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Least restrictive move-blocking model predictive control

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Abstract: The authors recently proposed an approach to enforce strong feasibility in move-blocking model predictive control problems. In this paper the approach is utilized to design strong model predictive control problems which generate least restrictive controllers. The domains of different move-blocking regimes are thus equal and the same as the non-move-blocking maximum. This allows comparison of different move-blocking regimes based on cost performance only, without considering domain size also. A numerical example and case study demonstrate that between input- and offset-move-blocking, the latter is generally superior.

Keywords: Predictive control; constrained control; move-blocking; feasibility; set invariance.

1. INTRODUCTION

Move-blocking is used in many practical online model predictive control (MPC) systems in order to reduce the computational complexity of the associated finite-horizon optimal control problem [Qin & Badgewell (1997); Cagienard *et al.* (2007)]. In move-blocking MPC problems a predicted control move sequence of N prediction steps is parameterized by a sequence of $\tilde{N} < N$ predicted control moves, where predicted control moves are held constant over sets of multiple prediction steps. This method is primarily targeted at complexity reduction in online MPC methods but was demonstrated to reduce complexity in offline MPC methods also [Tøndel & Johansen (2002)].

Strong feasibility describes the quality that the closed-loop state trajectory from any feasible initial state never reaches an infeasible state [Kerrigan (2000)]. In non-move-blocking MPC problems, strong feasibility can be enforced using terminal state constraints [Mayne *et al.* (2000)]. However, terminal constraints fail to enforce strong feasibility in move-blocking MPC problems [Cagienard *et al.* (2007)]. In Gondhalekar & Imura (2007a,b) the authors presented a novel approach to strengthening MPC problems, applicable to move-blocking and more general MPC problems. In this approach the state at the first prediction step is constrained to lie within the *maximal controlled invariant feasible* set of the given MPC problem. This results in the least-restrictive MPC control law for the given MPC problem, with the controller domain equal to the maximal controlled invariant feasible set. Depending on the move-blocking scheme, the maximal controlled invariant feasible set can be very small when compared to the non-move-blocking maximum, which is equal to the maximal controlled invariant set [Blanchini (1999); Kerrigan (2000); Vidal *et al.* (2000)]. This reduction from the maximum is unwelcome firstly because the controller has a restricted domain, and secondly because controller comparison and optimization is difficult when both domain size and cost performance are different [Gondhalekar & Imura (2007c)].

In this paper an approach is presented to relax the move-blocking MPC problem such that the resulting maximal controlled invariant feasible set is equal to that of the non-move-blocking maximum. This is achieved by relaxing the prediction state constraints beyond the first prediction step. The resulting relaxed MPC problem retains strong feasibility. For a given plant the controller domain is then equal to the maximal controlled invariant set regardless of prediction horizon length and move-blocking scheme used. This allows comparison of different move-blocking schemes based on the cost performance only. In the last part of this paper a numerical example and case-study are presented which compare input- and offset-move-blocking [Cagienard *et al.* (2007); Gondhalekar & Imura (2007a,b)] for a large number of different plants. The results indicate that offset-move-blocking performs better in general, as is predicted by the theory (see Section 2.3). The proposed approach is also shown to be relevant for offline MPC methods, where significant reductions in controller complexity are possible. For simplicity this paper limits itself to linear-quadratic MPC with polytopic constraint sets containing the origin.

Notation: The real number set is denoted by $\mathbb{R}$ and the set of non-negative integers by $\mathbb{N}$ ($\mathbb{N}_+ := \mathbb{N}\backslash 0$). Denote by $I_n \in \{0,1\}^{n\times n}$ the identity matrix, by $0_{\{n,m\}} \in 0^{n\times m}$ the zero matrix and by 0 without subscript the zero matrix with dimension deemed obvious by context. The Kronecker product of matrices A and B is denoted by $A \otimes B$. For set $\mathbb{X} \subset \mathbb{R}^n$, $\mathrm{Vol}(\mathbb{X}) := \int_{\mathbb{X}} \mathrm{d}x$ denotes the volume. A sequence of elements $x_i \in \mathbb{X}\ \forall i \in \{j,\ldots,k\}$ is denoted by $\{x_i \in \mathbb{X}\}_{i=j}^{k}$. If the elements' parent set is obvious by context the sequence is denoted by $\{x_i\}_{i=j}^{k}$. Let $i \in \mathbb{N}$ denote a system's actual step index, while $k \in \mathbb{N}$ denotes the $(i+k)^{\text{th}}$ step as predicted from actual step i. Thus, $\psi_{(i,k)}$ denotes the future value of variable ψ at step $i+k$, as predicted from step i. For compact notation $\psi_{(i,0)} \equiv \psi_i$.

2. MOVE-BLOCKING MPC

There are various kinds of move-blocking regime [Cagienard *et al.* (2007)]. This paper concerns itself with two of them; input- and offset-move-blocking [Gondhalekar & Imura (2007a,b)]. For simplicity the strongly feasible MPC problem formulation (Sections 2.1, 2.2) and the MPC problem relaxation (Section 3) are explained at hand of input-move-blocking only. The derivation for offset-move-blocking follows analogously. A discussion on the basic difference between input- and offset-move-blocking is provided in Section 2.3. In Sections 4 and 5 both input- and offset-move-blocking are investigated. Subscript 'I' is used to denote variables relevant to input-move-blocking, while subscript 'O' denotes those of offset-move-blocking.

2.1 Input-Move-Blocking MPC

Consider the linear time-invariant controlled system

$$x_{i+1} = Ax_i + Bu_i \tag{1}$$

with step index $i \in \mathbb{N}$, system state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, state transition matrix $A \in \mathbb{R}^{n\times n}$ and input distribution matrix $B \in \mathbb{R}^{n\times m}$. The pair (A, B) is assumed stabilizable. State and control input must satisfy

$$x_i \in \mathbb{X} \subseteq \mathbb{R}^n \quad \wedge \quad u_i \in \mathbb{U} \subseteq \mathbb{R}^m \quad \forall i \in \mathbb{N}, \tag{2}$$

with polytopic constraint sets $\mathbb{X} := \{x \in \mathbb{R}^n | G_\mathrm{x} x \le W_\mathrm{x}\}$, $G_\mathrm{x} \in \mathbb{R}^{g_\mathrm{x}\times n}$ and $\mathbb{U} := \{x \in \mathbb{R}^m | G_\mathrm{u} x \le W_\mathrm{u}\}$, $G_\mathrm{u} \in \mathbb{R}^{g_\mathrm{u}\times m}$, containing the origin within their interior. MPC achieves closed-loop control action of system (1) subject to constraints (2) by applying at each step i the first control input of an optimal predicted open-loop control input trajectory, determined by the solution of N-step ($N \in \mathbb{N}_+$, $N < \infty$) finite-horizon optimal control Problem 1.

Problem 1. Find:

$$\begin{aligned} \mathcal{U}_i^*(x_i) := \arg \min_{\mathcal{U}_i := \{u_{(i,k)} \in \mathbb{U}\}_{k=0}^{N-1}} & \Big\{ x_{(i,N)}^\mathrm{T} P x_{(i,N)} \\ & + \sum_{k=0}^{N-1} \left[x_{(i,k)}^\mathrm{T} Q x_{(i,k)} + u_{(i,k)}^\mathrm{T} R u_{(i,k)} \right] \Big\} \\ \text{s.t. } \; x_{(i,k)} \in \mathbb{X} \;\; & \forall k \in \{0, \ldots, N\} \quad \wedge \\ x_{(i,k+1)} = Ax_{(i,k)} + Bu_{(i,k)} \;\; & \forall k \in \{0, \ldots, N-1\} \; . \end{aligned} \tag{3}$$

Terminal cost matrix P solves the discrete-time Riccati equation $P = A^\mathrm{T}PA + Q - A^\mathrm{T}PB\left(B^\mathrm{T}PB + R\right)^{-1} B^\mathrm{T}PA$.

Assumption 2. $Q \succ 0$, $R \succ 0$.

Writing $\mathcal{U}_i$ in vector form $U_i := \left[u_{(i,0)}^\mathrm{T}, \ldots, u_{(i,N-1)}^\mathrm{T}\right]^\mathrm{T} \in \mathbb{U}^N$, MPC Problem 1 can be reformulated into matrix form [Bemporad *et al.* (2002); Borrelli (2003)].

Problem 3. Find:

$$\begin{aligned} U_i^*(x_i) &:= \arg \min_{U_i \in \mathbb{R}^{Nm}} \left\{ U_i^\mathrm{T} H_\mathrm{I} U_i + U_i^\mathrm{T} L_\mathrm{I} x_i \right\} \\ \text{s.t. } \; & G_\mathrm{I} U_i \le W_\mathrm{I} + E_\mathrm{I} x_i \end{aligned}$$

where H_I, L_I, G_I, W_I and E_I are constructed from A, B, Q, R, P, G_x, W_x, G_u and W_u. Suppose W_I is constructed as: $W_\mathrm{I} = \left[{W_\mathrm{x}}^\mathrm{T}, {W_\mathrm{u}}^\mathrm{T}, {W_\mathrm{x}}^\mathrm{T}, {W_\mathrm{u}}^\mathrm{T}, \cdots, {W_\mathrm{x}}^\mathrm{T}\right]^\mathrm{T} \in \mathbb{R}^{N(g_\mathrm{x}+g_\mathrm{u})+g_\mathrm{x}}$.

In move-blocking schemes, control input trajectory $U_i \in \mathbb{R}^{Nm}$ of N control moves is parameterized by a control input trajectory $\tilde{U}_i := \left[\tilde{u}_{(i,0)}^\mathrm{T}, \ldots, \tilde{u}_{(i,\tilde{N}-1)}^\mathrm{T}\right]^\mathrm{T} \in \mathbb{R}^{\tilde{N}m}$ of $\tilde{N} < N$ ($\tilde{N} \in \mathbb{N}_+$) control moves and a blocking matrix $M \in \{0,1\}^{N\times\tilde{N}}$ according to [Cagienard *et al.* (2007); Gondhalekar & Imura (2007a,b); Tøndel & Johansen (2002)]:

$$U_i = (M \otimes I_m)\tilde{U}_i = \tilde{M}\tilde{U}_i \quad , \quad \tilde{M} := M \otimes I_m \; . \tag{4}$$

Substituting input trajectory parameterization (4) into MPC Problem 3 yields move-blocking MPC Problem 4:

Problem 4. Find:

$$\begin{aligned} \tilde{U}_i^*(x_i) &:= \arg \min_{\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}} \left\{ \tilde{U}_i^\mathrm{T} \tilde{M}^\mathrm{T} H_\mathrm{I} \tilde{M} \tilde{U}_i + \tilde{U}_i^\mathrm{T} \tilde{M}^\mathrm{T} L_\mathrm{I} x_i \right\} \\ \text{s.t. } \; & G_\mathrm{I} \tilde{M} \tilde{U}_i \le W_\mathrm{I} + E_\mathrm{I} x_i \; . \end{aligned}$$

Note that in this work the blocking matrix M is fixed and does not change with system step i. Thus this work focuses on *time-invariant* move-blocking schemes.

2.2 Strong Feasibility in Input-Move-Blocking MPC

When $M = I_N$, move-blocking MPC Problem 4 is equivalent to MPC Problem 3, and is referred to as the *full degree of freedom MPC problem.* Strong feasibility [Kerrigan (2000)] of full degree of freedom MPC problems is enforced via terminal constraints [Mayne *et al.* (2000)]. However, terminal constraints fail to enforce strong feasibility of move-blocking MPC Problem 4 when $M \ne I_N$ [Cagienard *et al.* (2007)]. The authors developed a generalized method for enforcing strong feasibility in MPC problems, applicable to move-blocking and more general MPC problems. The method is stated here to enable further discussion. For a thorough discussion consult Gondhalekar & Imura (2007a,b). The strongly feasible move-blocking MPC problem based on MPC Problem 4 is as follows:

Problem 5. Find:

$$\begin{aligned} \tilde{U}_i^*(x_i) &= \arg \min_{\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}} \left\{ \tilde{U}_i^\mathrm{T} \tilde{M}^\mathrm{T} H_\mathrm{I} \tilde{M} \tilde{U}_i + \tilde{U}_i^\mathrm{T} \tilde{M}^\mathrm{T} L_\mathrm{I} x_i \right\} \\ \text{s.t. } \; & G_\mathrm{I} \tilde{M} \tilde{U}_i \le W_\mathrm{I} + E_\mathrm{I} x_i \quad \wedge \quad x_{(i,1)} \in \mathbb{F}_\mathrm{I} \end{aligned}$$

where maximal *controlled invariant feasible* set $\mathbb{F}_\mathrm{I}$ is defined as:

$$\begin{aligned} \mathbb{F}_\mathrm{I} := \Big\{ & x_0 \in \mathbb{X} | \exists \{\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}\}_{i=0}^{\infty} \;\text{ s.t. } \; \forall i \in \mathbb{N}, \\ & G_\mathrm{I} \tilde{M} \tilde{U}_i \le W_\mathrm{I} + E_\mathrm{I} x_i \quad \wedge \quad x_{i+1} = Ax_i + BF\tilde{M}\tilde{U}_i \in \mathbb{X} \Big\} \; , \\ F := & \left[\, I_m \;,\; 0_{\{m,(N-1)m\}} \,\right] \; . \end{aligned}$$

The optimal control law $\kappa_\mathrm{I} : \mathbb{R}^n \to \{\mathbb{U}, \emptyset\}$ resulting from strongly feasible MPC Problem 5 is given implicitly by $u_i^* = \kappa_\mathrm{I}(x_i) := \tilde{u}_{(i,0)}^*(x_i) = F\tilde{M}\tilde{U}_i^*(x_i)$, or explicitly [Bemporad *et al.* (2002); Borrelli (2003)] as a piecewise affine (PWA) function: $\kappa_\mathrm{I}(x_i) = K_\mathrm{I}^\mathrm{s} x_i + a_\mathrm{I}^\mathrm{s}$ if $x_i \in \mathcal{X}_\mathrm{I}^\mathrm{s}$, $\mathrm{s} \in \{1, \ldots, \mathrm{S}_\mathrm{I}\}$, where S_I denotes the number of regions in the PWA partition. If MPC Problem 5 is infeasible for x_i we write $\kappa_\mathrm{I}(x_i) = \emptyset$. The set $\{x \in \mathbb{R}^n | \kappa_\mathrm{I}(x) \ne \emptyset\} = \bigcup_{\mathrm{s}=1}^{\mathrm{S}_\mathrm{I}} \mathcal{X}_\mathrm{I}^\mathrm{s}$ is termed the *domain* of controller κ_I. The closed-loop state trajectory evolves according to $x_{i+1} = Ax_i + B\kappa_\mathrm{I}(x_i)$.

2.3 Offset-Move-Blocking MPC

In offset-optimization MPC [Rossiter *et al.* (1997)], the system is pre-regulated by unconstrained LQR state feedback such that $u_{(i,k)} = K_{\text{LQR}} x_{(i,k)} + c_{(i,k)}\ \forall (i,k) \in \mathbb{N} \times \{0,\dots,N-1\}$, $K_{\text{LQR}} := -\left(R + B^{\mathrm{T}}PB\right)^{-1} B^{\mathrm{T}}PA$. The terms $c_{(i,k)} \in \mathbb{R}^m$ are the control input offsets from the unconstrained optimal control inputs which take into account system constraints. The offset-move-blocking MPC problem formulation proceeds analogously to the above input-move-blocking case, where instead of parameterized predicted open-loop control input-trajectory $\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}$, the optimization variable is the parameterized predicted open-loop control input offset-trajectory $\tilde{C}_i := \left[\tilde{c}^{\mathrm{T}}_{(i,0)},\dots,\tilde{c}^{\mathrm{T}}_{(i,\tilde{N}-1)}\right]^{\mathrm{T}} \in \mathbb{R}^{\tilde{N}m}$. The resulting controller is denoted by κ_{O}. For details see Cagienard *et al.* (2007), Gondhalekar & Imura (2007b), Rossiter *et al.* (1997).

Remark 6. In the full degree of freedom case when $M = I_N$, input-move-blocking controller κ_{I} and offset-move-blocking controller κ_{O} are the same. When $M \neq I_N$ they are, in general, different. Surrounding the origin is a region where no constraint is active. In this region the constrained MPC problem is equivalent to the unconstrained MPC problem, i.e. when $\mathbb{U} = \mathbb{R}^m$ and $\mathbb{X} = \mathbb{R}^n$. In this region, offset-move-blocking controller κ_{O} is infinite-horizon optimal. It corresponds to the unconstrained LQR optimum, because the sequence of predicted control input offsets equal to zero is the optimum *for any* blocking matrix M. This cannot be said for input-move-blocking controller κ_{I}, which is not guaranteed to provide the LQR optimal solution for any state except the origin. Offset-move-blocking might therefore be expected to outperform input-move-blocking. However, the domain of controllers κ_{I} and κ_{O} are in general different. In Cagienard *et al.* (2007) it was reported that domains of offset-move-blocking MPC controllers may be significantly smaller than those of input-move-blocking ones.

3. PREDICTION CONSTRAINT RELAXATION

Maximal controlled invariant set $\mathbb{C} := \left\{x_0 \in \mathbb{X} | \exists \{u_i \in \mathbb{U}\}_{i=0}^{\infty} \text{ s.t. } x_{i+1} = Ax_i + Bu_i \in \mathbb{X}\ \forall i \in \mathbb{N}\right\}$ [Blanchini (1999); Vidal *et al.* (2000)] is the largest set of states from which a permissible control input trajectory exists such that the resulting closed-loop state trajectory satisfies all state constraints. Clearly it is not possible to design an MPC controller (or any) with a domain larger than the maximal controlled invariant set. If $M = I_N$ then $\mathbb{F}_{\text{I}} = \mathbb{C}$. However, for $M \neq I_N$ maximal controlled invariant feasible set $\mathbb{F}_{\text{I}}$ is in general a subset of the maximal controlled invariant set: $\mathbb{F}_{\text{I}} \subseteq \mathbb{C}$. In the numerical example of Section 4 the difference in size of the domain is considerable. The decrease in domain size of move-blocking MPC controllers was also reported in Cagienard *et al.* (2007) and Gondhalekar & Imura (2007a,b). The reason for this decrease is obvious. For the same prediction state constraints of Eq. (3), reducing the degrees of freedom of the predicted open-loop control input trajectory may cause states close to the boundary of the maximal controlled invariant set to become infeasible. Consider a state $x_0 \in \mathbb{D}$, $\mathbb{D} := \mathbb{C} \setminus \mathbb{F}_{\text{I}}$. There exists a sequence of inputs $\{u_i \in \mathbb{U}\}_{i=0}^{\infty}$ such that $x_{i+1} = Ax_i + Bu_i \in \mathbb{X}\ \forall i \in \mathbb{N}$. However, there does not exist a sequence of parameterized predicted open-loop input trajectories $\{\tilde{U}_i \in \mathbb{U}^{\tilde{N}m}\}_{i=0}^{\infty}$ such that $\forall i \in \mathbb{N}$, $G_{\text{I}} \tilde{M} \tilde{U}_i \leq W_{\text{I}} + E_{\text{I}} x_i \quad \wedge \quad x_{i+1} = Ax_i + BF\tilde{M}\tilde{U}_i \in \mathbb{X}$.

Enforcing strong feasibility by constraints on the state at the first prediction step, as opposed to terminal constraints, allows a degree of flexibility in choosing the prediction constraints. Both prediction state and control input constraints beyond the first prediction step affect only the predicted open-loop system's performance of the model inside the controller, which is a software issue. Only constraints on the first prediction step directly affect physical control inputs and the resulting closed-loop state trajectory. Prediction constraints which *must* be satisfied are: $u_{(i,0)} \in \mathbb{U} \wedge x_{(i,k)} \in \mathbb{X}\ \forall k \in \{0,1\}$. In this section a relaxation procedure of the prediction state[1] constraints $x_{(i,k)} \in \mathbb{X}_k\ \forall k \in \{2,\dots,N\}$ is developed, where prediction state constraint sets $\mathbb{X}_k := \{x \in \mathbb{R}^n | G_{\text{x}} x \leq W_{\text{x}} + \delta_k\} \supseteq \mathbb{X}$, $\delta_k \in \mathbb{R}^{g_{\text{x}}}$, $\delta_k \geq 0$ have been minimally increased in size from the actual state constraint set $\mathbb{X}$ such that maximal controlled invariant feasible set $\hat{\mathbb{F}}_{\text{I}}$, which takes into account the relaxed prediction state constraints, equals the maximal controlled invariant set: $\hat{\mathbb{F}}_{\text{I}} = \mathbb{C}$.

Assumption 7. Maximal controlled invariant set $\mathbb{C}$ can be defined as the convex hull of a finite number of vertices $v_s \in \mathbb{X}$, $s \in \{1,\dots,S\}$, $S \in \mathbb{N}_+$, $S < \infty$.

The aim is to design a minimal[2] prediction state constraint relaxation $\Delta := \left[\delta_2^{\mathrm{T}},\dots,\delta_N^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{(N-1)g_{\text{x}}}$ such that a feasible parameterized predicted open-loop control input trajectory $\tilde{U}_s \in \mathbb{U}^{\tilde{N}}$ exists for each vertex v_s.

Problem 8. Find:

$$\begin{aligned} \Delta^* := \arg &\min_{\Delta \in \mathbb{R}^{(N-1)g_{\text{x}}}} \Delta^{\mathrm{T}}\Delta \\ \text{s.t. } &\forall s \in \{1,\dots,S\} \quad \exists \tilde{U}_s \in \mathbb{R}^{\tilde{N}m} \text{ s.t.} \\ &\quad G_{\text{I}} \tilde{M} \tilde{U}_s \leq W_{\text{I}} + E_{\text{I}} v_s + Z\Delta \end{aligned} \tag{5}$$

where distribution matrix $Z \in \{0,1\}^{[N(g_{\text{x}}+g_{\text{u}})+g_{\text{x}}] \times (N-1)g_{\text{x}}}$ places the relaxations into their correct position:

$$Z := \left[\begin{array}{ccccc} \multicolumn{5}{c}{0_{\{2(g_{\text{x}}+g_{\text{u}}),(N-1)g_{\text{x}}\}}} \\ \hline I_{g_{\text{x}}} & 0_{\{g_{\text{x}},g_{\text{x}}\}} & 0_{\{g_{\text{x}},g_{\text{x}}\}} & \cdots & 0_{\{g_{\text{x}},g_{\text{x}}\}} \\ 0_{\{g_{\text{u}},g_{\text{x}}\}} & 0_{\{g_{\text{u}},g_{\text{x}}\}} & 0_{\{g_{\text{u}},g_{\text{x}}\}} & \cdots & 0_{\{g_{\text{u}},g_{\text{x}}\}} \\ 0_{\{g_{\text{x}},g_{\text{x}}\}} & I_{g_{\text{x}}} & 0_{\{g_{\text{x}},g_{\text{x}}\}} & \cdots & 0_{\{g_{\text{x}},g_{\text{x}}\}} \\ 0_{\{g_{\text{u}},g_{\text{x}}\}} & 0_{\{g_{\text{u}},g_{\text{x}}\}} & 0_{\{g_{\text{u}},g_{\text{x}}\}} & \cdots & 0_{\{g_{\text{u}},g_{\text{x}}\}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{\{g_{\text{x}},g_{\text{x}}\}} & 0_{\{g_{\text{x}},g_{\text{x}}\}} & 0_{\{g_{\text{x}},g_{\text{x}}\}} & \cdots & I_{g_{\text{x}}} \end{array}\right].$$

Problem 8 can be rewritten in standard form as a quadratic programming (QP) problem.

Problem 9. Find:

$$\begin{aligned} \bar{\Delta}^* := \arg &\min_{\bar{\Delta} \in \mathbb{R}^{(N-1)g_{\text{x}}+S\tilde{N}m}} \bar{\Delta}^{\mathrm{T}} \bar{H} \bar{\Delta} \\ \text{s.t. } &\bar{Z}\bar{\Delta} \leq \bar{W} \end{aligned}$$

[1] Both prediction state and control input constraints can be relaxed. For simplicity only prediction state constraints are considered here.
[2] Minimization is performed with respect to a quadratic measure. A linear (or other) measure could be used instead.

with $\bar{H} := \begin{bmatrix} I_{(N-1)g_x} & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \cdots 0 \end{bmatrix}$, $\bar{W} := \begin{bmatrix} W_I + E_I v_1 \\ W_I + E_I v_2 \\ \vdots \\ W_I + E_I v_S \end{bmatrix}$,

$$\bar{\Delta} := \begin{bmatrix} \Delta \\ \tilde{U}_1 \\ \vdots \\ \tilde{U}_S \end{bmatrix} \text{ and } \bar{Z} := \begin{bmatrix} -Z & G_I\tilde{M} & 0 & \cdots & 0 \\ -Z & 0 & G_I\tilde{M} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Z & 0 & 0 & \cdots & G_I\tilde{M} \end{bmatrix}.$$

The optimal prediction state constraint relaxation is then given as follows: $\Delta^* := \left[I_{(N-1)g_x} , 0_{\{(N-1)g_x, S\tilde{N}m\}} \right] \bar{\Delta}^*$.

Remark 10. Problem 9 is not strictly convex, because $\bar{H}$ is only positive semi-definite. This may cause problems for QP problem solvers, because the solution, if it exists, is not guaranteed to be unique. A non-unique solution, if it exists, is however guaranteed to be a global minimum. So, this issue is of little concern as long as the solver is able to terminate with the determination of some minimizing solution. In all cases considered by the authors a minimizing solution was determined without problems.

Lemma 11. A prediction state constraint relaxation Δ, $\Delta^T\Delta < \infty$ satisfying Eq. (5) exists.

Proof. For step i and initial state $x_i \in \mathbb{X}$, consider any predicted open-loop control input trajectory $\{u_{(i,k)} \in \mathbb{U}\}_{k=0}^{N-1}$ and the resulting predicted open-loop state trajectory: $x_{(i,k+1)} = Ax_{(i,k)} + Bu_{(i,k)}$ $\forall k \in \{0, \ldots, N-1\}$. Constraint sets $\mathbb{X}$ and $\mathbb{U}$ are bounded by assumption, therefore each member of the predicted state trajectory $\{x_{(i,k)}\}_{k=0}^{N}$ is bounded. For each step k it is thus possible to find a prediction state constraint relaxation δ_k, $\delta_k^T\delta_k < \infty$ such that $x_k \in \mathbb{X}_k$, $\mathbb{X}_k := \{x \in \mathbb{R}^n | G_x x \leq W_x + \delta_k\}$. Assume such prediction state constraint relaxations have been found: $\Delta = [\delta_2^T, \ldots, \delta_N^T]^T$. Then, the only constraints of Eq. (5) to be able to become active are: $x_{(i,k)} \in \mathbb{X}$ $\forall k \in \{0, 1\}$. The constraints are then equivalent to those defining the maximal controlled invariant set $\mathbb{C}$. □

The maximal controlled invariant feasible set taking into account the relaxed constraints is then defined as follows:

$$\begin{aligned}\hat{\mathbb{F}}_I := \Big\{ x_0 \in \mathbb{X} | \exists \{\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}\}_{i=0}^{\infty} \text{ s.t. } \forall i \in \mathbb{N}, \\ G_I\tilde{M}\tilde{U}_i \leq (W_I + Z\Delta^*) + E_I x_i \quad \wedge \\ x_{i+1} = Ax_i + BF\tilde{M}\tilde{U}_i \in \mathbb{X} \Big\} \\ = \mathbb{C}.\end{aligned}$$

Equality with $\mathbb{C}$ follows from the convexity of $\mathbb{F}_I$ (see Gondhalekar & Imura (2007a,b)) and Assumption 7. Define the relaxed MPC problem as follows.

Problem 12. Find:

$$\tilde{U}_i^*(x_i) = \arg \min_{\tilde{U}_i \in \mathbb{R}^{\tilde{N}m}} \left\{ \tilde{U}_i^T \tilde{M}^T H \tilde{M} \tilde{U}_i + \tilde{U}_i^T \tilde{M}^T L x_i \right\}$$

$$\text{s.t. } G_I\tilde{M}\tilde{U}_i \leq (W_I + Z\Delta^*) + E_I x_i \quad \wedge \quad x_{(i,1)} \in \hat{\mathbb{F}}_I$$

The optimal control law $\hat{\kappa}_I : \mathbb{R}^n \to \{\mathbb{U}, \emptyset\}$ based on MPC Problem 12 is then given by $u_i^* = \hat{\kappa}_I(x_i) := F\tilde{M}\tilde{U}_i^*(x_i)$, $\hat{\kappa}_I(x_i) = \hat{K}_I^s x_i + \hat{a}_I^s$ if $x_i \in \hat{\mathcal{X}}_I^s$, $s \in \{1, \ldots, \hat{S}_I\}$.

4. NUMERICAL EXAMPLE

The double integrator with sample-period $\tau = 0.2$s,

$$x_{i+1} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix} u_i$$

with constraint sets $\mathbb{U} = \{u \in \mathbb{R} | \|u\|_\infty \leq 1\}$, $\mathbb{X} = \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 2\}$ and prediction horizon length $N = 50$ is considered. Two blocking regimes were chosen:

$$M_1 = I_N \quad , \quad M_2 = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & 1 \cdots 1 \end{bmatrix}^T \in \{0,1\}^{N \times 2}. \quad (6)$$

The prediction horizon $N = 50$ was chosen long enough to ensure that the full degree of freedom MPC problem when $M = M_1$ results in the infinite-time optimal control law [Grieder *et al.* (2004); Mayne *et al.* (2000)]. Note that use of blocking matrix M_2 achieves an enormous reduction in computational complexity of the MPC problem, as the number of decision variables is reduced from $50m$ to $2m$.

Figure 1 shows the maximal controlled invariant set $\mathbb{C} = \mathbb{F}_I(M_1) = \mathbb{F}_O(M_1)$ within $\mathbb{X}$ as the outer, lighter set, the maximal controlled invariant feasible set $\mathbb{F}_O(M_2)$ as the middle, medium dark set, and the maximal controlled invariant feasible set $\mathbb{F}_I(M_2)$ as the inner, dark set. The difference in size is clearly visible: Vol($\mathbb{X}$) = 16, Vol($\mathbb{C}$) = 13.34, Vol($\mathbb{F}_O(M_2)$) = 12.57, Vol($\mathbb{F}_I(M_2)$) = 8.65.

The prediction state constraint relaxation Δ^* was computed for $M = M_2$. Denote the associated relaxed maximal controlled invariant feasible set by $\hat{\mathbb{F}}_I(M_2) = \hat{\mathbb{F}}_O(M_2) = \mathbb{C}$. Figure 2 shows the relaxed prediction state constraint sets $\mathbb{X}_k$ $\forall k \in \{2, \ldots 50\}$ for the input-move-blocking case. Those for the output-move-blocking case are similar. From Figure 2 it cannot be determined which set corresponds to which prediction step. Generally, prediction steps towards the end of the prediction horizon have larger relaxed prediction state constraint sets associated with them.

The closed-loop system performance with $\hat{\kappa}_I$ ($\hat{\kappa}_O$) due to relaxed MPC Problem 12 is investigated. Denote by

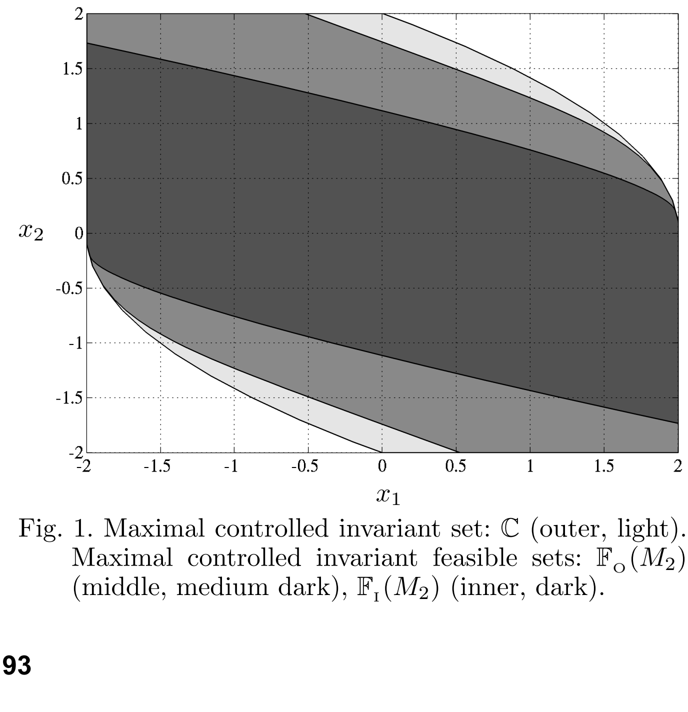

Fig. 1. Maximal controlled invariant set: $\mathbb{C}$ (outer, light). Maximal controlled invariant feasible sets: $\mathbb{F}_O(M_2)$ (middle, medium dark), $\mathbb{F}_I(M_2)$ (inner, dark).

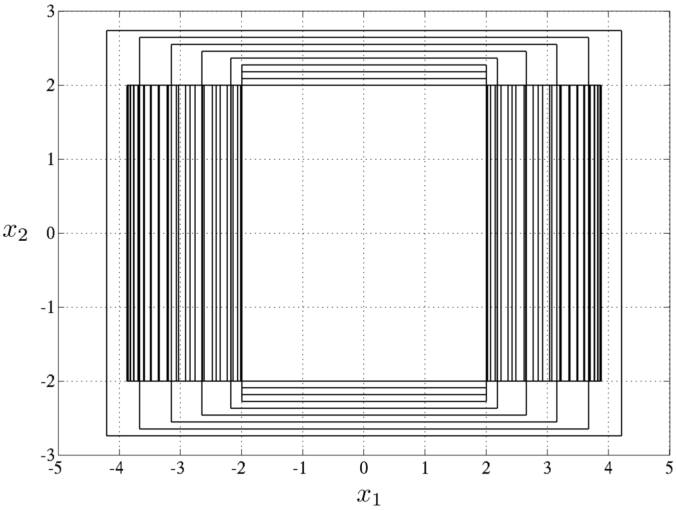

Fig. 2. Relaxed prediction state constraint sets $\mathbb{X}_k$ $\forall k \in \{2, \ldots, 50\}$ for the input-move-blocking case.

$$J(x, M) := \sum_{i=0}^{\infty} \left[x_i^{\mathrm{T}} Q x_i + \kappa^{\mathrm{T}}(x_i) R \kappa(x_i) \right] \quad , \quad x_0 = x$$

the running cost of a trajectory starting from initial state x, subject to control law κ determined by blocking-matrix M. Denote the average running cost over the entire controller domain $\mathbb{F}(M)$ of controller κ by

$$V(\mathbb{F}(M)) := \frac{1}{\mathrm{Vol}(\mathbb{F}(M))} \int_{\mathbb{F}(M)} J(x, M)\, \mathrm{d}x \ .$$

Methods introduced by the authors in Gondhalekar & Imura (2007c) were used to evaluate the average cost $V(\mathbb{F}(M))$ exactly. The average running cost was computed for both input- and offset-move-blocking, for use of blocking matrices M_1 and M_2. For use of M_2, the controller was generated by relaxed MPC Problem 12. Results are tabulated below. Cost matrices $Q = I_n$, $R = I_m$ were chosen. The average running cost for input-move-blocking is roughly 14% higher than for the optimal, full degree of freedom controller, whereas for offset-move-blocking there was no measurable performance difference.

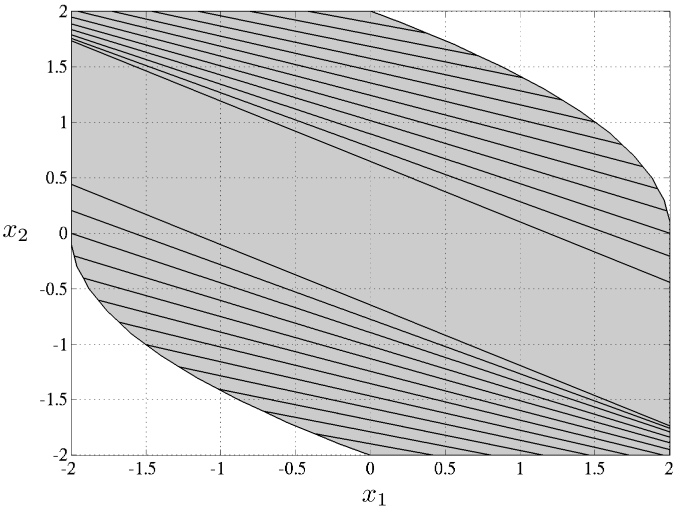

Fig. 3. Full degree of freedom controller partition ($M_1 = I_N$) for input- and offset-move-blocking: 25 regions.

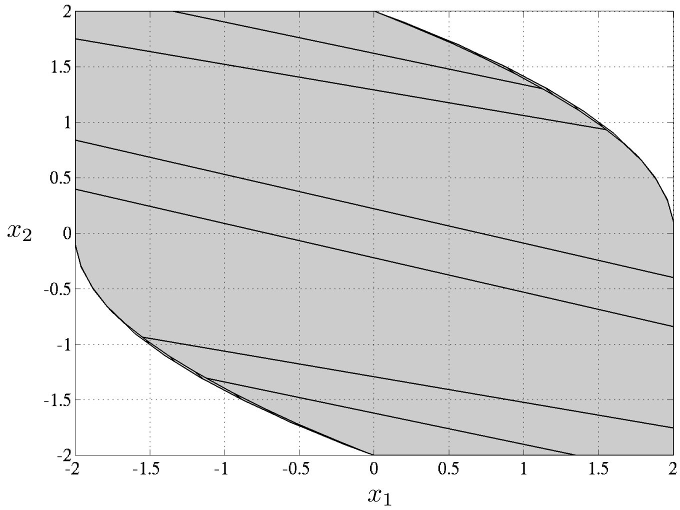

Fig. 4. Input-move-blocking controller partition using M_2: 31 regions. 99.6% volume contained in 7 regions.

Regime	V	V/V^*	Regions
$V^* = V(\mathbb{F}_{\mathrm{I}}(M_1)) = V(\mathbb{F}_{\mathrm{O}}(M_1))$	20.67	1	25
$V(\hat{\mathbb{F}}_{\mathrm{I}}(M_2))$	23.54	1.139	31
$V(\hat{\mathbb{F}}_{\mathrm{O}}(M_2))$	20.67	1.000	5

The right hand column of the above table contains the number of regions of the associated PWA controller partitions, which are plotted for controller $\kappa_{\mathrm{I}} = \kappa_{\mathrm{O}}$ (Fig. 3), constraint relaxed, input-move-blocking controller $\hat{\kappa}_{\mathrm{I}}$ (Fig. 4) and constraint relaxed, offset-move-blocking controller $\hat{\kappa}_{\mathrm{O}}$ (Fig. 5). The offset-move-blocking controller partition has only 5 regions, compared to the 25 regions of the full degree of freedom controller partition. The input-move-blocking controller partition has 31 regions, although many of them are extremely small and located at the fringe of the domain. In fact, 99.6% of the domain's volume is contained within the central 7 regions. These results indicate that the proposed least restrictive move-blocking method, intended mainly for complexity reduction in on-

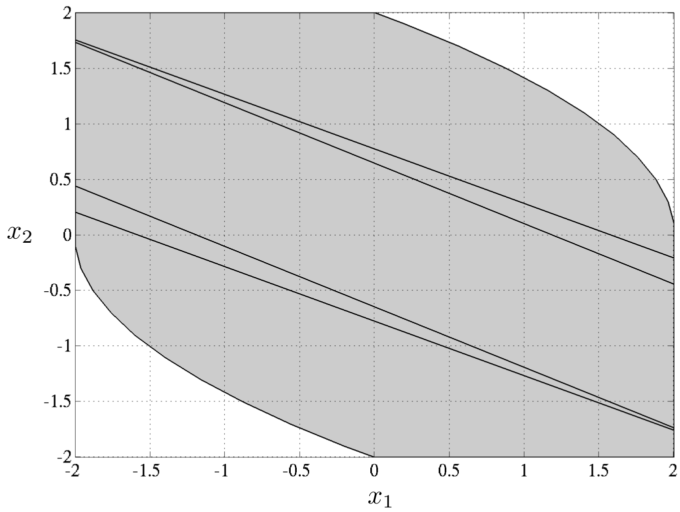

Fig. 5. Offset-move-blocking controller partition using M_2: 5 regions.

line MPC methods, can be a powerful method to reduce the complexity of offline MPC methods also, at no reduction in controller domain size.

A point of interest is the size of the region containing the origin. This origin-region is the set of states at which the constrained MPC problem is equivalent to the unconstrained MPC problem, i.e.: $\mathbb{U} = \mathbb{R}^m$, $\mathbb{X} = \mathbb{R}^n$. Suppose the origin is located in the region with index 1: $\hat{\mathcal{X}}^1_{\text{I}}$, $\hat{\mathcal{X}}^1_{\text{O}}$. Denote the origin-region for the full degree of freedom controller when $M = M_1$ by $\mathcal{X}^1$. Then, $\hat{\mathcal{X}}^1_{\text{O}} = \mathcal{X}^1$ always, although $\hat{\mathcal{X}}^1_{\text{I}} \subseteq \mathcal{X}^1$. This effect is again due to the fact that the predicted offset trajectory of only zeros results in the LQR optimal control input within this region for the offset-move-blocking controller, *for any* blocking matrix M. The difference in size of set $\hat{\mathcal{X}}^1_{\text{I}}$ of the input-move-blocking controller can be seen clearly in Fig. 4.

5. WIDE-RANGING CASE STUDY

The average running cost and number of controller regions are computed, as was done in Section 4 above, for a large number of different plants. As above, constraint sets $\mathbb{U} = \{u \in \mathbb{R} | \|u\|_\infty \leq 1\}$, $\mathbb{X} = \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 2\}$, prediction horizon length $N = 50$, and the two blocking matrices of Eq. (6) are chosen. Each plant has state dimension $n = 2$, control input dimension $m = 1$, with (A, B) controllable. Cost matrices $Q = I_n$ and $R = I_m$ are chosen. Twenty plants, ten open-loop stable, ten open-loop unstable, are generated randomly and analyzed. Each element ϵ of dynamics matrices A and B is generated randomly using the following probability density function:

$$f_{\text{pdf}}(\epsilon) = \begin{cases} 0.25 & \text{if } -2 \leq \epsilon \leq +2 \\ 0 & \text{otherwise} \end{cases} .$$

The prediction horizon is long enough such that the full degree of freedom MPC problem generates the constrained LQR optimal controller. Denote by V^* and S^* the average running cost and number of controller regions of the full degree of freedom controller, for each plant. Denote by V_{I} and S_{I} (V_{O}, S_{O}) the average running cost and number of controller regions of relaxed, input- (offset-) move-blocking controller $\hat{\kappa}_{\text{I}}$ ($\hat{\kappa}_{\text{O}}$). Tabulated below are the normalized average, maximum and minimum of these quantities.

	Average	Max.	Min.
V_{I}/V^*	1.11171	2.53616	1.00000
V_{O}/V^*	1.00000	1.00000	1.00000
S_{I}/S^*	1.39549	5.00000	0.40000
S_{O}/S^*	0.94286	1.00000	0.60000

The above results are for nineteen of the twenty plants. For one plant, input-move-blocking did not result in a stabilizing control law, whereas offset-move-blocking did. For this plant $V_{\text{O}}/V^* = 1.00014$.

Within the scope of this case study, these results indicate that offset-move-blocking, on average and in the worst case, achieves better cost performance and fewer controller regions than input-move-blocking. Input-move-blocking, in the best case, may result in fewer controller regions.

6. CONCLUSION

In this paper a method to design strongly feasible, move-blocking MPC problems which result in least-restrictive controllers was proposed. The controller domain is equal to the maximal controlled invariant set regardless of prediction horizon length or move-blocking scheme used. This allows comparison and optimization of the move-blocking scheme based on cost performance alone, without having to take controller domain size into consideration.

The results of a numerical example and case-study indicate that between input- and offset-move-blocking, the latter is very likely to be the method of choice. The proposed approach is demonstrated to be capable of significant complexity reduction in online as well as offline MPC methods at no compromise in controller domain size and little or no deterioration in control performance.

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