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Identifiability of Variable Coefficients for Vibrating Systems by Boundary Control and Observation in Finite Time Duration

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Abstract: The identifiability of spatial variable coefficients for vibration string and Euler-Bernoulli beam are considered. It is shown that the coefficients can be determined by means of boundary control and observation in finite time duration. These results can be considered as the generalization of infinite time coefficients identifiability through the application of Ingham-Beurling theorem.

Keywords: String equation; beam equation; identifiability; variable coefficients.

1. INTRODUCTION

Identifiability is one of the fundamental problems in parameter identification (see Banks and Kunisch (1989) for basic knowledge of identifiability). For coefficient identifiability problems of distributed parameter systems, only a few methods are available at present. One of them is to reduce the identifiability problems to some inverse spectral problems (see, e.g., Chang and Guo (2007); Kachalov et al. (2004)). This method can be applied to both one-dimensional parabolic and one-dimensional hyperbolic systems. For one-dimensional parabolic systems, by virtue of the theory of Dirichlet series, the coefficient identifiability with finite time observation can be directly obtained (see, e.g., Kravaris and Seinfeld (1986); Pierce (1979)). However, for one-dimensional hyperbolic systems, the coefficient identifiability can be specified only for those with infinite time observation (see, e.g., Udwadia and Sharma (1985)). Recently, this method was improved by authors in Chang and Guo (2007) and Chang (2008) to solve the identifiability of coefficients for one-dimensional vibrating systems including string and beam equations, and some new identifiability results with infinite time observation were obtained. One of the objectives of this paper is to generalize these results from infinite time observation to finite time observation.

It should be indicated that although some special identifiability results of coefficients with finite time observation for vibrating systems can be obtained directly by the theory of nonlinear integral equations (see, e.g., (Isakov, 1998, Section 8.1)), the common feasible way of establishing identifiability is first to investigate the identifiability with infinite time observation, and then to improve results by extending finite time data onto infinite time interval (Kachalov et al. (2004)). This is natural because in general, identifiability problems with infinite time observation are easier to be solved.

The main contribution of this paper is to propose a new simpler approach to extend the observation data in finite time interval to infinite time interval. This is realized by the help of the Ingham-Beurling type theorem. In addition, by this approach, the finite time coefficient identifiability for one-dimensional vibrating systems in some cases can be directly established as easily as for one-dimensional parabolic systems. This is because spectral data that are required to solve the associated inverse spectral problem can be uniquely determined from some observations in certain finite intervals without extending these observations to infinite time interval.

2. MAIN RESULTS

Let $\Omega = \{\omega_n\}_{n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers, where $\mathbb{Z}$ is an index subset of integers. Define the upper density $D^+(\Omega)$ of the sequence Ω by

$$D^+(\Omega) := \lim_{r\to\infty} \frac{n^+(r,\Omega)}{r},$$

where $n^+(r,\Omega)$ denotes the largest number of terms of the sequence Ω contained in an interval of length r (see, e.g., Avdonin and Moran (2001), (Komornik and Loreti, 2005, p.174)). If Ω is a separated set, that is

$$\inf_{m\neq n} |\omega_m - \omega_n| > 0$$

then $\{e^{i\omega_n t}\}_{n\in\mathbb{Z}}$ forms an $\mathcal{L}$-basis in $L^2(I)$, that is, a Riesz basis (Young (2001)) for the closed subspace of $L^2(I)$ spanned by itself, where I is any bounded interval of length $|I| > 2\pi D^+(\Omega)$ and hence for any nonharmonic Fourier series of the form

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\omega_n t}, \tag{1}$$

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there exist two constants $D_1, D_2 > 0$ such that

$$D_1 \sum_{n\in\mathbb{Z}} |a_n|^2 \le \int_I |f(t)|^2 dt \le D_2 \sum_{n\in\mathbb{Z}} |a_n|^2. \quad (2)$$

Consequently, f is uniquely determined by its restriction on I. The above result is usually referred to as the Ingham-Beurling theorem (Theorems 4.3, 9.2 of Komornik and Loreti (2005)). If Ω is not a separated set, the first inequality of (2) does not hold any more; but if Ω is a relatively separated set, that is, Ω is a union of finite separated sequences, the Ingham-Beurling theorem can be generalized in different types (see Theorem 9.4 of Komornik and Loreti (2005) or Proposition 1 of Avdonin and Moran (2001)).

As stated in introduction, the fact that two Dirichlet series which are equal to each other in any finite interval must have the same exponents and coefficients is key for establishing finite time identifiability of one-dimensional parabolic systems (see, e.g., Kravaris and Seinfeld (1986); Pierce (1979)). Here we give the similar uniqueness result for nonharmonic Fourier series so that the finite time identifiability of one-dimensional vibrating systems can also be achieved directly.

Theorem 1. Let $\Omega_1 = \{\mu_n\}_{n\in\mathbb{Z}}$ and $\Omega_2 = \{\nu_n\}_{n\in\mathbb{Z}}$ be any two strictly increasing sequences of real numbers, satisfying the gap condition

$$\mu_{n+1} - \mu_n > \gamma, \ \nu_{n+1} - \nu_n > \gamma, \forall\, n \in \mathbb{Z} \quad (3)$$

for some positive constant $\gamma > 0$. Suppose f is a function given by

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t} - \sum_{n\in\mathbb{Z}} b_n e^{i\nu_n t}, \quad (4)$$

where the complex coefficients a_n and b_n are square-summable. If I is a bounded interval of length $|I| > 4\pi D^+(\Omega_1)$, then

(i) f is uniquely determined by its restriction on I.

(ii) $\{a_n\}_{n\in\mathbb{Z}} = \{b_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\in\mathbb{Z}} = \{\nu_n\}_{n\in\mathbb{Z}}$ provided that $a_n \ne 0, b_n \ne 0$ for every $n \in \mathbb{Z}$, and $f = 0$ almost everywhere on I.

Now we consider the coefficient identification problem for a string equation given by

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0,\ 0 < x < 1,\ t > 0, \\ w(0,t) = 0,\ t \ge 0, \\ a(1)w_x(1,t) = u(t),\ t \ge 0, \\ w(x,0) = w_t(x,0) = 0,\ 0 \le x \le 1, \\ y(t) = w_t(1,t), t \ge 0, \end{cases} \quad (5)$$

where $a(x)$, unknown parameter, is the tension of the string, $u(t)$ is the known boundary input, and $y(t)$ is the observation to identify $a(x)$. Assume that $u(\cdot) \in L^2_{loc}(0,\infty)$ is not identical to zero. Furthermore, we assume that $a(\cdot)$ is in a parameter set Q given by

$$Q = \{a(x) \in C^2[0,1] :\ a(x) \ge a_0 > 0, \forall x \in [0,1]\}. \quad (6)$$

Define the operator $A : D(A)(\subset L^2(0,1)) \mapsto L^2(0,1)$ by

$$\begin{cases} Af = -(a(x)f')', \\ D(A) = \{f \in L^2(0,1) | (af')' \in L^2(0,1), \\ \qquad\qquad f(0) = f'(1) = 0\}. \end{cases} \quad (7)$$

It is well-known that the operator A is positive self-adjoint in $L^2(0,1)$. The state space of the system (5) is naturally chosen as Hilbert space $\mathbb{H} = D(A^{1/2}) \times L^2(0,1) = H^1_L(0,1) \times L^2(0,1)$, $H^1_L(0,1) = \{\varphi \in H^1(0,1) |\ f(0) = 0\}$, with the inner product induced norm

$$\|(\varphi,\psi)\|^2_{\mathbb{H}} = \int_0^1 [|\psi(x)|^2 + a(x)|\varphi'(x)|^2]dx,\ \forall\ (\varphi,\psi) \in \mathbb{H}.$$

The system (5) is then rewritten as (Guo and Luo (2002))

$$\begin{cases} w_{tt} + Aw = bu \ \text{ in } \ D(A^{1/2})', \\ y(t) = b^* w_t, \end{cases} \quad (8)$$

where

$$b = \delta(x-1) \in D(A^{1/2})',\ b^*\varphi = \varphi(1),\ \forall\, \varphi \in D(A^{1/2}) \quad (9)$$

with Dirac distribution $\delta(x-1)$.

The eigenvalue problem associated with (5) is

$$A\psi_n(x) = \mu_n^2 \psi_n(x), \quad (10)$$

where $\mu_n^2, n = 1,2,\ldots$, are eigenvalues of A and ψ_n is the eigenfunction corresponding to μ_n^2. $\{\psi_n\}_{n=1}^\infty$ forms an orthogonal basis for $L^2(0,1)$, which is normalized so that

$$\psi_n(1) > 0 \text{ and } \int_0^1 \psi_n^2(x)\, dx = 1.$$

Moreover, the following asymptotic expansions hold ((Ince, 1944, pp.270-273))

$$\begin{cases} \mu_n = L^{-1}(n-1/2)\pi + \mathcal{O}(n^{-1}), \\ \psi_n(1) = c + \mathcal{O}(n^{-1}), \end{cases} \quad (11)$$

where c is a positive constant and

$$L = \int_0^1 \frac{1}{\sqrt{a(x)}} dx.$$

By (11), it is known that b is an admissible input operator (Curtain (1997); Guo and Luo (2002); Ho and Russell (1983)), and so is b^* as an output operator. Moreover, it is shown in Proposition 2 of Chang (2008) that the system (8) is well-posed in the sense of D.Salamon and regular in the sense of G.Weiss in the state space $\mathbb{H}$ and input (output) space $\mathbb{C}$ (Curtain (1997)). Actually as it was indicated in Remark 2 of Chang (2008), the system considered in Guo and Luo (2002) must be regular if it is well-posed.

For the identification problem considered here, the coefficient $a(\cdot)$ is called identifiable by $\{(u(t), w_t(1,t)),\ 0 \le t \le T\}$ with respect to Q if for any $a(\cdot), \widetilde{a}(\cdot) \in Q$, $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost all $t \in [0,T]$ implies that $a(x) = \widetilde{a}(x)$ for any $x \in [0,1]$. (see e.g., (Banks and Kunisch , 1989, p.105)).

We have the following finite identifiability for string equation.

Theorem 2. Suppose there is a positive constant τ such that the input $u(\cdot)$ in (5) vanishes in $[\tau,\infty)$. Then the coefficient $a(\cdot)$ in (5) can be identified by $\{(u(t), w_t(1,t)), t \in [0,T]\}$, where $T \ge \tau + 4a_0^{-1/2}$.

The Corollary 3 below tells us that for the purpose of identification, the function of displacement and velocity makes no big difference theoretically, but the former is easier to be measured in practice. This is sharp contrast to stabilization.

Corollary 3. Suppose there is a positive constant τ such that the input $u(\cdot)$ in (5) vanishes in $[\tau,\infty)$. Then the coefficient $a(\cdot)$ in (5) can be identified by $\{(u(t), w(1,t)), t \in [0,T]\}$, where $T \ge \tau + 4a_0^{-1/2}$.

Remark 4. It should be pointed that the finite time coefficient identifiability for string equation with other boundary conditions can also be obtained by our approach due to well development of the associated inverse spectral theory (see e.g., Kravaris and Seinfeld (1986)).

Next we consider the coefficients identification for an Euler-Bernoulli beam equation described by

$$\begin{cases} \rho(x)w_{tt}(x,t) + (r(x)w_{xx}(x,t))_{xx} = 0,\ 0 < x < 1,\ t > 0, \\ w(0,t) = w_x(0,t) = 0,\ t \geq 0, \\ r(x)w_{xx}(x,t)|_{x=1} = 0,\ t \geq 0, \\ (r(x)w_{xx}(x,t))_x|_{x=1} = u(t),\ t \geq 0, \\ w(x,0) = 0, w_t(x,0) = 0,\ 0 \leq x \leq 1, \end{cases} \tag{12}$$

where $\rho(x)$ and $r(x)$, unknown parameters to be identified, are the mass density and the flexural rigidity of the beam, respectively, and $u(t)$ is the known boundary input. For system (12), we assume that $u(\cdot) \in L^2_{loc}(0,\infty)$ is not identical to zero and $(\rho(\cdot), r(\cdot))$ belongs to the parameter set

$$\begin{array}{l} Q = \{(\rho(\cdot), r(\cdot)) \in C^4[0,1] \times C^4[0,1] : \\ \rho(x) > 0, r(x) > 0, \forall\, x \in [0,1]\}. \end{array} \tag{13}$$

The formulation of identifiability of $(\rho(\cdot), r(\cdot)) \in Q$ is similar to that for system (5).

The eigenvalue problem associated with (12) is

$$\begin{cases} (r(x)\phi_n''(x))'' = \omega_n^2 \rho(x)\phi_n(x), 0 < x < 1, \\ \phi_n(0) = \phi_n'(0) = 0, \\ \phi_n''(1) = (r(x)\phi_n''(x))'|_{x=1} = 0, \end{cases} \tag{14}$$

where ω_n and $\omega_n^2, n = 1, 2, \ldots$, are eigenfrequencies and eigenvalues, respectively, ϕ_n is the eigenfunction corresponding to eigenvalue ω_n^2, which is normalized so that $\int_0^1 \rho(x)\phi_n^2(x)\,dx = 1$. Let $L^2_\rho(0,1)$ denote the space of square integrable functions over $[0,1]$ with weight ρ. It is well-known that $\{\phi_n\}_{n=1}^\infty$ forms an orthonormal basis for $L^2_\rho(0,1)$. We may assume without loss of generality that (Chang (2008))

$$\phi_n(1) > 0,\ \phi_n'(1) > 0. \tag{15}$$

It is also known that the following asymptotic properties hold (Chang (2008))

$$\begin{cases} \omega_n = M^{-1}(n - 1/2)^2\pi^2 + \mathcal{O}(n^{-1}), \\ \phi_n(1) = \tilde{c} + \mathcal{O}(n^{-1}),\ \tilde{c} > 0, \\ \phi_n'(1) = \mathcal{O}(n - 1/2), \end{cases} \tag{16}$$

where

$$M = \int_0^1 \left(\frac{\rho(x)}{r(x)}\right)^{\frac{1}{4}} dx.$$

Let $H = L^2_\rho(0,1)$. Define the operator $\mathcal{A} : D(\mathcal{A})(\subset H) \mapsto H$ by

$$\begin{cases} \mathcal{A}f = \dfrac{1}{\rho(x)}(r(x)f'')'',\ \forall\, f \in D(\mathcal{A}), \\ D(\mathcal{A}) = \big\{f \in H \big| (r(x)f'')'' \in L^2(0,1), \\ f(0) = f'(0) = f''(1) = (r(x)f''(x))'|_{x=1} = 0\big\}. \end{cases} \tag{17}$$

Such defined $\mathcal{A}$ is positive self-adjoint in H. Note that (14) is actually the eigenvalue problem of $\mathcal{A}$ and hence $\omega_n > 0$ for all $n \geq 1$. It is well-known that, in the state space $\mathbf{H} = D(\mathcal{A}^{1/2}) \times H = H^2_L(0,1) \times L^2(0,1)$, $H^2_L(0,1) = \{f \in H^2(0,1) | f(0) = f'(0) = 0\}$, the system (12) can be formulated as a SISO second order collocated system (Guo and Luo (2002)):

$$\begin{cases} w_{tt} + \mathcal{A}w + bu = 0 \text{ in } D(\mathcal{A}^{1/2})', \\ w_t(1,t) = b^* w_t, \end{cases} \tag{18}$$

where

$$b = \frac{1}{\rho(x)}\delta(x-1) \in D(\mathcal{A}^{1/2})',\ b^*g = g(1),\ \forall\, g \in D(\mathcal{A}^{1/2}).$$

From this formulation, it follows that (Chang (2008))

(i) there exists a unique solution to (12) such that $(w, w_t) \in C([0,\infty), \mathbf{H})$;

(ii) the system (18) is well-posed in the sense of D.Salamon and regular in the sense of G.Weiss in the sate space $\mathbf{H}$ and input (output) space $\mathbb{C}$ with zero feed-through operator.

Theorem 5. Suppose there is a constant $\tau > 0$ such that the input u in (12) satisfies

$$\begin{cases} u(t) \neq 0 & \text{for almost all } t \in (0,\tau); \\ u(t) = 0 & \text{for } t \geq \tau. \end{cases}$$

Then for each $T > \tau$, $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by $\{(u(t), w_t(1,t), w_x(1,t)), t \in [0,T]\}$.

Corollary 6. Suppose there is a constant $\tau > 0$ such that the input u in (12) satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then for each $T > \tau$, $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by $\{(u(t), w(1,t), w_x(1,t)), t \in [0,T]\}$.

Remark 7. Comparing with Theorem 6 and Theorem 3, we see that T in Theorem 3 has a positive lower bound while in Theorem 6, T can be taken as an arbitrary small number. This phenomenon is also observed in controllability and stabilization of wave and beam equations. It is caused essentially by the fact that the speed of wave propagation is finite, while that of beam is infinite.

3. PROOF OF MAIN RESULTS

Proof of Theorem 1. Set $\Omega = \{\lambda_n\}_{n\in\mathbb{Z}} = \{\mu_n, \nu_n\}_{n\in\mathbb{Z}}$ so that $\{\lambda_n\}$ is a strictly increasing sequence of real numbers. For any given integer $N > 0$, suppose $\{\lambda_n\}_{n=-K}^M = \{\mu_n, \nu_n\}_{n=-N}^N$. Then $M \geq N, K \geq N$ and

$$\sum_{n=-K}^{M} c_n e^{i\lambda_n t} = \sum_{n=-N}^{N} a_n e^{i\mu_n t} - \sum_{n=-N}^{N} b_n e^{i\nu_n t}.$$

Letting $N \to \infty$ gives

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{i\lambda_n t}, \tag{19}$$

where for $n \in \mathbb{Z}$,

$$c_n = \begin{cases} a_k, & \text{if } \lambda_n = \mu_k \neq \nu_m \text{ for some } k \in \mathbb{Z},\ \forall\, m \in \mathbb{Z}, \\ -b_k, & \text{if } \lambda_n = \nu_k \neq \mu_m \text{ for some } k \in \mathbb{Z},\ \forall\, m \in \mathbb{Z}, \\ a_k - b_m, & \text{if } \lambda_n = \mu_k = \nu_m \text{ for some } k, m \in \mathbb{Z}. \end{cases} \tag{20}$$

It is clear that $D^+(\Omega) \leq 2D^+(\Omega_1)$ since $D^+(\Omega_1) = D^+(\Omega_2)$.

By (3),

$$\lambda_{n+2} - \lambda_n > \gamma, \forall\, n \in \mathbb{Z}. \tag{21}$$

For any $n \in \mathbb{Z}$, denote by $D_{\lambda_n}(\gamma)$ the disk centered at λ_n with radius γ. Due to (21), we have, along the n's increasing direction, only two cases:

Case 1. $D_{\lambda_n}(\gamma)$ contains only λ_n. In this case, we denote $e_n(t) = e^{\lambda_n t}$.

Case 2. $D_{\lambda_n}(\gamma)$ contains $(\lambda_n, \lambda_{n+1})$. In this case, we denote $e_n(t) = e^{\lambda_n t}$ and $e_{n+1}(t) = \dfrac{e^{\lambda_n t} - e^{\lambda_{n+1} t}}{\lambda_n - \lambda_{n+1}}$. Then

$$\begin{aligned} &c_n e^{\lambda_n t} + c_{n+1} e^{\lambda_{n+1} t} \\ &= (c_n + c_{n+1}) e_n(t) - c_{n+1}(\lambda_n - \lambda_{n+1}) e_{n+1}(t). \end{aligned} \tag{22}$$

It was proved in Theorem 3 of Avdonin and Moran (2001) that any bounded interval I of length $|I| > 2\pi D^+(\Omega)$, $\{e_n(t)\}_{n\in\mathbb{Z}}$ forms an $\mathcal{L}$-basis in $L^2(I)$. Hence we can further write (19) as

$$f(t) = \sum_{n\in\mathbb{Z}} d_n e_n(t), \tag{23}$$

where

$$\begin{cases} d_n = c_n \text{ if } D_{\lambda_n}(\gamma) \cap \{\lambda_n\}_{n\in\mathbb{Z}} = (\lambda_n), \\ d_n = c_n + c_{n+1}, d_{n+1} = -c_{n+1}(\lambda_n - \lambda_{n+1}) \\ \text{ if } D_{\lambda_n}(\gamma) \cap \{\lambda_n\}_{n\in\mathbb{Z}} = (\lambda_n, \lambda_{n+1}). \end{cases} \tag{24}$$

Since $\{e_n(t)\}_{n\in\mathbb{Z}}$ is an $\mathcal{L}$-basis, there exist constants $C_1, C_2 > 0$ such that the Ingham-type inequality

$$C_1 \sum_{n\in\mathbb{Z}} |d_n|^2 \le \int_I |f(t)|^2 dt \le C_2 \sum_{n\in\mathbb{Z}} |d_n|^2 \tag{25}$$

holds (see e.g., Theorem 9.4 of Komornik and Loreti (2005) or directly Theorem 1.3 of Baiocchi et al (2002) from (19) and (21) to (25)).

If $f = 0$ almost everywhere on I, by (25), $d_n = 0$ for all $n \in \mathbb{Z}$ and hence $f \equiv 0$ by (23). This is (i). Moreover, it follows from (24) that $c_n = 0$ for all $n \in \mathbb{Z}$. Since $a_n \neq 0, b_n \neq 0$ for every $n \in \mathbb{Z}$, we have only third case in (20), which claims that $\{a_n\}_{n\in\mathbb{Z}} = \{b_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\in\mathbb{Z}} = \{\nu_n\}_{n\in\mathbb{Z}}$. ∎

In order to prove Theorem 2, we need several lemmas below.

Lemma 8. Suppose $\{\mu_n\}_{n=1}^{\infty}$ is given by the eigenvalue problem (7). Let $\Lambda = \{\mu_n, -\mu_n\}_{n=1}^{\infty}$. Then $D^+(\Lambda) \le (\sqrt{a_0}\,\pi)^{-1}$.

Proof. Since $a(x) \ge a_0$ for any $x \in [0,1]$, it has $L^{-1} \ge \sqrt{a_0}$. By (11), it follows that

$$\begin{aligned} \mu_n - \mu_{n-1} = L^{-1}\pi + \mathcal{O}(n^{-1}) \ge \sqrt{a_0}\,\pi + \mathcal{O}(n^{-1}) \\ \text{as } n \to +\infty. \end{aligned} \tag{26}$$

For any $\varepsilon > 0$ with $\varepsilon < \sqrt{a_0}\,\pi$, there exists a positive integer N such that

$$\mu_n - \mu_{n-1} \ge \sqrt{a_0}\,\pi - \varepsilon \quad \text{for } n > N.$$

Let I be an interval that contains $\{\mu_n, -\mu_n\}_{n=1}^{N}$. Then

$$D^+(\Lambda) = \lim_{r\to\infty} \frac{n^+(r,\Lambda)}{r} \le \lim_{r\to\infty} \frac{n^+(|I|,\Lambda) + 1 + r/(\sqrt{a_0}\,\pi - \varepsilon)}{r} = \frac{1}{\sqrt{a_0}\,\pi - \varepsilon},$$

showing $D^+(\Lambda) \le (\sqrt{a_0}\,\pi)^{-1}$. ∎

The following infinite time observation identifiability has been proven in Theorem 3 of Chang (2008).

Lemma 9. The coefficient $a(\cdot)$ in (5) can be identified by $\{(u(t), w_t(1,t)), t \ge 0\}$.

In view of Theorem 1(i), Lemmas 8 and 9, we can now prove the finite time identifiability for the coefficient $a(\cdot)$ in (5).

Proof of Theorem 2. Since the control u does not vanish only possibly in time interval $[0,\tau)$, from the time moment τ on, the system (5) will become a free system. From (8), we have

$$\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} \text{ for } t \ge \tau, \tag{27}$$

where

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad D(\mathbb{A}) = \{(\varphi,\psi) \in \mathbb{H} \,|\, \mathbb{A}(\varphi,\psi) \in \mathbb{H}\}.$$

It is easily seen that $\mathbb{A}$ is a skew-adjoint operator in $\mathbb{H}$, and hence generates a C_0-group by Stone's theorem. Moreover, $\mathbb{A}$ has eigenpairs $\{\pm i\mu_n, \Psi_{\pm n}\}_{n=1}^{\infty}$:

$$\mathbb{A}\Psi_{\pm n} = \pm i\mu_n \Psi_{\pm n},$$

where

$$\Psi_n = \begin{pmatrix} -i\mu_n^{-1}\psi_n \\ \psi_n \end{pmatrix}, \quad \Psi_{-n} = \begin{pmatrix} i\mu_n^{-1}\psi_n \\ \psi_n \end{pmatrix}.$$

It is well-known that $\{\Psi_{\pm n}\}_{n=1}^{\infty}$ forms an orthonormal basis for $\mathbb{H}$. Hence, the state of the system (5) at time τ can be represented as

$$\begin{pmatrix} w(\cdot,\tau) \\ w_t(\cdot,\tau) \end{pmatrix} = \sum_{n=1}^{\infty} a_n \Psi_n + \sum_{n=1}^{\infty} c_n \Psi_{-n},$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ are square-summable sequences. The solution of the system (5) for $t \ge \tau$ can be represented as

$$\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = \sum_{n=1}^{\infty} a_n e^{i\mu_n(t-\tau)} \Psi_n + \sum_{n=1}^{\infty} c_n e^{-i\mu_n(t-\tau)} \Psi_{-n},$$

which yields

$$w_t(1,t;a) = \sum_{n=1}^{\infty} a_n \psi_n(1) e^{i\mu_n(t-\tau)} + \sum_{n=1}^{\infty} c_n \psi_n(1) e^{-i\mu_n(t-\tau)}$$

for $t \ge \tau$. By (26) and the Ingham theorem, the above expression makes sense because by (11), both $\{a_n\psi_n(1)\}_{n=1}^{\infty}$ and $\{c_n\psi_n(1)\}_{n=1}^{\infty}$ are square-summable.

In what follows, we write the solution of (5) as $w(\cdot,\cdot;a)$ instead of w for showing the dependence of w on $a(\cdot)$.

By Lemma 9, we need only show that for any $a(\cdot), \widetilde{a}(\cdot) \in Q$, $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost every $t \in [0, \tau + 4a_0^{-1/2}]$ can imply that $w_t(1,t;a) = w_t(1,t;\widetilde{a})$ for almost all $t \in [0,\infty)$. To do this, let $\widetilde{\mu}_n^2$ and $\widetilde{\psi}_n$ be the eigenpairs corresponding to $\widetilde{a}(\cdot)$. Then we have

$$w_t(1,t;\widetilde{a}) = \sum_{n=1}^{\infty} \widetilde{a}_n \widetilde{\psi}_n(1) e^{i\widetilde{\mu}_n(t-\tau)} + \sum_{n=1}^{\infty} \widetilde{c}_n \widetilde{\psi}_n(1) e^{-i\widetilde{\mu}_n(t-\tau)}$$

for $t \ge \tau$, where $\{\widetilde{a}_n\widetilde{\psi}_n(1)\}_{n=1}^{\infty}$ and $\{\widetilde{c}_n\widetilde{\psi}_n(1)\}_{n=1}^{\infty}$ are square-summable. Now set

$$f(t) = w_t(1,t;a) - w_t(1,t;\widetilde{a}).$$

Since by (26), $\Omega_1 = \{\mu_n, -\mu_n\}_{n\in\mathbb{Z}}$ and $\Omega_2 = \{\widetilde{\nu}_n, -\widetilde{\nu}_n\}_{n\in\mathbb{Z}}$ satisfy (3), it follows from Theorem 1(i) and Lemma 8 that $\{f(t), t \in [\tau,\infty)\}$ is uniquely determined by $\{f(t), t \in [\tau, \tau + 4a_0^{-1/2}]\}$. In particular, if $f(t) = 0$ for almost every $t \in [\tau, \tau + 4a_0^{-1/2}]$, then $f(t) = 0$ for almost every $t \in [\tau, \infty)$. This completes the proof. ∎

Proof of Corollary 3. It was found in Chang (2008) that the solution of (5) can be represented as

$$w(x,t) = \sum_{n=1}^{\infty} \psi_n(1)\psi_n(x) \int_0^t \frac{\sin \mu_n(t-\tau)}{\mu_n} u(\tau)\, d\tau,$$

and

$$w_t(x,t) = \sum_{n=1}^{\infty} \psi_n(1)\psi_n(x) \int_0^t \cos \mu_n(t-\tau) u(\tau)\, d\tau,$$

by which we have

$$w(1,t) = \sum_{n=1}^{\infty} \psi_n^2(1) \int_0^t \frac{\sin \mu_n(t-\tau)}{\mu_n} u(\tau)\, d\tau, \quad (28)$$

$$w_t(1,t) = \sum_{n=1}^{\infty} \psi_n^2(1) \int_0^t \cos \mu_n(t-\tau) u(\tau)\, d\tau. \quad (29)$$

As it was indicated before, the system (5) is well-posed in the sense of D.Salamon, so $w_t(1,t) \in L^2_{loc}(0,\infty)$ for any $u \in L^2_{loc}(0,\infty)$. From (28) and (29), it is easily checked that for any $h > 0$,

$$\int_0^h w_t(1,t)\phi dt = -\int_0^h w(1,t)\phi_t dt, \ \forall\, \phi \in C_0^\infty(0,h). \quad (30)$$

This shows that $w_t(1,t)$ is the weak derivative of $w(1,t)$ and is uniquely determined by $w(1,t)$. By Theorem 2, we then conclude that $a(\cdot)$ can be uniquely determined by $\{(u(t), w(1,t)), 0 \le t \le \tau + 4a_0^{-1/2}\}$. ∎

Now we turn to the proof of Theorem 5. To this purpose, we need several preliminary lemmas. The following inverse spectral result is due to Barcilon (see Chang and Guo (2007) for more details).

Lemma 10. $(\rho(\cdot), r(\cdot)) \in Q$ can be uniquely determined by $\{\omega_n, \phi_n(1), \phi_n'(1)\}_{n=1}^\infty$.

If follows from Chang (2008), the solution of (12) can be represented as

$$w(x,t) = -\sum_{n=1}^{\infty} \phi_n(1)\phi_n(x) \int_0^t \frac{\sin \omega_n(t-\tau)}{\omega_n} u(\tau) d\tau, \quad (31)$$

and

$$w_t(x,t) = -\sum_{n=1}^{\infty} \phi_n(1)\phi_n(x) \int_0^t \cos \omega_n(t-\tau) u(\tau)\, d\tau. \quad (32)$$

The following Lemma 11 comes from Lemma 3 of Chang (2008).

Lemma 11. $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\}_{n=1}^\infty$ are uniquely determined by $\{(u(t), w_t(1,t)), t \ge 0\}$.

Lemma 12. Suppose there is a positive constant τ such that the input $u(\cdot)$ in (12) vanishes in $[\tau, \infty)$. Then for each $T > \tau$, $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\}_{n=1}^\infty$ are uniquely determined by $\{(u(t), w_t(1,t)), t \in [0,T]\}$.

Proof. Analogous to the proof of Theorem 2, we can obtain

$$w_t(1,t;\rho,r) = \sum_{n=1}^{\infty} a_n \varphi_n(1) e^{i\omega_n(t-\tau)} + \sum_{n=1}^{\infty} c_n \varphi_n(1) e^{-i\omega_n(t-\tau)}$$

for $t \in [\tau, \infty)$, where $\{a_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ are square-summable, and so are $\{a_n\psi_n(1)\}_{n=1}^\infty$ and $\{c_n\psi_n(1)\}_{n=1}^\infty$ by (16). By (16), one can easily show that $D^+(\Omega) = 0$ for $\Omega = \{\omega_n, -\omega_n\}_{n=1}^\infty$. This together with *(i)* of Theorem 1 shows that for each $T > \tau$, $\{w_t(1,t), t \in [\tau, \infty)\}$ is uniquely determined by $\{w_t(1,t), t \in [\tau, T]\}$, that is, $\{w_t(1,t), t \in [0,\infty)\}$ is uniquely determined by $\{w_t(1,t), t \in [0,T]\}$. Therefore, it follows from Lemma 11 that $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\}_{n=1}^\infty$ can be uniquely determined by $\{(u(t), w_t(1,t)), 0 \le t \le T\}$. ∎

Analogous with Corollary 3, we can get Lemma 13 below from Lemma 12.

Lemma 13. Suppose there is a positive constant τ such that the input $u(\cdot)$ in (12) vanishes in $[\tau,\infty)$. Then for each $T > \tau$, $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\}_{n=1}^\infty$ are uniquely determined by $\{(u(t), w(1,t)), t \in [0,T]\}$.

In order to obtain another spectral sequence $\{\phi_n'(1)\}_{n=1}^\infty$ from finite time observation, we need the following Lemma 14.

Lemma 14. (Titchmarsh, 1948, Theorem 151) Assume that $P, g \in L^1(0,T)$, and for a positive constant $\tau < T$,

$$g(t) \neq 0 \text{ for almost all } t \in (0,\tau).$$

If

$$\int_0^t P(t-s)g(s)ds = 0 \text{ for almost all } t \in [0,T],$$

then $P(t) = 0$ for almost all $t \in [0, T-\tau]$.

Corollary 15. Assume that $g \in L^1(0,T)$, and for a positive constant $\tau < T$,

$$g(t) \neq 0 \text{ for almost all } t \in (0,\tau).$$

If the integral equation

$$\int_0^t P(t-s)g(s)ds = \varphi(t)$$

admits a solution $P \in L^1(0, T-\tau)$ for almost all $t \in [0,T]$, then P is unique.

The proof of the following Lemma 16, unlike Lemma 12, tells us how to directly obtain spectral data from some observation in finite time interval without replying on the infinite time observation.

Lemma 16. Suppose there is a constant $\tau > 0$ such that the input u in (12) satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then for every $T > \tau$, $\{\omega_n\}_{n=1}^\infty$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^\infty$ can be uniquely determined by $\{(u(t), w_x(1,t)), 0 \le t \le T\}$.

Proof. Since $D^+(\Omega) = 0$ for $\Omega = \{\omega_n, -\omega_n\}_{n=1}^\infty$, by the Ingham-Beurling theorem, for any $h > 0$, $\{e^{i\omega_n t}, e^{-i\omega_n t}\}_{n=1}^\infty$ forms an $\mathcal{L}$-basis in $L^2(0,h)$. Since

$$\sin \omega_n t = \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}, \quad (33)$$

the sequence

$$\left\{ \int_0^t \sin \omega_n(t-s) u(s) ds \right\}_{n=1}^\infty$$

is square-summable for each $t \ge 0$. This together with (16) implies that the series

$$\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(x)}{\omega_n} \int_0^t \sin \omega_n(t-s) u(s)\, ds$$

is uniformly convergent in x, which guarantees, from (31), that

$$w_x(x,t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(x)}{\omega_n} \int_0^t \sin \omega_n(t-s) u(s)\, ds.$$

Hence

$$w_x(1,t) = -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \int_0^t \sin\omega_n(t-s)u(s)\,ds.$$

It is easily shown that the above series is uniformly convergent on any finite time interval and $w_x(1,t)$ is a continuous function.

Next, set

$$\begin{aligned} P(t) &= -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \sin\omega_n(t) \\ &= -\sum_{n=1}^{\infty} \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i}. \end{aligned} \tag{34}$$

By basis property of $\{e^{i\omega_n t}, e^{-i\omega_n t}\}_{n=1}^{\infty}$ and (33), $P(\cdot)$ is well-defined, and so $P(\cdot) \in L^2_{loc}(0,\infty)$. We claim that

$$w_x(1,t) = \int_0^t P(t-s)u(s)ds. \tag{35}$$

In fact, set

$$P_n(t) = -\sum_{k=1}^{n} \frac{\phi_k(1)\phi_k'(1)}{\omega_n} \sin\omega_n(t).$$

Then for any $t > 0$, one has

$$\begin{aligned} &\left| \int_0^t P_n(t-s)u(s)\,ds - \int_0^t P(t-s)u(s)\,ds \right| \\ &\le \int_0^t |P_n(t-s) - P(t-s)|\,|u(s)|ds \\ &\le \left(\int_0^t |u(s)|^2\,ds \right)^{1/2} \left(\int_0^t |P_n(t-s) - P(t-s)|^2\,ds \right)^{1/2} \\ &\le \left(\int_0^t |u(s)|^2\,ds \right)^{1/2} \left(\int_0^t |P_n(s) - P(s)|^2\,ds \right)^{1/2}. \end{aligned}$$

This yields

$$\left| \int_0^t P_n(t-s)u(s)ds - \int_0^t P(t-s)u(s)ds \right| \to 0 \text{ as } n \to \infty$$

for $P_n(\cdot)$ converges to $P(\cdot)$ in $L^2(0,t)$. We thus have proved (35).

Finally, by Corollary 15, (35) implies that $\{P(t), t \in (0, T-\tau)\}$ is uniquely determined by $\{(u(t), w_x(1,t)), t \in [0,T]\}$. Since $D^+(\Omega) = 0$ for $\Omega = \{\omega_n, -\omega_n\}_{n=1}^{\infty}$, it follows from (34) and the Ingham-Beurling theorem that

$$\{\omega_n\}_{n=1}^{\infty} \quad \text{and} \quad \left\{ \frac{\phi_n(1)\phi_n'(1)}{\omega_n} \right\}_{n=1}^{\infty}$$

are uniquely determined by $\{P(t), t \in (0, T-\tau)\}$, which implies that $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ is uniquely determined by $\{P(t), t \in (0, T-\tau)\}$. Therefore, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(u(t), w_x(1,t)), t \in [0,T]\}$. The proof is complete. ∎

Proof of Theorem 5. By Lemma 12, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ are uniquely determined by $\{(u(t), w_t(1,t)), t \in [0,T]\}$. By Lemma 16, $\{\phi_n(1)\phi_n'(1)\}_{n=1}^{\infty}$ can be uniquely determined by $\{(u(t), w_x(1,t)), 0 \le t \le T\}$. Thus, $\{(\omega_n, \phi_n(1), \phi_n'(1))\}_{n=1}^{\infty}$ can be uniquely determined by $\{(u(t), w_t(1,t), w_x(1,t)), 0 \le t \le T\}$. By virtue of Lemma 10, this shows that $(\rho(\cdot), r(\cdot)) \in Q$ can be identified by $\{(u(t), w_t(1,t), w_x(1,t)), t \in [0,T]\}$. ∎

Lemma 17. Suppose there is a constant $\tau > 0$ such that the input u in (12) satisfies $u(t) \neq 0$ for almost all $t \in (0,\tau)$. Then for every $T > \tau$, $\{\omega_n\}_{n=1}^{\infty}$ and $\{\phi_n(1)\}_{n=1}^{\infty}$ can be uniquely determined by $\{(u(t), w(1,t)), 0 \le t \le T\}$.

Proof of Corollary 6. It is a consequence of combination of Lemma 10, Lemmas 16 and 17. ∎

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