t) + \mathbf{B}_j\mathbf{u}(t).
\end{aligned} \tag{5}$$

The EKFPE starts from initial guesses for the five unknowns $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\theta}}$, **P, k** and **σ** at $(t|t\text{-}1)$ for $t = 0$. Note that the value of the estimate $\hat{\boldsymbol{\rho}}$ of the process noise variance parameter is not defined in $\mathbf{Q}(\hat{\boldsymbol{\rho}}(t \mid t))$ in the equation for $\mathbf{P}(t+1|t)$.

For the case in which there is no process noise, i. e. $\mathbf{Q}(\boldsymbol{\rho}) = \mathbf{0}$, Ljung (1979) proved that the EKFPE is globally convergent under the conditions considered here.

4. A SIMPLE FIX-UP

This section heuristically fixes up the EKFPE, proofs are in the Appendices. First we find $\hat{\boldsymbol{\rho}}$, by incorporating another variable, **W,** which will be updated along with updates for $\hat{\boldsymbol{\rho}}$, its gain **h,** and its variance **π**, similar to the updates for **θ**, etc. Using the orthogonality principle (Anderson and Moore, 1979), $\hat{\boldsymbol{\rho}}$ satisfies an equation in which the conditional expectation of $\tilde{\boldsymbol{\rho}}$ times its data equals zero. Since **ρ** is the process noise variance parameter, its data is past values of $\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t)$. So $\mathrm{E}\{\tilde{\boldsymbol{\rho}}_i(t|t-1)\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t)\}$ must be driven to zero. Because v(t) can be solved for in terms of $\tilde{\mathbf{x}}$ in the following (7), equivalently drive to zero the n x n matrix $\mathbf{W}_i$,

$$\mathbf{W}_i(t \mid t-1) \approx \mathrm{E}\{\tilde{\boldsymbol{\rho}}_i(t \mid t-1)\tilde{\mathbf{x}}(t \mid t-1)\tilde{\mathbf{x}}^{\mathrm{T}}(t \mid t-1)\} \tag{6}$$

Before we derive the update for $\mathbf{W}_i$, as in (4) we find updates

$$\begin{aligned}
&\tilde{\mathbf{x}}(t \mid t) = \tilde{\mathbf{x}}(t \mid t-1) - \mathbf{K}(t)\boldsymbol{\varepsilon}(t) \\
&\tilde{\mathbf{x}}(t+1 \mid t) = \hat{\mathbf{A}}(t)\tilde{\mathbf{x}}(t \mid t) + \sum_j^{np}\boldsymbol{\psi}_j(t)\tilde{\boldsymbol{\theta}}_j(t \mid t) + \mathbf{v}(t) \\
&\hat{\boldsymbol{\rho}}_i(t \mid t) = \hat{\boldsymbol{\rho}}_i(t \mid t-1) + \mathbf{h}_i^{\mathrm{T}}(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t) \\
&\hat{\boldsymbol{\rho}}_i(t+1 \mid t) = \hat{\boldsymbol{\rho}}_i(t \mid t) \\
&\boldsymbol{\pi}_i(t \mid t) = \boldsymbol{\pi}_i(t \mid t-1) - \mathbf{h}_i^{\mathrm{T}}(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\mathbf{C}\mathbf{h}_i(t \mid t-1) \\
&\boldsymbol{\pi}_i(t+1 \mid t) = \boldsymbol{\pi}_i(t \mid t),
\end{aligned} \tag{7}$$

where $\mathbf{h}_i$ is the estimate of $\mathrm{E}(\tilde{\mathbf{x}}\tilde{\boldsymbol{\rho}}_i)$. To find the time update of **h**, use (7) in its definition to obtain

$$\mathbf{h}_i(t+1 \mid t) = \hat{\mathbf{A}}(t)\mathbf{h}_i(t \mid t). \tag{8}$$

The measurement update of **h** uses the best linear unbiased estimate of $\mathbf{h}(t|t)$ given the innovations $\boldsymbol{\varepsilon}(t)$, computed assuming all random variables are Gaussian, so as to incorporate the effect of third order moments.

$$\begin{aligned}
\mathbf{h}_i(t \mid t) = {} & \mathrm{E}[\tilde{\boldsymbol{\rho}}_i(t \mid t)\tilde{\mathbf{x}}(t \mid t)] + \\
& + \mathrm{E}[\tilde{\boldsymbol{\rho}}_i(t \mid t)\tilde{\mathbf{x}}(t \mid t)\tilde{\boldsymbol{\varepsilon}}^{\mathrm{T}}(t)]\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t).
\end{aligned} \tag{9}$$

Substituting the updates for errors in **x** and **ρ** from (7) gives

$$\begin{aligned}
\mathbf{h}_i(t \mid t) = {} & [\mathbf{I} - \mathbf{K}(t)\mathbf{C}][\mathbf{h}_i(t \mid t-1) + \\
& + \mathbf{W}_i(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t)].
\end{aligned} \tag{10}$$

where all other third order moments except $\mathbf{W}_i$ are ignored. Similarly, (7) can be used in the definition of $\mathbf{W}_i$ to get

$$\mathbf{W}_i(t \mid t) = [\mathbf{I} - \mathbf{K}(t)\mathbf{C}]\mathbf{W}_i(t \mid t-1)[\mathbf{I} - \mathbf{K}(t)\mathbf{C}]^{\mathrm{T}} \tag{11}$$

$$\mathbf{W}_i(t+1 \mid t) = \hat{\mathbf{A}}(t)\mathbf{W}_i(t \mid t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + \mathrm{E}[\tilde{\boldsymbol{\rho}}_i(t \mid t)\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t)]. \tag{12}$$

Instead of the square of $\mathbf{v}(t)$, use $\mathbf{Q}(\boldsymbol{\rho})$ and (3) to get

$$\mathbf{W}_i(t+1 \mid t) = \hat{\mathbf{A}}(t)\mathbf{W}_i(t \mid t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + \mathbf{Q}_i\boldsymbol{\pi}_i(t \mid t). \tag{13}$$

The un-projected version of FEKFPE is (4), last part of (7), (8), (10), (11) and (13). These equations are the same as FEKFPE2 in Appendix A modified to make FEKFPE by omitting third order moments associated with **θ**. But FEKFPE must be projected.

5. PROJECTION

In a recursive parameter estimator, projection returns a computed estimate of an updated variable from a forbidden region to some value inside a permissible region. Because the innovations $\boldsymbol{\varepsilon}(t)$ can take any value, all the measurement update equations in which **ε** appears need to be projected. For example, an un-projected update of the estimate of a variance might be computed to be negative. Variances are never negative, so such an estimated value should be projected into the positive real numbers. See (Ljung, 1987).

The permissible region for **θ** is DS, intersection with DB, the *a priori* bounded region imposed by physical limitations on the parameter values. Mathematical averaging theory used to prove convergence of FEKFPE2 requires DB to be bounded. Similarly **ρ** must be in $DN \cap DB$.

More projections than on **θ** and **ρ** must be in FEKFPE. The innovations representation must exist for the *extended* state **ξ**. The continuous time nonlinear optimal filter (Wiberg and DeWolf, 1993) for ξ has a solution that is an infinite sequence of equations for the moments, each of which having an input that is one order higher moment. The 3-OM parameter estimator, upon which FEKFPE2 is based, closes this infinite sequence of equations by approximating the fourth order moment input to the third order moment equation. Because the optimal nonlinear filter is driven by innovations, then the optimal nonlinear filter, 3-OM, FEKFPE2, and so FEKFPE require the existence of the nonlinear filter extended state innovations. For the innovations representation to exist, the estimate of the variance of the extended state ξ should be kept positive definite. Thus for $j = 1, 2, \ldots np$ and $i = 1, 2, \ldots nq$,

$$\begin{pmatrix} \mathbf{P} & \mathbf{k}_j \\ \mathbf{k}_j^{\mathrm{T}} & \boldsymbol{\sigma}_j \end{pmatrix} > 0 \text{ and } \begin{pmatrix} \mathbf{P} & \mathbf{h}_i \\ \mathbf{h}_i^{\mathrm{T}} & \boldsymbol{\pi}_i \end{pmatrix} > 0. \tag{14}$$

Therefore **π** must be projected as

$$\boldsymbol{\pi}_i(t \mid t) > \kappa / t \tag{15}$$

for κ some small positive number, and the parameter estimate update gain n-vector $\mathbf{h}$ must be projected such that its magnitude is bounded as

$$[\mathbf{h}_i^{\mathrm{T}}(t\,|\,t)\mathbf{h}_i(t\,|\,t)]^2 < \boldsymbol{\pi}_i(t\,|\,t)\mathbf{h}_i^{\mathrm{T}}(t\,|\,t)\mathbf{P}(t\,|\,t)\mathbf{h}_i(t\,|\,t). \quad (16)$$

Similarly, $\boldsymbol{\sigma}$ and $\mathbf{k}$ must be projected as in (15) and (16) also. Incorporation of all these projections in the above FEKFPE gives the complete algorithm.

Convergence of the FEKFPE and FEKFPE2 is proven using the averaging technique of (Ljung, 1987) on the un-projected versions. Projection is accounted for by proving convergence assuming no projection limit is hit, and requiring that if a projection limit is hit, then the projection is into the interior of the permissible region, and then the algorithm is restarted from that interior point. After perhaps many restarts, then the algorithm must either converge to a limit point in the interior of the permissible region (i.e., it becomes the un-projected algorithm because no limit points are hit) or interminably bounce off of the projection limits. This is the sense of convergence in (Ljung, 1987) and here. Convergence is global to a local maximum of the likelihood function.

6. EXAMPLES

The one-dimensional state case was simulated for examples of systems whose parameters were estimated by both FEKFPE and FEKFPE2. In all cases, it made little difference whether FEKFPE or FEKFPE2 was used. Several stable linear systems were simulated, whose results agree with the examples of (Wiberg, et al., 2000) and (Wiberg and DeWolf, 1993). More interestingly, first consider the slightly unstable first order linear system

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) + v(t) \\ y(t) &= x(t) + w(t) \end{aligned} \quad (17)$$

with true parametric values $a = 1.01$, $b = 2$, $Q = 0.5$, $R = 1$, all unknown including R, and with known input $u(t)$ as zero-mean white noise with unity variance. A typical run of FEKFPE for the system (17) is shown (Fig.1).

The second example is the first order nonlinear system

$$\begin{aligned} x(t+1) &= x(t) + ax^3(t) + bu(t) + v(t) \\ y(t) &= x(t) + w(t) \end{aligned} \quad (18)$$

with true parametric values a = -0.02, b = 2, Q = 0.5, R = 1, all unknown including R, and with known input $u(t)$ as zero-mean white noise with unity variance. The state equations (18) were adjoined to $\boldsymbol{\theta} = (a, b)$ and $\rho = Q$, and the FEKFPE formed analogously to the derivation given above. A typical run of FEKFPE for (18) is shown (Fig.2).

If the magnitude of the true value of the parameter a is increased slightly in both the above examples, the EKF and consequently the FEKFPE did not work well.

The estimate for R was obtained recursively as follows. Note that R does not appear in the FEKFPE explicitly, only $V(t)$.

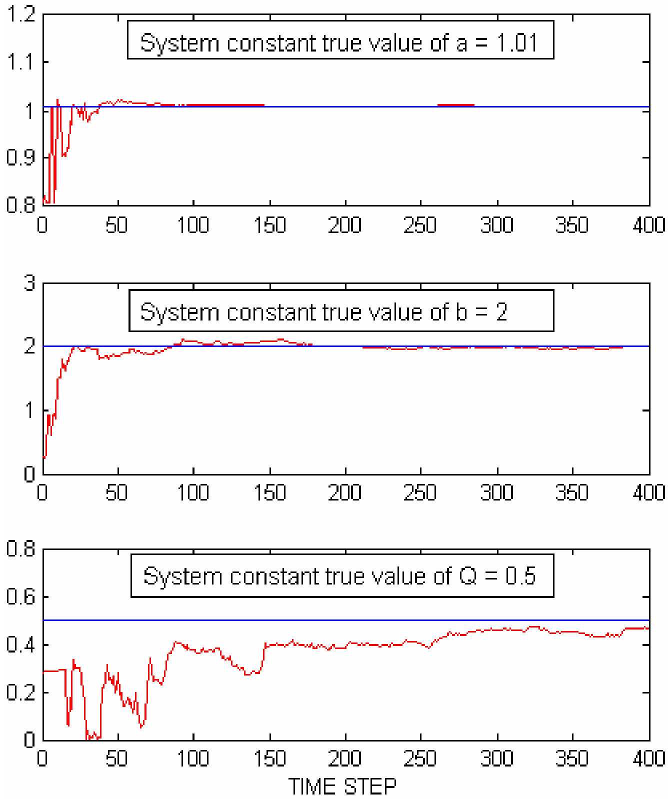

Fig. 1. Estimates of parameters (a, b, Q) in an unstable first-order linear system.

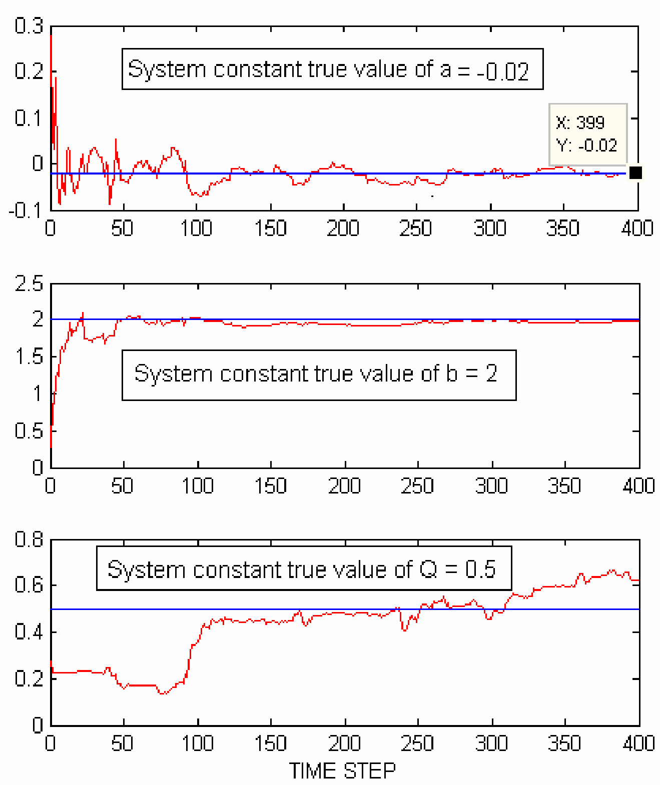

Fig. 2. Estimates of parameters (a, b, Q) in the nonlinear system $x(t+1) = x(t) + ax^3(t) + bu(t) + v(t)$

Estimate $V(t)$ as the time average of the residuals.

$$\hat{V}(t) = (\sum_{\tau=0}^{t} \varepsilon^2(\tau))/(t+1) = \hat{R}(t) + CP(t \mid t-1)C^T \quad (19)$$

But $V(t)$ can be estimated recursively starting from $V(-1) = 0$.

$$\hat{V}(t) = \hat{V}(t-1) + [\varepsilon^2(t) - \hat{V}(t-1)]/(t+1) \quad (20)$$

So if $\hat{V}(t) < CP(t \mid t-1)C^T$, set $V(t)$ in the FEKFPE equations to $CP(t \mid t-1)C^T$, if not set $V(t) = \hat{V}(t)$. Then the estimate for R is obtained from (19).

7. CONCLUSIONS.

The FEKFPE has $(n+np+nq)$ updates for $\hat{x}$, $\hat{\theta}$ and $\hat{\rho}$, an $n(n+1)/2$ dimensional Riccati update for P, the $n(np+nq)$ updates for the gains of $\hat{\theta}$ and $\hat{\rho}$, the $(np+nq)$ updates for the variance of $\hat{\theta}$ and $\hat{\rho}$, and nq times $n(n+1)/2$ linear symmetric updates for W_i. Further simplification that retains global convergence is not possible.

Although a continuous time version of FEKFPE has been proven globally convergent only in the one-dimensional case, it is now reasonable to proceed with the programming of a corresponding n-dimensional discrete-time version of FEKFPE for practical use. As a back-up, if FEKFPE converges in too small a region of parameter space, then FEKFPE2 can be programmed and global convergence guaranteed even in the n-dimensional case, although many more W_i equations would also need to be updated. The asymptotic averaged dynamic behaviour of FEKFPE and FEKFPE2 are analyzed and compared for a simple case in Appendix B. This simple case leads us to conjecture that FEKFPE2 makes FEKFPE more stable at the cost of a slower rate of convergence.

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Appendix A. DERIVATION OF FEKFPE2

For only this Appendix A, the parameter p-vector $\boldsymbol{\theta}$ includes both structural $\boldsymbol{\theta}$ and noise $\boldsymbol{\rho}$ parameters, so $p = np + nq$. See (Wiberg *et al,* 2000) for a description of the complicated 3-OM parameter estimator. There, equations (6.10) – (6.26) are proven globally convergent to a local maximum of the likelihood function. FEKFPE2 is a reformulation of 3-OM resulting from redefining variables in (6.10) – (6.26) as:

(1) The parameter p-vector $\boldsymbol{\theta}$ is written for one element $\boldsymbol{\theta}_i$ at a time for $i = 1, 2, \ldots, p$. Then the daunting Kronecker products collapse to merely matrix products.

(2) Each element $\boldsymbol{\theta}_i$ is assumed independent of $\boldsymbol{\theta}_j$ for i not equal to j. Then it can be shown that estimates of the parameter variance become scalar $\boldsymbol{\sigma}_i$ for each i.

(3) The estimate of third-order moment Kronecker products denoted $\mathbf{M}$ in 3-OM becomes a sequence of n x n matrices denoted $\mathbf{W}_i$ here, for $i = 1, 2, \ldots p$.

(4) The Kronecker products involving the permutation matrix $\mathbf{U}$ are easily computed by realizing that the expressions containing $\mathbf{U}$ must be permutations of the matrices in the products.

(5) For simplicity, the cross correlation matrix $\mathbf{S}$ between process and measurement noise is taken to be zero.

(6) In 3-OM, the state $\mathbf{x}(t)$ is excited by white noise $\mathbf{v}(t)$, so $\mathbf{P}(t)$ does not decay, but parameter gain $\mathbf{k}(t)$ and parameter variance $\boldsymbol{\sigma}(t)$ decay as $O(1/t)$. FEKFPE2 equations omit terms that decay faster than other terms in any equation.

The resulting FEKFPE2 projected equations are the measurement updates, for $i = 1, 2, \ldots, p$, with projection operations $\xrightarrow{project}$ from forbidden to permitted region:

$$\begin{aligned}
&\hat{\mathbf{x}}(t \mid t) = \hat{\mathbf{x}}(t \mid t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t) \\
&\hat{\boldsymbol{\theta}}_i(t \mid t) = \xrightarrow{project} \{\hat{\boldsymbol{\theta}}_i(t \mid t-1) + \\
&\qquad + \mathbf{k}_i(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t)\} \\
&\mathbf{P}(t \mid t) = [\mathbf{I} - \mathbf{K}(t)\mathbf{C}]\mathbf{P}(t) \\
&\mathbf{k}_i(t \mid t) = \xrightarrow{project} \{[\mathbf{I} - \mathbf{K}(t)\mathbf{C}][\mathbf{k}_i(t \mid t-1) + \\
&\qquad + \mathbf{W}_i(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\boldsymbol{\varepsilon}(t)]\} \\
&\boldsymbol{\sigma}_i(t \mid t) = \xrightarrow{project} \{\boldsymbol{\sigma}_i(t \mid t-1) - \\
&\qquad - \mathbf{k}_i^{\mathrm{T}}(t \mid t-1)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}(t)\mathbf{C}\mathbf{k}_i(t \mid t-1)\} \\
&\mathbf{W}_i(t \mid t) = [\mathbf{I} - \mathbf{K}(t)\mathbf{C}]\mathbf{W}_i(t \mid t-1)[\mathbf{I} - \mathbf{K}(t)\mathbf{C}]^{\mathrm{T}}
\end{aligned} \quad (A.1)$$

With initial conditions $\mathbf{W}_i(0 \mid -1) = 0$. Time up date is:

$$\hat{\mathbf{x}}(t+1 \mid t) = \hat{\mathbf{A}}(t)\hat{\mathbf{x}}(t \mid t) + \hat{\mathbf{B}}(t)\mathbf{u}(t)$$

$$\hat{\boldsymbol{\theta}}_i(t+1 \mid t) = \hat{\boldsymbol{\theta}}_i(t \mid t)$$

$$\mathbf{k}_i(t+1 \mid t) = \hat{\mathbf{A}}(t)\mathbf{k}_i(t \mid t) + \boldsymbol{\sigma}_i(t \mid t)\boldsymbol{\psi}_i(t)$$

$$\boldsymbol{\sigma}_i(t+1 \mid t) = \boldsymbol{\sigma}_i(t \mid t)$$

$$\mathbf{P}(t+1\,|\,t) = \hat{\mathbf{A}}(t)\mathbf{P}(t\,|\,t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + \sum_{j=1}^{p}[\boldsymbol{\psi}_i(t)\mathbf{k}_i^{\mathrm{T}}(t\,|\,t)\hat{\mathbf{A}}^{T}(t) + \hat{\mathbf{A}}(t)\mathbf{k}_i(t\,|\,t)\boldsymbol{\psi}_i^{\mathrm{T}}(t) + \boldsymbol{\sigma}_i(t\,|\,t)\boldsymbol{\psi}_i(t)\boldsymbol{\psi}_i^{\mathrm{T}}(t)] + \hat{\mathbf{Q}}(t) \quad \text{(A.2)}$$

$$\mathbf{W}_i(t+1\,|\,t) = \hat{\mathbf{A}}(t)\mathbf{W}_i(t\,|\,t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + + \boldsymbol{\sigma}_i(t\,|\,t)[\mathbf{A}_i\mathbf{P}(t\,|\,t)\hat{\mathbf{A}}^{\mathrm{T}}(t) + \hat{\mathbf{A}}(t)\mathbf{P}(t\,|\,t)\mathbf{A}_i^{\mathrm{T}} + \mathbf{Q}_i].$$

Appendix B. ASYMPTOTIC DYNAMICS OF FEKFPE AND FEKFPE2

In this Appendix B, we analyze and compare the asymptotic averaged dynamic behavior of FEKFPE and FEKFPE2 in continuous-time, rather than the discrete-time, and only for a one-dimensional example. However, this example gives a reason to choose either FEKFPE or FEKFPE2. Consider

$$dx = \theta x dt + dv \qquad dy = cxdt + dw. \quad \text{(B. 1)}$$

Here, the parameter θ is to be estimated, and $Q > 0$ and $r > 0$ are known scalar variances associated with the independent Brownian motions $v(t)$ and $w(t)$ Now apply the continuous-time limit of un-projected FEKFPE2. Define $d\varepsilon = dy - c\hat{x}dt$ and $\Lambda = \theta - Pc^2/r$, so FEKFPE2 is

$$d\hat{x} = \hat{\theta}\hat{x}dt + Pcd\varepsilon/r \quad \text{(B. 2)}$$

$$d\hat{\theta} = ckd\varepsilon/r \quad \text{(B. 3)}$$

$$dP/dt = 2\hat{\theta}P - c^2P^2/r + Q + 2\hat{x}k \quad \text{(B. 4)}$$

$$dk = (\hat{\theta}k + \sigma\hat{x})dt + Wcd\varepsilon/r \quad \text{(B. 5)}$$

$$d\sigma/dt = -c^2k^2 \quad \text{(B. 6)}$$

$$dW/dt = 2\Lambda W + 2\sigma P. \quad \text{(B. 7)}$$

FEKFPE consists of the same equations except $W = 0$. In this example, FEKFPE is the same as the EKFPE. Using Ito calculus, change variables in (B. 2) - (B. 7) as

$$\mu(t) = 1/t\sigma(t),\ s(t) = tk(t)\mu(t), L(t) = t\mu(t)W(t).$$

Then (B. 2) – (B. 7) become

$$\begin{aligned} d\hat{x} &= (\hat{\theta}\hat{x} + s/t)dt + Pcd\varepsilon/r \\ d\hat{\theta} &= scd\varepsilon/tr\mu \\ dP/dt &= 2\hat{\theta}P - P^2c^2/r + Q + 2(\hat{x}s + L)/t\mu \\ ds &= (\hat{\Lambda}s + s^3c^2/tr\mu + \hat{x})dt + Lcd\varepsilon/r \\ d\mu/dt &= (s^2c^2/r - \mu)/t \\ dL/dt &= 2\hat{\Lambda}L + 2P + s^2c^2L/tr\mu + 2s^2/t\mu. \end{aligned} \quad \text{(B. 8)}$$

Define

$$\tau = e^t,\ \Pi = \lim_{t\to\infty} P(t),\ \overline{L} = \lim_{t\to\infty} L(t),$$
$$\overline{\theta}(\tau) = \mathrm{E}\{\hat{\theta}(\ln\tau)\},\ \overline{\mu}(\tau) = \mathrm{E}\{\mu(\ln\tau)\}.$$

Then the averaged trajectories $\overline{\theta}(\tau)$ of $\hat{\theta}$ and $\overline{\mu}(\tau)$ of $1/\sigma$ obey (Wiberg and DeWolf, 1993).

$$d\overline{\theta}/d\tau = \mathrm{E}\{s\tilde{x}\}c^2/r\overline{\mu} \qquad d\overline{\mu}/dt = \mathrm{E}\{s^2\}c^2/r - \overline{\mu}. \quad \text{(B. 9)}$$

The two expected quantities in the above are found forming

$$d\boldsymbol{\xi} = \boldsymbol{\Psi}\boldsymbol{\xi}dt + \mathbf{G}dv + \mathbf{H}dw, \quad \text{(B. 10)}$$

from (B. 8) for $\hat{\theta} = \theta$ and (B. 1) for $\theta = \theta_T$, the true value of θ, in which we have defined

$$\boldsymbol{\xi}^{\mathrm{T}} = (x, \tilde{x}, s),$$
$$\boldsymbol{\Psi} = \begin{pmatrix} \theta_T & 0 & 0 \\ \theta_T - \theta & \Lambda(\theta) & 0 \\ 1 & \overline{L}c^2/r - 1 & \Lambda(\theta) \end{pmatrix}, \quad \text{(B. 11)}$$
$$\mathbf{G} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{H} = \begin{pmatrix} 0 \\ -\Pi c/r \\ \overline{L}c/r \end{pmatrix}.$$

where Π and $\overline{L}$ obey the steady state of (B. 8) as

$$0 = 2\hat{\theta}\Pi - \Pi^2c^2/r + Q \qquad 0 = 2\hat{\Lambda}\overline{L} + 2\Pi. \quad \text{(B. 12)}$$

The steady state variance Liapunov equation for (B. 10) is

$$0 = \boldsymbol{\Psi}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{\Psi}^{\mathrm{T}} + \mathbf{G}Q\mathbf{G}^{\mathrm{T}} + \mathbf{H}r\mathbf{H}^{\mathrm{T}}, \quad \text{(B. 13)}$$

where $\boldsymbol{\Sigma} = \mathrm{E}\{\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}\}$. The solution is

$$\begin{aligned} &\mathrm{E}\{x^2\} = -Q/2\theta_T \\ &\mathrm{E}\{\tilde{x}^2\} = Q[\theta^2 - \theta_T\Lambda(\theta)]/[\{2\theta_T\Lambda(\theta)[\Lambda(\theta) + \theta_T]\} \\ &\quad - \Pi^2(\theta)c^2/2r\Lambda(\theta)] \\ &\mathrm{E}\{s\tilde{x}\} = -[\alpha(\theta,\theta_T) + \overline{L}\beta(\theta,\theta_T)](\theta - \theta_T) \end{aligned} \quad \text{(B. 14)}$$

where $\alpha(\theta, \theta_T)$, $\beta(\theta, \theta_T)$ and $\gamma(\theta, \theta_T)$ are the positive constants

$$\begin{aligned} &\alpha(\theta,\theta_T) = (\mathrm{E}\{x^2\} - \theta_T\Pi/2)\Pi c^2/r\gamma(\theta,\theta_T) \\ &\beta(\theta,\theta_T) = -(\theta + \theta_T)\mathrm{E}\{x^2\}c^2/r\gamma(\theta,\theta_T) \\ &\gamma(\theta,\theta_T) = -2\Lambda^2/(\Lambda + \theta_T), \end{aligned} \quad \text{(B. 15)}$$

in which the θs and Λ are negative by stability and Π is positive as a variance. Substitution of (B. 14) into (B. 9) and using $\overline{\theta}$ for θ then gives

$$d\overline{\theta}/d\tau = [\alpha(\overline{\theta},\theta_T) + \overline{L}\beta(\overline{\theta},\theta_T)]c^2(\theta_T - \overline{\theta})/r\overline{\mu}. \quad \text{(B. 16)}$$

Thus $\overline{\theta}$ converges only to θ_T for all permissible (positive) values of $\overline{L}$ of FEKFPE2 and for $\overline{L} = 0$, i.e., for FEKFPE. This simple example leads to the conjecture that FEKFPE2 is more stable than FEKFPE, but converges slower on average.

Appendix C. ACKNOWLEDGMENT

This work was supported in part by Laboratory for Adaptive Optics, funded by the Gordon and Betty Moore Foundation; by a research grant from NSF Center for Adaptive Optics, managed by the University of California at Santa Cruz under cooperative agreement No. AST-9876783 and performed as a Visiting Professor at the School of Electrical and Computer Engineering, Ajou University, Suwon, Rep. of Korea.