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Affine Parameterization of Cascade Control with Time Delays

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Abstract: The cascade control architecture is a standard solution in control engineering practice for industrial plants with considerable time delays. In this paper, the set of all stabilizing cascade controllers in parametric form is presented for such plants. The parameterization is based on the use of the so-called RQ meromorphic functions expressing the time delay character of the plant and the controller. The parameter enters any closed-loop transfer function in an affine manner. The performance is specified in terms of disturbance rejection and the "slave" controller gain is considered as the key parameter of the affine parameterization. Unlike in most literature on the subject, the primary controlled output is not considered to be directly dependent on the secondary controlled variable. The disturbance rejection potentials are discussed for various options of controlled plants with delays and the varying role of the secondary controlled output option is investigated.

1. INTRODUCTION

The cascade control strategy is widely used by control practitioners to overcome the crucial problem of time-delay system control, namely the corrective action to attenuate the disturbance impact does not begin until after the delayed controlled variable deviates from the set point. The key idea of cascade control configuration consists in finding and feeding back a secondary plant output that is located as close as possible to the source of disturbance. This secondary closed loop is designed so as to recognize much sooner the upset conditions in the plant and to compensate for them. A well selected auxiliary measured output for the secondary loop can yield not only a better disturbance rejection but also an enhanced stability and a better robustness to model uncertainties (Shinskey, 1998). A specific suitability of cascade control for the time-delay systems was anticipated already by Morari and Zafiriou (1989) where it was shown that the non-minimum phase character of the plant is an essential condition of an efficient application of the cascade control scheme. Skogestad and Postlethwaite (1996) proved that for the minimum phase plant the input-output controllability for both the primary and secondary outputs are the same and hence no fundamental benefit is to be expected from applying the secondary control loop. In view of this fact the cascade control is a specifically suitable approach for time delay plants. Basically the primary and the secondary outputs of the plant do not depend on each other. Although a dependence of both outputs is assumed by most authors, this constraining assumption is not justified in general and will not be applied here. In general, the cascade control could also be considered as a special case of multivariable control, but we will keep the single-variable framework for the sake of clarity.

The algebraic approach to the design of time-delay system control was developed during the three last decades (Vidyasagar, 1985). The factorization techniques using the Bézout ring of quasi-polynomials were summarized and further worked out by Loiseau (2000). The affine parameterization of stabilizing controllers has become one of the main tools in the design of linear control systems described not only by rational transfer functions but also for a class of time delay systems (Hlava J., 2001; Zhang et al., 2006; Zítek and Kučera, 2003). It was shown (Mirkin, 2003; Mirkin and Raskin, (2003) that H2 and H∞ optimal controllers have a model-based dead-time compensator structure. The parameterization interpreted as the internal model control of linear time-delay systems was investigated in Zítek and Hlava (2001) and Zítek and Vyhlídal (2004).

The paper is structured as follows. After the introduction to cascade control strategy in Section 2, a meromorphic model of double-input-double-output time-delay plant is introduced and its application to parameterize the cascade control configuration is developed in Section 3. The affine parameterization based design of the slave controller is presented in Section 4 while the whole cascade control synthesis is given in Section 5. Concluding remarks are in Sections 6.

2. CASCADE CONTROL PRELIMINARIES

The cascade control is considered in the configuration given by the block diagram in Fig. 1, where $G(s)$, $G_S(s)$ and $G_D(s)$,

$G_Z(s)$ are the plant transfer functions for the primary and secondary controlled outputs y and z with respect to the control variable u and the disturbance d, respectively. This system can be viewed as a special case of two-variable control system with the input $\mathbf{v} = [u, d]^T$ and the output $\mathbf{y} = [y, z]^T$ where, actually, only the primary output variable y is to be controlled. The four transfer functions are then considered to make up the transfer function matrix of the plant

$$\mathbf{G}(s) = \begin{bmatrix} G(s), & G_D(s) \\ G_S(s), & G_Z(s) \end{bmatrix}. \tag{1}$$

Even though the secondary output z is utilized, the entire control system has the same degree of freedom as a single-loop control.

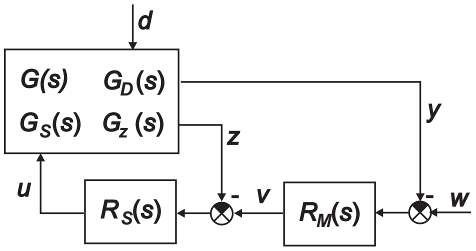

Fig. 1. Cascade control configuration

Let the so-called *master* and *slave* controllers be represented by their transfer functions $R_M(s)$ and $R_S(s)$, respectively, see Fig. 1. Then the complementary sensitivity function of the whole cascade control system for the primary controlled variable is as follows

$$T(s) = \frac{y(s)}{w(s)} = \frac{R_M(s)\, R_S(s)\, G(s)}{1 + R_S(s) G_S(s) + R_M(s) R_S(s) G(s)}. \tag{2}$$

The cascade control is efficient at rejecting the disturbances on the plant input only, as in Fig. 1, where the secondary measured variable z is located, in line with the idea of cascade control, *between* d and y. The disturbance sensitivity function is then of the form

$$\begin{aligned} S_D(s) &= \frac{y(s)}{d(s)} = \\ &= \frac{G_D(s)[1 + R_S(s) G_S(s)] - R_S(s) G_Z(s) G(s)}{1 + R_S(s) G_S(s) + R_M(s) R_S(s) G(s)} = \\ &= \frac{G_D(s) - R_S(s) \det \mathbf{G}(s)}{1 + R_S(s) G_S(s) + R_M(s) R_S(s) G(s)}. \end{aligned} \tag{3}$$

If $\mathbf{G}(s)$ is nonsingular, then the presence of $\det\mathbf{G}(s)$ in the numerator of (3) points out the potential of the cascade configuration to minimize the impact of the disturbance on the system response through the secondary controller $R_S(s)$. The following Lemma holds.

Lemma 1. Consider a stable plant controlled as in Fig. 1, given by transfer function matrix (1), where $G(s)$, $G_D(s)$ correspond to the primary control loop while $G_S(s)$, $G_Z(s)$ are subject to the choice of the secondary output z. If all these functions have finite and non-zero limits for $s \to 0$, then the secondary controller $R_S(s)$ alone can cancel the impact of the disturbance d on the controlled variable y in the low frequency band whenever the determinant

$$\det \mathbf{G}(0) = \gamma$$

has a non-zero value, $\gamma \neq 0$, and both γ and $G_D(0)$ are of *the same sign*.

Proof. The choice of z for the secondary control loop and the corresponding γ may result in three different possibilities

a) if $\gamma = 0$ then it is apparent from (3) that the secondary controller $R_S(s)$ *cannot affect* the numerator of the transfer function (3) at all;

b) if both $G_D(0) > 0$ and $\gamma > 0$ or both $G_D(0) < 0$ and $\gamma < 0$ then according to (3) the term $R_S(s)\det\mathbf{G}(s)$ *diminishes* the $S_D(j\omega)$ numerator value in the low frequency band;

c) if γ is of the opposite sign than $G_D(0)$ then the disturbance response is getting worse in comparison with the single loop scheme as soon as the secondary loop is applied, regardless of $R_S(s)$ setting. □

In spite of other effects of the secondary loop, the option c) for the selection of z is to be considered as *unfavorable* for a cascade control application since it is a more efficient disturbance rejection than it is the primary concern of the cascade control.

Corollary 1. Consider the same stable plant as in Lemma 1. If $\det\mathbf{G}(0)$ is of the same sign as $G_D(0)$ – the case b) of Lemma 1 – then there exists a possibility to render the disturbance transfer function (3) zero in the low frequency band if the secondary controller fulfils the condition

$$R_S(0) = G_D(0)\,\gamma^{-1}. \tag{4}$$

Proof. Follows directly from (3).

Remark. As regards the option a) it is worth noting that if the plant itself is of "cascade" structure too, i.e., if $y(s) = G_M(s)\,z(s)$ where $G(s) = G_M(s) G_S(s)$, then the matrix $\mathbf{G}(s)$ is always singular, $\det\mathbf{G}(s) \equiv 0$, and the possibility to put down the $S_D(s)$ numerator is excluded. Unfortunately, these conditions are often encountered in common cascade control applications. For example, if u is a control valve opening, d is the pressure change and z is the resulting rate of flow then apparently $G(s)/G_S(s) = G_D(s)/G_Z(s)$ and hence $\mathbf{G}(s)$ is singular. The effect of cascade control then consists in accelerating the response in disturbance rejection, thus keeping the control error down.

3. MEROMORPHIC EXTENSION OF AFFINE PARAMETERIZATION

In extending the class of admissible functions from rational to meromorphic, the obvious requirements of causality and feasibility for both the plant and the controller have to be respected in the control system implementation. To satisfy these conditions in *rational* function algebraic design, one constrains the plant and controller models to be *proper* rational functions. An equivalent restriction is to be introduced for *meromorphic functions* as well. Similarly, the four plant transfer functions in $\mathbf{G}(s)$ are assumed to be stable. This assumption is appropriate for the common area of cascade control applications. The admissible time-delay systems are supposed containing *lumped delays* only and with the so-called *retarded* structure (Hale and Verduyn Lunel, 1993). This class of systems is defined below.

Definition 1. (*RQ meromorphic function*) A ratio of quasi-polynomials $B(s)/A(s)$ is said to be a retarded quasi-polynomial (RQ) meromorphic function if

- $A(s)$ is a retarded quasi-polynomial of the generic form
  $$A(s) = s^n + \sum_{i=0}^{n-1} \sum_{j=1}^{h} a_{ij}\, s^i \exp(-\vartheta_{ij} s)$$
  where s^n represents a delay free term and ϑ_{ij} are non-negative delays,
- $B(s)$ can be factorized as $B(s) = \widetilde{B}(s)\exp(-s\tau)$
  where $\tau > 0$ and $\widetilde{B}(s)$ is a retarded quasi-polynomial
  $$\widetilde{B}(s) = s^m + \sum_{i=0}^{m-1} \sum_{j=1}^{h} b_{ij}\, s^i \exp(-\tau_{ij} s)$$
- the fraction is strictly proper, i.e. it holds for the highest powers that $n - 1 \geq m.$

In order to keep the utmost analogy between the conventional algebraic approach and the time delay system design, let the *ring* of all *stable* functions given by Definition 1 be denoted by R_{MS}. The Mikhaylov criterion holds for the quasi-polynomials $A(s)$ and therefore the stability of the introduced RQ-meromorphic function can be checked by the Mikhaylov hodograph of $A(j\omega)$. Accordingly, the plant transfer functions $G(s)$, $G_S(s)$ and $G_D(s)$, $G_Z(s)$ are supposed to belong to R_{MS}. In order to avoid impulsive modes in closed loop system's responses the so-called *internal stability* condition is adopted.

Definition 2 (*Internal stability*) A feedback control loop is said to be internally stable if its four sensitivity functions that relate the reference and control inputs with the control error and the output respectively, are stable.

Since the cascade control system in Fig. 1 is a single-output control system with a *stable* plant, the parameterization of all stabilizing cascade controllers is based on a representation of the complementary sensitivity function of the control system (2) in the form $T(s)=C(s)G(s)$, where $C(s)$ is a stable *parameterizing controller function* and thus $T(s)$ is affine in $C(s)$ (Goodwin et al., 2001). As the ideal value of $T(s)$ is one, the product $C(s)G(s)$ highlights the fundamental idea of $G(s)$ inversion in control design.

Due to the nested configuration of both the loops of cascade control scheme the inner secondary loop may be regarded as a "precompensated plant" with the inputs v and d. This separately considered subsystem is then described as follows

$$y = H(s)v + H_D(s)d \;=\; \frac{G(s)R_S(s)}{1+R_S(s)G_S(s)}v + \frac{G_D(s) - R_S(s)\left[G(s)G_Z(s) - G_D(s)G_S(s)\right]}{1+R_S(s)G_S(s)}d \tag{5}$$

where $H(s), H_D(s)$ are the transfer functions of the "precompensated plant" for v and d respectively.

Two control loops with the plant $H(s), H_D(s)$ are now to be parameterized, namely the inner secondary loop with the slave controller $R_S(s)$ and the primary loop with the master controller $R_M(s)$. Accordingly two parameterizing controller functions $C_S(s)$ and $C_M(s)$ are to be introduced in order to achieve the following factorizations

$$y = G(s)C_S(s)v = H(s)C_M(s)w \tag{6}$$

for parameterizing the secondary and the primary control loops respectively.

Corollary. If the parameterizing controller function of the cascade control system in Fig. 1 is considered factorized as in (5), then the following relationships hold for the slave and master controllers, respectively

$$R_S(s) = \frac{C_S(s)}{1 - C_S(s)G_S(s)} \tag{7a}$$

$$R_M(s) = \frac{C_M(s)}{1 - C_S(s)C_M(s)G(s)} \tag{7b}$$

Proof. The inner secondary loop may be regarded as a self-contained control circuit with the complementary sensitivity function for input v

$$T_S(s) = \frac{R_S(s)G(s)}{1+R_S(s)G_S(s)} = \frac{y}{v}$$

which is to be equivalent to the factorization in the inner loop $T_S(s) = C_S(s)G(s)$. Therefore it holds for $C_S(s)$

$$C_S(s) = \frac{R_S(s)}{1+R_S(s)G_S(s)}\,. \tag{8}$$

and that is the inverse form of equation (7a). The relationship between the transfer functions $R_M(s)$ and $C_M(s)$ result from substituting $H(s)$ for the pre-compensated plant in the primary control loop and the complementary sensitivity function $T(s)$ is obtained in the form

$$T(s) = \frac{R_M(s)H(s)}{1+R_M(s)H(s)} = \frac{y}{w}$$

The parameterizing function $C_M(s)$ is introduced to achieve the factorized form $T(s)=C_M(s)H(s)$ and since $H(s) = = C_S(s)G(s)$ the following equality holds

$$R_M(s) = C_M(s)\left[1 + R_M(s)C_S(s)G(s)\right]$$

the relationship (7b) is obtained. □

The secondary control loop serves as an "accelerating core" in the cascade system. In accordance with the equality

$T_S(s)=C_S(s)G_S(s)$, the design of $C_S(s)$ is to be based on the properties of this loop only, i.e., on the inversion of $G_S(s)$ in fact. This inversion implied by (7a), however, is not feasible and therefore the usual *inner-outer factorization*

$$G_S(s) = G_{S0}(s)D_S(s) \tag{9}$$

is to be considered where $D_S(s)$ is *inner* and absorbs the delay as well as the unstable zeros of $G_S(s)$, and where $G_{S0}(s)$ is *outer* with numerator order $m_{S0} \le m_S$.

4. AFFINE PARAMETERIZATION-BASED DESIGN OF THE SLAVE CONTROLLER

The well known aim of efficient cascade control application is to achieve as high as possible gain in the slave controller loop. Due to the presence of delays ($D_S(s)$ and $D(s)$), however, both $G_S(s)$ and $G(s)$ are non-minimum phase transfer functions. To point out this aspect, the following type of slave controller is proposed.

Proposition 1. Suppose the slave controller $R_S(s)$ is in the form

$$R_S(s) = \frac{r}{G_{S0}(s)F_S(s)} \tag{10}$$

where r is a gain parameter, and $F_S(s)$ is a conditioning ν-th order factor polynomial, where ν is the relative degree of $G_{S0}(s)$, $\nu = n_S - m_{S0}$. Using relation (8) the secondary parameterizing controller function is then given as follows

$$C_S(s) = \frac{R_S(s)}{1+R_S(s)G_S(s)} = \frac{r}{G_{S0}(s)\left[F_S(s)+rD_S(s)\right]}\ . \tag{11}$$

Remark. The primary purpose of selecting z is to obtain an as prompt as possible response to the critical disturbances. Having this in mind, rather a low-order $G_S(s)$ is to be expected, sometimes with a negligible dead time. In view of this character of $G_S(s)$, its inner factor will be further supposed to be a delay term $D_S(s) = \exp(-\tau_S s)$ only.

Proposition 2. To provide $C_S(s)$ with as high as possible degree of stability regarding the condition given below, the conditioning rational function $[F_S(s)]^{-1}$ needs to be endowed with a single pole markedly dominating the others. In order to achieve this spectral property, the following polynomial is proposed

$$F_S(s) = (\kappa\tau_S s+1)(\tau_S s+1)^{\nu-1}$$

where $\kappa > 1$ is a selected ratio coefficient. Apparently the higher κ the more the pole $s_1 = (-\kappa\tau_S)^{-1}$ dominates the other repeated pole $s_2 = (-\tau_S)^{-1}$. This form of $F_S(s)$ helps to fulfill the condition given in the following Lemma.

Lemma 2. Consider the following retarded quasi-polynomial

$$L_S(s) = (\kappa\tau_S s+1)(\tau_S s+1)^{\nu-1} + r\exp(-\tau_S s)\,.$$

If, for the chosen r, κ and for the frequency ω_C complying with the equation

$$\left|\kappa\tau_S\omega_C+1\right|\left|\tau_S\omega_C+1\right|^{\nu-1} = r\,, \tag{12}$$

the following condition is satisfied

$$\operatorname{atan}(\kappa\tau_S\omega_C) + (\nu-1)\operatorname{atan}(\tau_S\omega_C) + \tau_S\omega_C < \pi \tag{13}$$

for any $\omega \in \langle 0, \omega_C\rangle$, then $[L_S(s)]^{-1}$ is analytic in the closed right half s-plane, i.e., it is *stable*.

Proof. Since the quasi-polynomial $L_S(s)$ is of the *retarded type* (s^ν is without delay factor) the argument increment rule of Mikhaylov holds for its stability proof. The first term $F_S(j\omega)$ of $L_S(s)$ is a stable polynomial and therefore, for $s = j\omega$, the argument of $F_S(j\omega)$ increases monotonously from zero to $\nu\pi/2$ for $\omega \to \infty$, while the argument $-\tau_S\omega$ of $r\exp(-\tau_S s)$ is negatively increasing unlimitedly as $\omega \to \infty$. The intersection point of the separately considered hodographs of $F_S(j\omega)$ and $r\exp(-j\tau_S\omega)$, at a frequency ω_C given by

$$\left|\kappa\tau_S\omega_C+1\right|\left|\tau_S\omega_C+1\right|^{\nu-1} = r$$

determines the boundary until which $F_S(j\omega)$ is not dominating in determining arg $L_S(j\omega_C)$ and the intersection point satisfies the equality $\left|F_S(j\omega)\right| = r$. If it holds for the argument φ_C = arg$F_S(j\omega_C)$ of this intersection point that $\varphi_C < \pi - \tau_S\omega_C$, then the Mikhaylov hodograph $L_S(j\omega)$ obviously cannot leave the first quadrant for $\omega \in \langle 0, \omega_C\rangle$. On the other hand, for $\omega > \omega_C$, the polynomial $F_S(j\omega)$ definitely determines the value of arg $L_S(j\omega)$ and thus fulfills the argument increment condition.□

Theorem 1. (*Internal stability of the secondary loop*) Consider the time delay plant model $G_S(s)$ as in (9). Then the secondary loop is internally stable if and only if $F_S(s) + rD_S(s)$ is a stable quasi-polynomial, i.e., the parameters κ, r satisfy the conditions (12), (13).

Proof. It follows from the affine parameterization that the secondary loop is internally stable if and only if the slave controller parameterizing function $C_S(s)$ is stable. Then the claim follows from (11). □

Example 1. Find a slave controller $R_S(s)$ for the following secondary plant transfer function

$$G_S(s) = \frac{K_S\exp(-s\tau_S)}{T_S s+\exp(-s\vartheta_S)}\,.$$

Using (10), we obtain an anisochronic modification of PD controller – with delayed proportional action – of the form

$$R_S(s) = \frac{r\left[T_S s+\exp(-s\vartheta_S)\right]}{K_S(\kappa\tau_S s+1)}\,. \tag{14}$$

The stability conditions (12) and (13) result in the following boundary for the slave controller gain

$$\omega_K = \frac{\sqrt{r^2-1}}{\kappa\tau_S}$$

$$\pi - \operatorname{atan}(\sqrt{r^2-1}) > \frac{\sqrt{r^2-1}}{\kappa}$$

For a practical controller setting, this condition can be approximately reduced to the following simplified one: $r < \kappa\pi/2$. In other words, the higher gain r to be achieved the higher κ to be chosen for the given delay τ_S . Theorem 1 provides a way to select r and κ that result in a stable secondary loop. But if the secondary output z is selected so that the option b) holds in Lemma 1, i.e., if det**G**(0) and $G_D(0)$ are non-zero and of the same sign, then it is possible to render zero the disturbance sensitivity function (3) in the low frequency band by means of setting $R_S(0) = r$. In this case the gain factor r is given by (4).

5. INTERNAL MODEL PRINCIPLE IN MASTER CONTROLLER DESIGN

Having designed the slave controller $R_S(s)$, a complete parameterization of the entire cascade control system can be accomplished. In search of the master controller $R_M(s)$ for the main control loop, we begin with a $C(s)$ in the product form $C(s) = C_M(s)C_S(s)$, where $C_S(s)$ is given by (11).

Theorem 2. (*Internal stability of the primary loop*) Consider the plant model for the controlled variable in a form analogous to (9),

$$G(s) = G_0(s)D(s)\ , \tag{15}$$

and suppose a secondary control loop has been designed with the slave controller $R_S(s)$ according to (10). Then the master controller

$$R_M(s) = \frac{G_{S0}(s)\left[F_S(s) + rD_S(s)\right]}{rG_0(s)\left[F_M(s) - D(s)\right]}\ , \tag{16}$$

where $F_M(s)$ is a stable polynomial or quasi-polynomial of degree equal to the relative degree of $G_0(s)$ with the property $F_M(0) = 1$, internally stabilizes the cascade control system in Fig. 1.

Proof. The parameterizing controller function $C(s)$ for the complete system can be designed using the internal model inversion principle (Goodwin *et al*., 2001), i.e., in the generic form

$$C(s) = \frac{1}{G_0(s)F_M(s)} = C_M(s)C_S(s)$$

Comparing the relationships $C(s) = C_M(s)C_S(s)$ and (11) the following form of the parameterizing master controller function is obtained

$$C_M(s) = \frac{1}{C_S(s)G_0(s)F_M(s)} = \frac{G_{S0}(s)\left[F_S(s) + rD_S(s)\right]}{r\,G_0(s)F_M(s)} \tag{17}$$

where the pair of functions $G_0(s)$ and $G_{S0}(s)$ may have common factors that are cancelled in forming (17).

For the internal stability of the entire cascade control system, it is necessary and sufficient that the parameterizing controller function $C(s)$ is stable. Considering the factorization $C(s) = C_M(s)\ C_S(s)$ and since the stability of $C_S(s)$ has already been proved, it remains to prove that $C_M(s)$ is a stable RQ-meromorphic transfer function. With respect to (15) and the stability of the numerator of $G_0(s)$, this condition is satisfied by any choice of stable polynomial or quasi-polynomial $F_M(s)$. Thus the whole cascade control system is internally stable.

Lemma 2. Suppose the plant is given by (15) and the slave and master controllers are given by (10) and (16), respectively. Then the steady state control error, i.e., the limit value of $w - y$ as time increases, is zero for any stable $F_M(s)$ whenever $F_M(0) = 1$.

Proof. This property of (15) is apparent from the fact that

$$\lim_{s\to 0}\left[F_M(s) - D(s)\right] = 0 \tag{18}$$

and therefore $s = 0$ is a pole of $R_M(s)$. □

Proposition 3. Write the plant model (15) in the form

$$G(s) = G_0(s)D(s) = \frac{B_0(s)D(s)}{A(s)}\ .$$

Suppose that the denominator quasi-polynomial $A(s)$ can be factorized as $A(s) = \overline{A}(s)f_0(s)$, where $f_0(s)$ is a quasi-polynomial of degree $n - m - 1$ and further suppose that $D(s) = \exp(-s\tau)$. Then selecting the conditioning polynomial

$$F_M(s) = T_F s\, f_0(s) + 1$$

results in master controller $R_M(s)$ in which the frequency transfer function representing the integration in $R_M(j\omega)$ can be approximated as follows

$$g(j\omega) = \frac{f_0(j\omega)}{F_M(j\omega) - D(j\omega)} \cong \frac{1}{T_I(\omega)\,j\omega} \tag{19}$$

being determined by the following asymptotes

a) for $\omega \to 0$: $T_I(\omega) = T_F + \tau / f_0(0)$

b) for $\omega \to \infty$: $T_I(\omega) = T_F$

Proof. To evaluate the effective integration investigate the following limits for $\omega \to 0$ a $\omega \to \infty$ of the product

$$j\omega g(j\omega) = \frac{j\omega f_0(j\omega)}{T_F\, j\omega f_0(j\omega) + 1 - \exp(-j\omega\tau)} \tag{20}$$

Since $1 - \exp(-j\omega\tau)$ is negligible compared to $T_F\, j\omega\, f_0(j\omega)$ for $\omega \to \infty$ one obtains

$$\lim_{\omega\to\infty} j\omega g(j\omega) = \lim_{\omega\to\infty} \frac{j\omega f_0(j\omega)}{T_F\, j\omega f_0(j\omega) + 1 - \exp(-j\omega\tau)} = \frac{1}{T_F} \tag{21}$$

As regards the other limit the L'Hospital's rule is to be applied

$$\lim_{\omega\to 0} j\omega\, g(j\omega) = \lim_{\omega\to 0} \frac{f_0(j\omega) + j\omega\, f_0'(j\omega)}{T_F f_0(j\omega) + T_F\, j\omega\, f_0'(j\omega) + \tau\exp(-j\omega\tau)} = \frac{1}{T_F + \tau / f_0(0)}. \tag{22}$$

and the proof is completed. □

Example 2. Let the plant be given by the following four transfer functions

$$G_S(s) = \frac{K_S \exp(-s\tau_S)}{T_S s + \exp(-s\vartheta_S)}\ ,\ G(s) = \frac{K\exp(-s\tau)}{\left(Ts + \exp(-s\vartheta)\right)\left(T_1 s + 1\right)}$$

$$G_Z(s) = \frac{K_Z \exp(-s\tau_Z)}{T_S s + \exp(-s\vartheta_S)}, G_D(s) = \frac{K_D \exp(-s\tau_D)}{(Ts + \exp(-s\vartheta))(T_1 s + 1)}$$

where the disturbance d transfer functions have the same denominators as for u, only the gains K_Z, K_D and the dead times τ_Z, τ_D are different. For the time constants it holds $T >> T_S$, $\tau >> \tau_S$, $\vartheta >> \vartheta_S$ and $T_1 < T$. The slave controller has been already designed in Example 1. Before applying the formula (16) a suitable conditioning polynomial $F_M(s)$ (with the steady state gain equal to one) is to be selected. With the aim to cancel the factor $(T_1 s + 1)$ in the lowest frequency band, the following form is advantageous

$$F_M(s) = T_I s(T_1 s + 1) + 1$$

and the master controller transfer function results as follows

$$R_M(s) = \\ = \frac{K_S(Ts + \exp(-s\vartheta))(T_1 s + 1)(\kappa\tau_S s + 1 + r\exp(-s\tau_S))}{rK(T_S s + \exp(-s\vartheta_S))(T_I s(T_1 s + 1) + 1 - \exp(-s\tau))}. \quad (23)$$

This result may seem to be rather complicated but taking into account the approximation (19) it is easy to see that (23) is an anisochronic PID controller providing a steady state disturbance rejection.

In Fig. 2, the set-point response and disturbance rejection of the proposed cascade control scheme is shown for the parameters of the system: K_S=2, T_S=4, τ_S=0.5, ϑ_S=1.5, K_Z=2.5, τ_Z=0.8, K=1.5, T=30, τ=7, ϑ=10, T_1=5, K_D=1.2, τ_D=5 and the parameters of the controllers: T_i=15, κ =7, $r = 0.5, 1, 1.5, 2, 2.5$. As can be seen, the higher is the value of the parameter r, the more effective is the disturbance rejection. However, the parameter r is bounded by stability conditions (12), (13), here by the value r=11.64.

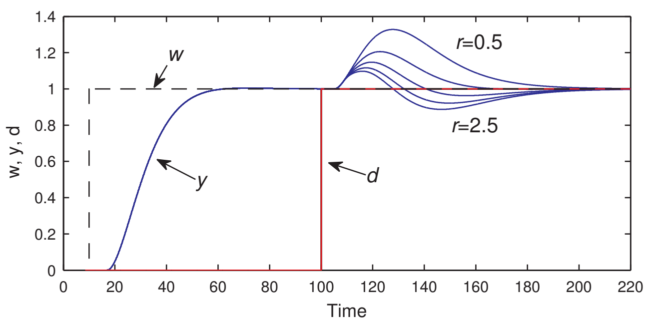

Fig. 2. Set-point response and disturbance rejection of the cascade control scheme for $\kappa = 7$, $T_i = 15$, $r = 0.5, 1, 1.5, 2, 2.5$.

6. CONCLUDING REMARKS

The cascade control configuration is a particularly efficient solution in the plants where the disturbance-to-output response is much faster than that of control variable. The delay effect in the feedback loop then renders the single-loop control incapable of efficiently rejecting the disturbance impact. The closer the secondary measured variable z is located to the potential source of disturbances, the better effect in control dynamics yields the application of the cascade control strategy. *In the presented parameterization anisochronic versions of PD and PID controllers are assumed for the slave and master controllers respectively.* It is also worth noting that the presented cascade control design results in a system that goes to show the properties of a *finite spectrum assignment*: The eigenvalues of the control system are given by the polynomial $F_M(s)$ only; of course, on the assumption that the plant and its model are perfectly identical. Then the characteristic equation of the whole entire control system is the algebraic equation $F_M(s) = 0$.

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