Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Filtering of Discrete-time Markov Jump Linear Systems through Linear Matrix Inequalities ⋆

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Abstract: This paper addresses the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering design problems of discrete-time Markov jump linear systems. First, under the assumption that the Markov parameter is measured, the main contribution is on the LMI characterization of all filters such that the estimation error remains bounded by a given norm level, yielding the complete solution of the mode-dependent filtering design problem. Based on this result, a robust filter design to deal with convex bounded parameter uncertainty is considered. Second, from the same LMI characterization, a procedure for mode-independent filtering design is proposed. An example is solved for illustration and comparisons with the methods available in the literature.

Keywords: Markov models; Discrete-time systems; Kalman filters; Linear Matrix Inequalities.

1. INTRODUCTION

Dynamic systems that present sudden changes on their structures or parameters have been the subject of several studies in the last decades. Among the several ways to model such a dynamic system, one of increasing interest is the Markovian jump linear system (MJLS). One of the first works in the literature dealing with this class of models was presented in Blair and Sworder (1975). After that, a large amount of theory and design procedures have been developed in order to extend the concepts of the deterministic systems to this special class, namely stability concepts and testable conditions in Costa and Fragoso (1993), Ji et al. (1991), Ji and Chizeck (1988); optimal state feedback control in Ji and Chizeck (1990); state feedback $\mathcal{H}_2$ optimization via convex programming in Costa et al. (1997); state feedback $\mathcal{H}_2$ optimization via Linear Matrix Inequalities (LMIs) in do Val et al. (2002) and state feedback $\mathcal{H}_\infty$ optimization via LMIs in Costa and Marques (1998), Souza (2005) just to stay with a few of them.

An important assumption to consider for MJLS design problems is if the Markov chain state, often called mode, is available or not to the controller or filter at every instant of time $k \geq 0$. Based on that information, the design is said to be either mode-dependent or mode-independent, respectively.

The problem of determining a strictly proper optimal $\mathcal{H}_2$ mode-dependent filter was solved in Costa and Tuesta (2004) with the use of Coupled Algebraic Riccati Equations (CARE). For the mode-independent problem, a filter design using augmented matrices based on the Kronecker product was proposed in Costa and Guerra (2002a) and Costa and Guerra (2002b) where they design optimal mode-independent filters through CARE and LMIs, respectively. The filters obtained using this method are of order Nn, where N is the number of modes of the Markov chain and n is the order of the plant. In all cases, only strictly proper filters have been considered. Another aspect of the problem solved by Costa and Guerra (2002b) is that the output to be estimated must be independent of the system mode and input noise. It is important to stress that these are remarkable results as far as the optimality of the $\mathcal{H}_2$ mode-independent filter is concerned.

In addition, the problem of determining a strictly proper optimal $\mathcal{H}_\infty$ mode-dependent filter was solved in Souza and Fragoso (2003) with the use of LMIs. It is important to notice that in that paper only the gains of an observer-based filter are obtained, and the imposition of this particular filter structure based on the internal model of the plant makes the result useless for robust filter design. For the mode-independent problem, a filter design was proposed in Souza (2003), by solving sufficient conditions to impose a guaranteed level to the $\mathcal{H}_\infty$ norm of the estimation error, considering once again only strictly proper filters.

In this paper, initially the set of all full order mode-dependent proper filters with bounded $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the estimation error is obtained. This set is expressed in terms of LMIs, allowing the optimal filtering design problem to be solved in one shot rather than iteratively. The use of LMIs also allow us to add additional constraints to the basic problem, as for instance to design robust filters able to face parameter uncertainty. Another additional constraint to the main problem allows us to design mode-independent filters.

⋆ This research was supported by grants from "Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq" and "Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP", Brazil.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used. For real matrices or vectors $(')$ indicates transpose. For the sake of easing the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. The set of natural numbers is denoted by $\mathbb{N}$, while the finite set of the first N natural numbers $\{1, \cdots, N\}$ is denoted by $\mathbb{N}_N$. Given N^2 nonnegative real numbers p_{ij} satisfying $p_{i1} + \cdots + p_{iN} = 1$ for all $i \in \mathbb{N}_N$ and N real matrices X_j for all $j \in \mathbb{N}_N$, the convex combination of these matrices with weights p_{ij} is denoted by $X_{pi} = \sum_{j=1}^{N} p_{ij} X_j$. The symbol $\mathcal{E}\{\cdot\}$ denotes mathematical expectation of $\{\cdot\}$. For any stochastic signal $z(k)$, defined in the discrete time domain $k \in \mathbb{N}$, the quantity $\|z\|_2^2 = \sum_{k=0}^{\infty} \mathcal{E}\{z(k)'z(k)\}$ is its squared norm.

2. PROBLEM FORMULATION

A discrete-time Markovian jump linear system (MJLS) is described by the following stochastic equations

$$\mathbb{G} : \begin{cases} x(k+1) = A(\theta_k)x(k) + J(\theta_k)w(k) \\ z(k) = C_z(\theta_k)x(k) + E_z(\theta_k)w(k) \\ y(k) = C_y(\theta_k)x(k) + E_y(\theta_k)w(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^m$ is the external perturbation, $z(k) \in \mathbb{R}^r$ is the output to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output. It is assumed that the system evolves from $x(0) = 0$. The state space matrices (1) depend on a Markov chain taking values in the finite set $\mathbb{N}_N$ with the associated transition probability matrix $\mathbb{P} \in \mathbb{R}^{N\times N}$ whose elements are given by $p_{ij} = Prob(\theta_{k+1} = j \mid \theta_k = i)$, which satisfies the normalized constraints $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for each $i \in \mathbb{N}_N$. To ease the presentation, the following notations $A(\theta_k) := A_i$, $J(\theta_k) := J_i$, $C_z(\theta_k) := C_{zi}$, $E_z(\theta_k) := E_{zi}$, $C_y(\theta_k) := C_{yi}$ and $E_y(\theta_k) := E_{yi}$ whenever $\theta_k = i \in \mathbb{N}_N$ are used.

The next two propositions show how both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms can be calculated, respectively.

Lemma 1. The $\mathcal{H}_2$ norm of system $\mathbb{G}$ (with input w and output z) defined in (1) can be calculated by (see Costa et al. (1997)):

$$\|\mathbb{G}\|_2^2 = \inf_{(W_i,P_i)\in\Phi} \sum_{i=1}^{N} \mu_i \mathrm{Tr}\,(W_i) \tag{2}$$

where $\mu_i = Prob(\theta_0 = i \in \mathbb{N}_N)$ and Φ is the set of all positive definite matrices $(W_i, P_i) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{n\times n}$ for $i \in \mathbb{N}_N$ such that the following matrix inequalities

$$\begin{bmatrix} W_i & \bullet & \bullet \\ J_i & P_{pi}^{-1} & \bullet \\ E_{zi} & 0 & I \end{bmatrix} > 0 \tag{3}$$

and

$$\begin{bmatrix} P_i & \bullet & \bullet \\ A_i & P_{pi}^{-1} & \bullet \\ C_{zi} & 0 & I \end{bmatrix} > 0 \tag{4}$$

are satisfied for all $i \in \mathbb{N}_N$. Moreover, if we define a new scalar variable σ, impose N additional constraints $\mathrm{Tr}\,(W_i) < \sigma$ for all $i \in \mathbb{N}_N$ in the set Φ and calculate $\|\mathbb{G}\|_2^2 = \inf_{(\sigma,W_i,P_i)\in\Phi} \sigma$, we achieve the worst-case $\mathcal{H}_2$ norm, called this way because it equals the maximum value of the right hand side of (2) with respect to unknown initial probabilities $\mu_i, i \in \mathbb{N}_N$.

Lemma 2. The $\mathcal{H}_\infty$ norm of system $\mathbb{G}$ (with input w and output z) defined in (1) can be calculated by (see Seiler and Sengupta (2003)):

$$\|\mathbb{G}\|_\infty^2 = \inf_{(\gamma,P_i)\in\Phi} \gamma \tag{5}$$

where Φ is the set of all positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and $\gamma \in \mathbb{R}$ such that the following matrix inequality

$$\begin{bmatrix} P_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ A_i & J_i & P_{pi}^{-1} & \bullet \\ C_{zi} & E_{zi} & 0 & I \end{bmatrix} > 0 \tag{6}$$

holds for each $i \in \mathbb{N}_N$.

It is interesting to observe that in the deterministic case, characterized by $N = 1$, the previous definitions reduce to the usual $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the LTI discrete-time system $\mathbb{G}$. Also, the reason to define the set Φ through the matrices P_{pi}^{-1} will become clear in the sequence.

We are now in position to state the filtering design problem to be dealt with in the rest of this paper. Associated to (1), consider the full order mode-dependent linear filter

$$\mathbb{F} : \begin{cases} x_f(k+1) = A_f(\theta_k)x_f(k) + B_f(\theta_k)y(k) \\ z_f(k) = C_f(\theta_k)x_f(k) + D_f(\theta_k)y(k) \end{cases} \tag{7}$$

where $x_f(k) \in \mathbb{R}^n$, $x_f(0) = 0$ and the matrices A_{fi}, B_{fi}, C_{fi} and D_{fi} for all $i \in \mathbb{N}_N$ are of compatible dimensions. The goal is to determine these matrices in such a way that some norm of the estimation error is minimized. Connecting the filter (7) to the system (1), the estimation error satisfies

$$\mathbb{E} : \begin{cases} \tilde{x}(k+1) = \tilde{A}(\theta_k)\tilde{x}(k) + \tilde{J}(\theta_k)w(k) \\ e(k) = \tilde{C}(\theta_k)\tilde{x}(k) + \tilde{E}(\theta_k)w(k) \end{cases} \tag{8}$$

where the indicated matrices are given by

$$\tilde{A}_i := \begin{bmatrix} A_i & 0 \\ B_{fi}C_{yi} & A_{fi} \end{bmatrix} \qquad \tilde{J}_i := \begin{bmatrix} J_i \\ B_{fi}E_{yi} \end{bmatrix} \tag{9}$$

$$\tilde{C}_i := [C_{zi} - D_{fi}C_{yi} \;\; -C_{fi}] \qquad \tilde{E}_i := E_{zi} - D_{fi}E_{yi} \tag{10}$$

hence, the problem to be solved is written in the form

$$\min_{A_{fi},B_{fi},C_{fi},D_{fi}} \|\mathbb{E}\|_2^2 \tag{11}$$

for the $\mathcal{H}_2$ norm filtering design, and the correspondent $\mathcal{H}_\infty$ norm case is given by

$$\min_{A_{fi},B_{fi},C_{fi},D_{fi}} \|\mathbb{E}\|_\infty^2 \tag{12}$$

It is important to make clear that the above formulations of the filtering design problem are highly non convex and difficult to solve, that is, in these forms generally it is not possible to calculate their global optimal solutions. The reason is that the calculation of the objective functions depend on auxiliary variables that multiply the filter variables producing, consequently, non convex problems. The way to circumvent this difficulty is to introduce a one-to-one change of variables able to linearize the nonlinear constraints to be handled. This point, the main result of this paper, will be fully addressed in the next section.

3. MODE-DEPENDENT FILTERING DESIGN

Based on the previous results, our main purpose in this section is to calculate the global optimal solution of both the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ mode-dependent filtering design problem, given by (11) and (12) respectively.

Notice that for these problems, matrices A_i, J_i, C_{zi} and E_{zi} appearing in (3), (4) and (6) are replaced by matrices $\tilde{A}_i$, $\tilde{J}_i$, $\tilde{C}_i$ and $\tilde{E}_i$ respectively. Accordingly, matrices P_i and $P_{pi} = \sum_{i=1}^N p_{ij} P_j$ are $2n \times 2n$ real symmetric matrices partitioned as follows:

$$P_i = \begin{bmatrix} X_i & U_i \\ U_i' & \hat{X}_i \end{bmatrix},\ P_i^{-1} = \begin{bmatrix} Y_i & V_i \\ V_i' & \hat{Y}_i \end{bmatrix},\ T_i = \begin{bmatrix} I & I \\ V_i' Y_i^{-1} & 0 \end{bmatrix} \quad (13)$$

where all blocks are $n \times n$ real matrices. It is immediately verified that

$$T_i' P_i T_i = \begin{bmatrix} Z_i & Z_i \\ Z_i & X_i \end{bmatrix} \quad (14)$$

where $Z_i = Y_i^{-1}$, for all $i \in \mathbb{N}_N$. It is a well known fact that if the matrix in (14) is constrained to be definite positive then it is always possible to determine the matrix blocks in (13) in order to get $P_i > 0$. Moreover, this can be accomplished even if matrix U_i or matrix V_i is arbitrarily fixed and all the other ones are determined in such a way that $P_i P_i^{-1} = I$. Now, we proceed by considering $P_i > 0$ and adopting a similar reasoning to the convex combination of these matrices. From (13), the same partition yields

$$P_{pi} = \sum_{j=1}^N p_{ij} P_j = \begin{bmatrix} X_{pi} & U_{pi} \\ U_{pi}' & \hat{X}_{pi} \end{bmatrix} \quad (15)$$

and denoting

$$P_{pi}^{-1} = \begin{bmatrix} R_{1i} & R_{2i} \\ R_{2i}' & R_{3i} \end{bmatrix},\ Q_i = \begin{bmatrix} R_{1i}^{-1} & X_{pi} \\ 0 & U_{pi}' \end{bmatrix} \quad (16)$$

it is verified that

$$Q_i' P_{pi}^{-1} Q_i = \begin{bmatrix} R_{1i}^{-1} & R_{1i}^{-1} \\ R_{1i}^{-1} & X_{pi} \end{bmatrix} \quad (17)$$

It is important to stress that the four block matrices which define the inverse P_{pi}^{-1} depend nonlinearly on the four block matrices of P_{pi}. However, since $R_{1i}^{-1} = X_{pi} - U_{pi}\hat{X}_{pi}^{-1}U_{pi}'$, the partitioned matrix in (17) is linearized by setting U_i such that $U_i = -\hat{X}_i$, which implies that $U_{pi} = -\hat{X}_{pi}$ and

$$Q_i' P_{pi}^{-1} Q_i = \begin{bmatrix} X_{pi} + U_{pi} & X_{pi} + U_{pi} \\ X_{pi} + U_{pi} & X_{pi} \end{bmatrix} \quad (18)$$

From the above discussion, we mention again that the particular choice $U_i = -\hat{X}_i$ can be made with no loss of generality and constrains matrix U_i to be symmetric and negative definite. Furthermore, (13) provides $U_i = -\hat{X}_i = Y_i^{-1} - X_i = Z_i - X_i$ which enables us to rewrite (18) in the final form

$$Q_i' P_{pi}^{-1} Q_i = \begin{bmatrix} Z_{pi} & Z_{pi} \\ Z_{pi} & X_{pi} \end{bmatrix} \quad (19)$$

Surprisingly enough is to see that the proposed parametrization of matrices P_i and P_{pi}^{-1} converts (19) into the convex combination of (14). Moreover, in the general case, without taking into account the particular choice $U_i = \hat{X}_i$, matrices R_{1i} and Z_{pi} satisfy

$$\begin{aligned} R_{1i}^{-1} &= X_{pi} - U_{pi}\hat{X}_{pi}^{-1}U_{pi}' \\ &\geq \sum_{j=1}^N p_{ij}(X_j - U_j \hat{X}_j^{-1} U_j') \\ &\geq \sum_{j=1}^N p_{ij} Y_j^{-1} \\ &\geq Z_{pi} \end{aligned} \quad (20)$$

for all $i \in \mathbb{N}_N$. The relations (19) and (20) are the key results to be used afterwards for the filtering design. The next theorems, the main results of this paper, show that these matrix properties can be successfully used to linearize the matrix inequality of the filtering design problems (11) and (12).

Theorem 3. There exists a mode-dependent filter of the form (7) satisfying the constraint $\|\mathbb{E}\|_2^2 < \gamma$ if and only if there exist symmetric matrices X_i, Z_i, W_i and matrices M_i, L_i, F_i, K_i of compatible dimensions such that

$$\sum_{i=1}^N \mu_i \mathrm{Tr}(W_i) < \gamma \quad (21)$$

and the following LMIs

$$\begin{bmatrix} W_i & \bullet & \bullet & \bullet \\ Z_{pi} J_i & Z_{pi} & \bullet & \bullet \\ X_{pi} J_i + F_i E_{yi} & Z_{pi} & X_{pi} & \bullet \\ E_{zi} - K_i E_{yi} & 0 & 0 & I \end{bmatrix} > 0 \quad (22)$$

and

$$\begin{bmatrix} Z_i & \bullet & \bullet & \bullet & \bullet \\ Z_i & X_i & \bullet & \bullet & \bullet \\ Z_{pi} A_i & Z_{pi} A_i & Z_{pi} & \bullet & \bullet \\ X_{pi} A_i + F_i C_{yi} + M_i & X_{pi} A_i + F_i C_{yi} & Z_{pi} & X_{pi} & \bullet \\ C_{zi} - K_i C_{yi} + L_i & C_{zi} - K_i C_{yi} & 0 & 0 & I \end{bmatrix} > 0 \quad (23)$$

are satisfied $\forall i \in \mathbb{N}_N$. In the affirmative case, a suitable mode-dependent filter is given by $A_{fi} = (Z_{pi} - X_{pi})^{-1} M_i$, $B_{fi} = (Z_{pi} - X_{pi})^{-1} F_i$, $C_{fi} = -L_i$ and $D_{fi} = K_i$.

Proof. For the necessity, assume that inequalities (3) and (4) hold, with A_i, J_i, C_{zi} and E_{zi} replaced by $\tilde{A}_i$, $\tilde{J}_i$, $\tilde{C}_i$ and $\tilde{E}_i$ respectively. Partitioning P_{pi}^{-1} as in (16), multiplying (3) to the right by $\mathrm{diag}[I, Q_i, I]$ and to the left by its transpose, multiplying (4) to the right by $\mathrm{diag}[T_i, Q_i, I]$ and to left by its transpose and adopting the change of variables $M_i = U_{pi} A_{fi} V_i' Z_i$, $F_i = U_{pi} B_{fi}$, $L_i = -C_{fi} V_i' Z_i$ and $K_i = D_{fi}$ we obtain the LMIs (22) and (23) with all Z_{pi} replaced by R_{1i}^{-1}. Finally, multiplying (3) to the right by $\mathrm{diag}[I, R_{1i} Z_{pi}, I, I]$ and to left by its transpose and (4) to the right by $\mathrm{diag}[I, I, R_{1i} Z_{pi}, I, I]$ and to left by its transpose, and taking into account that (20) implies $Z_{pi} \geq Z_{pi} R_{1i} Z_{pi}$ for all $i \in \mathbb{N}_N$, we get inequalities (22) and (23), proving thus the necessity.

For the sufficiency, assume that inequalities (22) and (23) hold, which implies that $X_{pi} > Z_{pi} > 0$. Hence, defining $U_i = Z_i - X_i$, the matrix $U_{pi} = Z_{pi} - X_{pi}$ is nonsingular and we can define $A_{fi} = U_{pi}^{-1} M_i$, $B_{fi} = U_{pi}^{-1} F_i$, $C_{fi} = -L_i$ and $D_{fi} = K_i$. On the other hand, taking into account that this choice provides

$$P_{pi} = \begin{bmatrix} X_{pi} & \bullet \\ Z_{pi} - X_{pi} & X_{pi} - Z_{pi} \end{bmatrix} > 0 \quad (24)$$

it is immediately verified that (19) holds and that $R_{1i}^{-1} = Z_{pi}$. The conclusion is that inequality (22) can be rewritten as

$$\begin{bmatrix} W_i & \bullet & \bullet \\ Q_i'\tilde{J}_i & Q_i'P_{pi}^{-1}Q_i & \bullet \\ \tilde{E}_i & 0 & I \end{bmatrix} > 0 \tag{25}$$

which multiplied to the right by $\mathrm{diag}[I, Q_i^{-1}, I]$ and to the left by its transpose provides the inequality (3). Besides, (23) can be rewritten as

$$\begin{bmatrix} T_i'P_iT_i & \bullet & \bullet \\ Q_i'\tilde{A}_iT_i & Q_i'P_{pi}^{-1}Q_i & \bullet \\ \tilde{C}_iT_i & 0 & I \end{bmatrix} > 0 \tag{26}$$

which provides (4) after multiplication the above inequality to the right by $\mathrm{diag}[T_i^{-1}, Q_i^{-1}, I]$ and to the left by its transpose. This concludes the proof of the proposed theorem. □

The most important consequence of Theorem 3 is that the optimal global solution of the $\mathcal{H}_2$ filtering design problem (11) can be alternatively determined from

$$\inf_{(W_i,X_i,Z_i,M_i,F_i,L_i,K_i)\in\Omega} \sum_{i=1}^{N} \mu_i \mathrm{Tr}\,(W_i) \tag{27}$$

where Ω is the set of all feasible solutions of LMIs (22) and (23). In other words, the optimal filtering design problem under consideration has been converted into a convex programming problem expressed in terms of LMIs, which enables the use of efficient numerical methods for its solution. At this moment, a relevant point to be discussed is on the existence of a feasible solution to the set Ω. A necessary condition for that is obtained by applying the Schur Complement to the first three rows and columns of inequality (23) yielding

$$X_i > Z_i > 0\ ,\ A_i'Z_{pi}A_i - Z_i < 0 \tag{28}$$

which requires the system $\mathbb{G}$ be stable. This is a necessary assumption when dealing with robust or mode-independent filtering, two important design problems to be analyzed in the next section. In the present case, this assumption can be eliminated and problem (27) can be further simplified as indicated in the next corollary of Theorem 3.

Corollary 4. Problem (27) is equivalent to

$$\inf_{(W_i,X_i,F_i,K_i)\in\Psi} \sum_{i=1}^{N} \mu_i \mathrm{Tr}\,(W_i) \tag{29}$$

where Ψ is the set of all matrices W_i, X_i, F_i, K_i satisfying the LMIs

$$\begin{bmatrix} W_i & \bullet & \bullet \\ X_{pi}J_i + F_iE_{wi} & X_{pi} & \bullet \\ E_{zi} - K_iE_{yi} & 0 & I \end{bmatrix} > 0 \tag{30}$$

and

$$\begin{bmatrix} X_i & \bullet & \bullet \\ X_{pi}A_i + F_iC_{yi} & X_{pi} & \bullet \\ C_{zi} - K_iC_{yi} & 0 & I \end{bmatrix} > 0 \tag{31}$$

for all $i \in \mathbb{N}_N$. The state space realization of the optimal mode-dependent filter is given by $A_{fi} = A_i + X_{pi}^{-1}F_iC_{yi}$, $B_{fi} = -X_{pi}^{-1}F_i$, $C_{fi} = C_{zi} - K_iC_{yi}$ and $D_{fi} = K_i$.

Proof. Considering the optimal filter design problem (27), we notice that the matrix variables M_i and L_i appear only in the LMI (23) and consequently can be eliminated. To this end, applying the Finsler Lemma as indicated in Boyd et al. (2005), it is seen that (23) is satisfied for some M_i and L_i if and only if the LMIs

$$\begin{bmatrix} Z_i & \bullet & \bullet \\ Z_i & X_i & \bullet \\ Z_{pi}A_i & Z_{pi}A_i & Z_{pi} \end{bmatrix} > 0 \tag{32}$$

and

$$\begin{bmatrix} X_i & \bullet & \bullet & \bullet \\ Z_{pi}A_i & Z_{pi} & \bullet & \bullet \\ X_{pi}A_i + F_iC_{yi} & Z_{pi} & X_{pi} & \bullet \\ C_{zi} - K_iC_{yi} & 0 & 0 & I \end{bmatrix} > 0 \tag{33}$$

are simultaneously satisfied. The three inequalities (22), (32) and (33) have a common characteristic, namely at the optimal solution of problem (27) we must have $Z_i > 0$ and also $Z_{pi} > 0$ arbitrarily small. Hence, setting $Z_i \to 0$ for all $i \in \mathbb{N}_N$ the optimal solution located arbitrarily close to the boundary of the LMI constraints must satisfy (30) and (31). Finally, taking into account that $Z_{pi} \to 0$, the feasibility of (23) imposes $M_i = -X_{pi}A_i - F_iC_{yi}$ and $L_i = -C_{zi} + K_iC_{yi}$. The claim follows from the filter state space realization provided by Theorem 3. □

The same steps can be made for the $\mathcal{H}_\infty$ filtering problem, yielding the next results.

Theorem 5. There exists a mode-dependent filter of the form (7) satisfying the constraint $\|\mathbb{E}\|_\infty^2 < \gamma$ if and only if there exist symmetric matrices X_i, Z_i and matrices M_i, L_i, F_i, K_i of compatible dimensions satisfying the LMIs

$$\begin{bmatrix} Z_i & \bullet & \bullet & \bullet & \bullet & \bullet \\ Z_i & X_i & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \gamma I & \bullet & \bullet & \bullet \\ Z_{pi}A_i & Z_{pi}A_i & Z_{pi}J_i & Z_{pi} & \bullet & \bullet \\ \Xi_{1i} & X_{pi}A_i + F_iC_{yi} & X_{pi}J_i + F_iE_{yi} & Z_{pi} & X_{pi} & \bullet \\ \Xi_{2i} & C_{zi} - K_iC_{yi} & E_{zi} - K_iE_{yi} & 0 & 0 & I \end{bmatrix} > 0 \tag{34}$$

where $\Xi_{1i} = X_{pi}A_i + F_iC_{yi} + M_i$ and $\Xi_{2i} = C_{zi} - K_iC_{yi} + L_i$ for all $i \in \mathbb{N}_N$. In the affirmative case, a suitable mode-dependent filter is given by $A_{fi} = (Z_{pi} - X_{pi})^{-1}M_i$, $B_{fi} = (Z_{pi} - X_{pi})^{-1}F_i$, $C_{fi} = -L_i$ and $D_{fi} = K_i$.

Proof. Follows the same pattern of the proof of Theorem 3, and thus it is omitted. □

Again, the important consequence of Theorem 5 is that the optimal global solution of the $\mathcal{H}_\infty$ filter design problem (12) can be alternatively determined from

$$\inf_{(\gamma,X_i,Z_i,M_i,F_i,L_i,K_i)\in\Omega} \gamma \tag{35}$$

where Ω is the set of all feasible solutions of the LMIs (34).

As in the $\mathcal{H}_2$ filtering case, this problem can be further simplified by the use of the Finsler Lemma as stated in the next corollary.

Corollary 6. Problem (35) is equivalent to

$$\inf_{(\gamma,X_i,F_i,K_i)\in\Psi} \gamma \tag{36}$$

where Ψ is the set of all matrices X_i, F_i, K_i and the scalar $\gamma \in \mathbb{R}$ such that the LMI

$$\begin{bmatrix} X_i & \bullet & \bullet & \bullet \\ 0 & \gamma I & \bullet & \bullet \\ X_{pi}A_i + F_iC_{yi} & X_{pi}J_i + F_iE_{yi} & X_{pi} & \bullet \\ C_{zi} - K_iC_{yi} & E_{zi} - K_iE_{yi} & 0 & I \end{bmatrix} > 0 \tag{37}$$

holds for each $i \in \mathbb{N}_N$. The state space realization of the optimal mode-dependent filter is given by $A_{fi} = A_i + X_{pi}^{-1} F_i C_{yi}$, $B_{fi} = -X_{pi}^{-1} F_i$, $C_{fi} = C_{zi} - K_i C_{yi}$ and $D_{fi} = K_i$.

Proof. Again, it follows the same pattern of the proof of Corollary 4, being thus omitted. □

Corollary 4 and 6 deserve some comments. First, for $N = 1$ they reduce to the well known Kalman filter and $\mathcal{H}_\infty$ central filter for LTI systems, respectively. Second, defining the gain $G_i = -X_{pi}^{-1} F_i$, the optimal mode-dependent $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters are given by

$$\begin{aligned} x_f(k+1) &= A(\theta_k)x_f(k) + G(\theta_k)(y(k) - C_y(\theta_k)x_f(k)) \\ z_f(k) &= C_z(\theta_k)x_f(k) + K(\theta_k)(y(k) - C_y(\theta_k)x_f(k)) \end{aligned} \tag{38}$$

which exhibits the classical form of an observer based on the internal model of the plant.

Clearly, if the plant parameters or the state of the Markov chain are not exactly known then the solutions provided by Corollaries 4 and 6 can not be attained. However, in these two important cases, the solutions provided by Theorems 3 and 5 still apply. These particular features of the filter design problem will be fully addressed in the next section.

4. ROBUST AND MODE-INDEPENDENT FILTERING

In this section we consider two problems for which the results of Corollaries 4 and 6 do not apply. In other words, for these problems, the global optimal solution can not be attained with $Z_i \to 0$ due to the additional constraints one has to take into account. It is important to stress that, in the present context, only the sufficient part of the previous results remains true.

4.1 Robust Filter

If the system matrices appearing in (1) are not exactly known, but belongs to the convex polytope

$$\mathcal{P} = \text{co}\left\{ \begin{bmatrix} A_i^l & J_i^l \\ C_{zi}^l & E_{zi}^l \\ C_{yi}^l & E_{yi}^l \end{bmatrix}, \ l = 1, \cdots, N_p \right\} \tag{39}$$

defined by the convex hull of N_p vertices then a minimum guaranteed cost is given from the optimal solution of problem (27) or (35), with $\Omega = \bigcap_{l=1}^{N_p} \Omega_l$ where Ω_l is the set defined by respective LMIs calculated at each vertex of the uncertain polytope (39). As it can be easily verified, this claim follows from the linear dependence of those LMIs with respect to the plant state space matrices.

4.2 Mode-independent Filter

Theorems 3 and 5 can be generalized to cope with mode-independent filtering. The main goal is to get a filter such that the state space representation does not depend on the Markov mode $i \in \mathbb{N}_N$. We first observe that imposing to Ω the additional constraints

$$M_i = M, \ F_i = F, \ L_i = L, \ K_i = K \tag{40}$$

for all $i \in \mathbb{N}_N$ then from the Theorems in question, the filter will be mode-independent if the transition probability matrix satisfies $p_{ij} = p_j$ for all $(i, j) \in \mathbb{N}_N \times \mathbb{N}_N$ (see Seiler and Sengupta (2005)), since in this particular case $Z_{pi} = \sum_{j=1}^N p_j Z_j$ and $X_{pi} = \sum_{j=1}^N p_j X_j$ are constant matrices for all $i \in \mathbb{N}_N$. In the general case, that is, if the previous condition $p_{ij} = p_j$ does not hold, the mode-independent filter design needs the additional constraints (40) together with

$$X_{pi} - Z_{pi} = S \tag{41}$$

for all $i \in \mathbb{N}_N$, where $S \in \mathbb{R}^{n\times n}$ is a matrix variable such that $S > 0$ as required for the feasibility of Ω. From Theorems 3 and 5, the optimal solution of problems (27) and (35), with the additional convex constraints (40) and (41), yields the mode-independent filter state space realization $A_{fi} = -S^{-1}M$, $B_{fi} = -S^{-1}F$, $C_{fi} = -L$ and $D_{fi} = K$ assuring the norm of the estimation error be such that $\|\mathbb{E}\|_2^2 \le \inf \sum_{i=1}^N \mu_i \text{Tr}(W_i)$ or $\|\mathbb{E}\|_\infty^2 \le \inf \gamma$, respectively.

The constraint (41) admits an interesting interpretation. Since $\sum_{j=1}^N p_{ij} = 1$, it is satisfied whenever $X_j - Z_j = S$ for all $j \in N_N$. In this case, from Theorems 3 and 5, we conclude that the corresponding solution to the Lyapunov inequalities has the particular form

$$P_i = \begin{bmatrix} X_i & \bullet \\ -S & S \end{bmatrix} \tag{42}$$

for all $i \in \mathbb{N}_N$. This is quite expected in the sense that only the plant but not the filter may be dependent on the actual mode.

In the next section, an illustrative example is solved for numerical comparison and performance evaluation.

5. EXAMPLES AND COMPARISONS

In this section, a simple example is considered. It consists of two masses coupled by a spring and a damper. The first mass is also connected to a wall through another spring. The problem is to estimate the position of the second mass measuring the velocity of the first one with an associated error. Furthermore, this information is delivered to the estimator through a Markovian channel, which can insert error to each information package. We assume that the estimator can detect but not correct the defected packages, and in this case it will discard them. The probability that a good package is received just after another good one is given by $p_{R,R}$, whereas the probability that a bad package is received after another bad one is given by $p_{L,L}$, (see Seiler and Sengupta (2005) and Seiler (2001)).

The state space realization of this continuous-time system is described as usual with the specific data given below. Notice that the system dynamics and the output to be estimated are the same for the two modes, while the measured output is mode-dependent.

$$\left[\begin{array}{c|c} A_i & J_i \\ \hline C_{zi} & E_{zi} \\ \hline C_{y1} & E_{y1} \\ \hline C_{y2} & E_{y2} \end{array}\right] = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -30.0 & 10.0 & -0.36 & 0.36 & 0 & 0 \\ 5.0 & -5.0 & 0.18 & -0.18 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0.1 \end{array}\right] \tag{43}$$

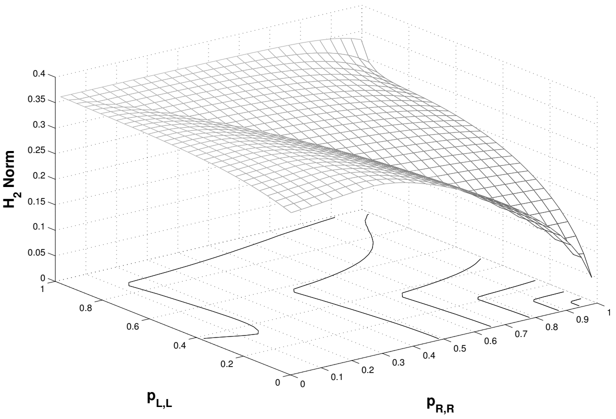

Fig. 1. Performance of the $\mathcal{H}_2$ proper filter versus packet loss rate : Two state Markov channel.

This system was in sequence discretized with a zero order hold in each input and sample time $T_s = 0.5$ *sec* providing the discrete-time model (1). All the worst-case $\mathcal{H}_2$ mode-dependent proper filters for a grid of transition probability matrices were calculated, as it can be seen in Figure 1.

Finally, for the sake of comparison, we considered a high-quality channel ($p_{R,R} = 0.98$) subjected to burst errors ($p_{L,L} = 0.95$). A model with these kind of characteristics might be adequate to explain a channel that most of the time has a very reliable communication, but the errors come in chunks, for example, due to an external environmental interference that damages the transmission packets.

For this part we have calculated the suboptimal, strictly proper $\mathcal{H}_\infty$ norm mode-independent filter given by (35) with the additional constraints (40) and (41), and we achieved a filter with a guaranteed cost of $\|\mathbb{E}\|_\infty \leq 2.0141$, while the filter proposed in Souza (2003) achieved a guaranteed cost of $\|\mathbb{E}\|_\infty \leq 2.1403$.

6. CONCLUSION

In this paper, a LMI design method for mode-dependent $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters was presented. Comparing to the design procedures in the literature, the one proposed here allows the treatment of a wider class of problems of theoretical and practical importance, by simply including additional linear constraints to the basic design problem. This is precisely the case for robust and mode-independent filter design. For the mode-independent filtering design problem, it was shown by means of a simple example that our method can outperform the one proposed in Souza (2003). Moreover, the fact that we are able to handle proper filters instead of strictly proper ones reduces conservatism. We have also shown that for systems with the particular structure of the transition probability matrix $p_{ij} = p_j$ for all $i \in \mathbb{N}_N$ a mode-independent filter is obtained fairly easily. This simplified probability matrix has been adopted in Seiler and Sengupta (2005) to address the $\mathcal{H}_\infty$ control problem.

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