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PID Controllers for Robots Equipped with Brushed DC-Motors Revisited

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Abstract: In this note we are concerned with controller design for robot manipulators equipped with brushed DC-motors in the case when the electric dynamics of these actuators is not neglected. We present, for the first time, stability proofs which show that PD control with desired gravity compensation and the classical PID controller presented previously in the literature under the assumption that no actuator dynamics exists can also be designed in this case. In the case of the classical PID controller we show that design can be done without the exact knowledge of neither robot nor actuator parameters. We present, for the first time, a theoretical justification for use of torque control, a strategy commonly used in industrial practice to control brushed DC-motors.

1. INTRODUCTION

Although many robots use electric motors (brushed DC-motors in particular) as actuators, most controllers for robot manipulators existing in the literature are designed under the assumption that the actuator dynamics can be neglected. The main reason for this, as stated by Ailon et al. [2000], is that the introduction of an electrical system between the control input and the torque actually applied to the robot links complicates the controller design in robotics. However, some studies as those reported by Tarn et al. [1991] and Eppinger and Seering [1987] have shown that neglecting the actuator dynamics may result in closed loop performance degradation. This has motivated lots of works on robot control taking into account the dynamics of the brushed DC-motors used as actuators (see Ailon et al. [1997], Ailon et al. [2000], Burg et al. [1996], Mahmoud [1993], Colbaugh and Glass [1995], Oya et al. [2004], and references therein). However, these works introduce control schemes that are, in fact, more complicated than their counterparts designed under the assumption that the actuator dynamics can be neglected. Some disadvantages of complex strategies are the sensibility to numerical errors, input voltage saturation and noise amplification, as recognized by Ortega et al. [1998] pp. 257, 395, 403.

On the other hand, robot control theory based on the assumption that the actuator dynamics can be neglected has succeeded to design simple PID control strategies. For instance, a PD control strategy was proposed by Takegaki and Arimoto [1981] which achieves global asymptotic stability and only requires the exact knowledge of a reduced number of the robot mechanical parameters (the desired gravity compensation term). Later, Kelly [1995], Ortega et al. [1995], Alvarez et al. [2000], Kelly et al. [2005] Ch. 9 and Meza et al. [2007] have presented local and semiglobal stability proofs for the classical PID controller. An interesting advantage of this PID controller is that it does not require the exact knowledge of any of the robot mechanical parameters. Further, this control scheme is widely used in industrial practice.

Two are the main contributions of the present note. First, we show that the stability proofs of the fore mentioned PD and PID controllers can be extended to the case when the dynamics of the brushed DC-motors used as actuators is taken into account. Thus, we present for the first time a theoretical justification for use of PID controllers in industrial robots equipped with brushed DC-motor actuators which, on the other hand, is a common practice. We stress the importance of this result: we present detailed stability analysis showing that simple PID controllers can be designed in this case whereas most works reported until now have been forced to design complex nonlinear controllers because of complications originated by the actuator electric dynamics even for regulation tasks (see Colbaugh and Glass [1995], for instance). Second we present, for the first time, a theoretical justification for use of torque control (see Parker Automation [1998], for instance), a common strategy in industrial practice to control brushed DC-motors when they are used to actuate rigid robots among lots of other mechanical systems. This result is a refinement of ideas reported by Hernández-Guzmán et al. [2007].

Our proposals have the following features: *i*) the exact knowledge of neither any robot parameter nor any actuator parameter is required in the case of classical PID control and *ii*) contrary to the common assumption, the electric actuator dynamics does not need to be fast com-

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pared to robot dynamics. Hence, contrary to Kokotovic et al. [1986], for instance, we solve the problem without relying on singular perturbations. On the other hand, we stress that the methodology presented by Astolfi and Ortega [2003] is not useful for our purposes. This is because that paper needs the exact knowledge of both the robot and the actuator dynamics to compute acceleration and to complete an error equation for the actuator electrical subsystem. Our results rely on the following items: 1) torque constant equals back electromotive constant in a brushed DC-motor (which, although a well know fact, has not been exploited until now in robot control analysis and design) and 2) a novel error variable is introduced to describe the electric dynamics of actuators. These ideas have their origin in the work by Hernández-Guzmán et al. [2007] where a simple linear adaptive controller is designed.

Finally, we stress the following. Nowadays, it is widely recognized that use of brushless DC-motors presents a number of advantages with respect to use of brushed DC-motors. However, we believe that study of brushed DC-motors as actuators in robotics is still important. From a practical point of view lots of robots equipped with brushed DC-motors already exist which require a controller to be redesigned in many instances. From a theoretical point of view, the study that we present in this note has not been presented before and, given the linear model of these actuators, it represents a first step towards the solution of this control problem for robots equipped with actuators whose models are more complex, i.e. brushless DC-motors.

This paper is organized as follows. In section 2 we present the dynamic model of rigid robots that we consider as well as some useful properties. Section 3 is devoted to present our main results in two propositions and some concluding remarks are given in section 4. Finally, some remarks on notation. We use $\lambda_{min}(A(x))$ and $\lambda_{max}(A(x))$ to represent, respectively, the smallest and the largest eigenvalues of the symmetric positive definite matrix $A(x)$, for any $x \in \mathcal{R}^n$. Given an $x \in \mathcal{R}^n$ and a matrix $A(x)$ the norm of x is defined as $\|x\| = \sqrt{x^T x}$ and the induced norm of $A(x)$ is defined as $\|A\| = \sqrt{\lambda_{max}(A^T A)}$ which implies $\|A\| = \lambda_{max}(A(x))$ if $A(x)$ is a symmetric positive definite matrix. Symbol $p = (d/dt)$ denotes the differential operator.

2. THE DYNAMIC MODEL OF RIGID ROBOTS

The dynamic model of an n degrees of freedom rigid robot equipped only with revolute joints and with n brushed DC-motors as actuators is given as (see Ailon et al. [2000]):

$$L\frac{di}{dt} + r\,i + K_b\dot{q} = u \tag{1}$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + K_v\dot{q} = K_m i \tag{2}$$

where $q \in \mathcal{R}^n$ represents the link positions, $C(q,\dot{q})\dot{q}$ is known as the Coriolis and centrifugal effects term, K_v is the $n \times n$ diagonal positive definite matrix of viscous friction coefficients and $g(q)$ is the gravitational effect term, given as $g(q) = \frac{\partial U(q)}{\partial q}$, where $U(q)$ is the potential energy due to gravity. We define the gear ratio constant diagonal matrix as $\theta = Nq$, where $\theta \in \mathcal{R}^n$ represents the actuator positions. $D(q)$ is the inertia matrix which is $n \times n$ symmetric and positive definite. Variables $i, u \in \mathcal{R}^n$ represent, respectively, the electric current and voltage in the brushed DC-motors armature circuits, while L, r, K_e and K_a are $n \times n$ diagonal and positive definite matrices representing the inductance, resistance, back-electromotive constant and torque constant, respectively. We define $K_b = K_e N$ and $K_m = NK_a$. Torque applied to robot links is given as $\tau = K_m i$. Throughout this note we use $q_d \in \mathcal{R}^n$ and $\tilde{q} \in \mathcal{R}^n$ to represent, respectively, the constant desired link positions and the links position error defined as $\tilde{q} = q_d - q$.

As it is by now well known, some important properties of this model are the following.

*P*roperty 1. (Koditschek [1984], Kelly [1995]). Matrices $\dot{D}(q)$ and $C(q,\dot{q})$ satisfy:

$$\dot{q}^T\left(\frac{1}{2}\dot{D}(q) - C(q,\dot{q})\right)\dot{q} = 0, \quad \forall \dot{q} \in \mathcal{R}^n \tag{3}$$

$$\dot{D}(q) = C(q,\dot{q}) + C^T(q,\dot{q}) \tag{4}$$

*P*roperty 2. (Kelly [1995], Tomei [1991b], Kelly et al. [2005] pp. 101). There exist positive constants k_g and k_c such that for all $w, y, z, q \in \mathcal{R}^n$, we have:

$$\|C(w,y)z\| \le k_c\|y\|\|z\| \tag{5}$$

$$\left\|\frac{\partial g(q)}{\partial q}\right\| \le k_g \tag{6}$$

$$\|g(w) - g(y)\| \le k_g\|w - y\| \tag{7}$$

An important property of brushed DC-motors is related to power conservation. The electric power transformed into mechanical power is given in terms of the back-electromotive force, $e_{bef} = K_e\dot{\theta}$, and the electric current through the armature circuits $P_e = e_{bef}^T i$ whereas the resulting mechanical power is given in terms of velocity and the electromagnetic torque $P_m = \dot{\theta}^T\tau_{em}$, where $\tau_{em} = K_a i$. From power conservation $P_e = P_m$, we obtain $K_e = K_a$ which implies, because of the diagonal property of all the involved matrices, that $K_b = K_m$ (Hernández-Guzmán et al. [2007]).

Finally, we list some well known properties of norms. Let $w, y \in \mathcal{R}^n$ be two vectors and let $B(x)$ and $M(x)$ be two $n \times n$ matrices the former being symmetric and positive definite $\forall x \in \mathcal{R}^n$, then:

$$\pm y^T M(x)w \le \|y\|\|M(x)\|\|w\| \tag{8}$$

$$\pm y^T B(x)w \le \|y\|\|B(x)\|\|w\| = = \lambda_{max}(B(x))\|y\|\|w\| \tag{9}$$

$$y^T B(x)y \ge \lambda_{min}(B(x))\|y\|^2 \tag{10}$$

3. MAIN RESULT

Our first main result is related to controller introduced for the first time by Takegaki and Arimoto [1981] for the case of no actuator dynamics (also see Tomei [1991b], Tomei [1991a], Kelly [1997], Kelly et al. [1994]).

Proposition 1. (PD with desired gravity compensation) Consider plant (1), (2) together with controller:

$$u = -r_a\, i + \overline{K_P}\, \tilde{q} - \overline{K_D}\, \dot{q} + RK_m^{-1} g(q_d) \tag{11}$$

where $\overline{K_P}$ and r_a are $n \times n$ diagonal positive definite matrices such that:

$$\lambda_{min}\left(K_m R^{-1} \overline{K_P}\right) > k_g \tag{12}$$
$$R = r + r_a$$

Under this condition it is always possible to find a $n \times n$ diagonal positive definite matrix $\overline{K_D}$ such that the closed loop system (1), (2), (11) has an unique equilibrium point, where $\tilde{q} = 0$ and $\dot{q} = 0$, which is globally asymptotically stable.

Proof. Replacing control law (11) in (1) and defining:

$$\rho = i - R^{-1}[\overline{K_P}\, \tilde{q} + RK_m^{-1} g(q_d)]$$

yields:

$$L\dot{\rho} = -R\rho - K_b\, \dot{q} - \overline{K_D}\, \dot{q} + LR^{-1}\overline{K_P}\, \dot{q}$$

Let K_D be an arbitrary diagonal positive definite matrix. Choose:

$$\overline{K_D} = LK_D + LR^{-1}\overline{K_P} \tag{13}$$

to write:

$$L\dot{\rho} = -R\rho - K_b\, \dot{q} - LK_D\, \dot{q} \tag{14}$$

Due to the diagonal property of all matrices involved we obtain from this:

$$\rho_j = \frac{-\frac{K_{bj}}{L_j}}{p + \frac{R_j}{L_j}}\, \dot{q}_j + \frac{-K_{Dj}}{p + \frac{R_j}{L_j}}\, \dot{q}_j, \quad j = 1, \ldots, n$$

where subindex j represents the j-th diagonal entry if a matrix or the $j-$th component if a vector. Define $\rho = \varrho + \sigma$, where:

$$\varrho_j = \frac{-\frac{K_{bj}}{L_j}}{p + \frac{R_j}{L_j}}\, \dot{q}_j, \quad \sigma_j = \frac{-K_{Dj}}{p + \frac{R_j}{L_j}}\, \dot{q}_j, \quad j = 1, \ldots, n$$

This allows to write:

$$L\dot{\varrho} = -R\varrho - K_b\, \dot{q}$$
$$L\dot{\sigma} = -R\sigma - LK_D\, \dot{q}$$

Note that these expressions are equivalent to (14). Use of this and defining $K_P = K_m R^{-1}\overline{K_P}$ yields the closed loop system:

$$\frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \dot{q} \\ \\ \varrho \\ \sigma \end{bmatrix} = \begin{bmatrix} -\dot{q} \\ D(q)^{-1}[-C(q,\dot{q})\dot{q} - g(q) + K_m\varrho \\ +K_m\sigma + K_P\, \tilde{q} + g(q_d) - K_v\, \dot{q}] \\ -L^{-1}R\varrho - L^{-1}K_b\, \dot{q} \\ -L^{-1}R\sigma - K_D\, \dot{q} \end{bmatrix} \tag{15}$$

Proceeding as Tomei [1991b] it is easy to see that the equilibrium point $(\tilde{q}, \dot{q}, \varrho, \sigma) = (0, 0, 0, 0)$ is unique and the following scalar function:

$$w(\tilde{q}, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} + U(q) - U(q_d) + \tilde{q}^T g(q_d) + \frac{1}{2}\tilde{q}^T K_P \tilde{q}$$

is positive definite and radially unbounded if (12) is satisfied. Thus, we can use:

$$W(\tilde{q}, \dot{q}, \varrho, \sigma) = \frac{1}{2}\varrho^T L\varrho + \frac{1}{2}\sigma^T K_m K_D^{-1}\sigma + w(\tilde{q}, \dot{q}) \tag{16}$$

as a Lyapunov function candidate. The time derivative of W along the trajectories of the closed loop system (15) is given as:

$$\dot{W} = -\varrho^T R\varrho - \sigma^T K_m K_D^{-1} L^{-1} R\sigma - \dot{q}^T K_v\, \dot{q} \tag{17}$$

where (3) and $K_m = K_b$ have been used. Hence, $\dot{W}$ is a globally negative semidefinite function. This, together with the fact that W is a positive definite radially unbounded function allow to use the LaSalle invariance principle to ensure global asymptotic stability of $(\tilde{q}, \dot{q}, \varrho, \sigma) = (0, 0, 0, 0)$. Note that this result is also possible even if $K_v = 0$. Thus proposition 1 has been proven.

Remark 2. Matrices $\overline{K_P}$ and $\overline{K_D}$ in controller (11) have to satisfy (12) and (13). Being K_D an arbitrary positive definite matrix it is always possible to satisfy (13) without requiring the exact knowledge of neither L nor R. Further, use of approximate values of L and R suffices to compute a lower bound for $\overline{K_D}$ such that (13) is satisfied.

Remark 3. Variable ρ is introduced to represent the actuator electric dynamics and it can be seen as a kind of electric current error. Using σ we show that filtering of the velocity feedback term $\overline{K_D}\, \dot{q}$ succeeds to introduce suitable damping. This is done similarly as position filtering is shown to introduce suitable damping by Kelly et al. [1994].

Remark 4. Matrices K_m and R, i.e. r, have to be exactly known because of the feedforward compensation of gravity. Recall that $g(q_d)$ is a torque produced through an electric current which is generated by voltage applied to actuators. Thus, constants K_m and R must be known in order to compute the required voltage.

Now we present our second main result which is concerned with classical PID control. The main motivation for this controller is to avoid the requirement of an exact knowledge of both K_m and r. Our proposal is based on the version without actuator dynamics presented by Kelly et al. [2005] Ch. 9 and Meza et al. [2007].

Proposition 5. (Classical PID control) Consider plant (1), (2) together with controller:

$$u = -r_a\, i + \overline{K_P}\, \tilde{q} - \overline{K_D}\, \dot{q} + \overline{K_I}\int_0^t \tilde{q}(s)ds \tag{18}$$

There always exist $n \times n$, diagonal and positive definite matrices r_a, $\overline{K_P}$, $\overline{K_D}$, $\overline{K_I}$ such that the closed loop system (1), (2), (18) has an unique equilibrium point, where $\tilde{q} = 0$ and $\dot{q} = 0$, which is asymptotically stable.

Proof. Let $\underline{K_D}$ be an $n \times n$ arbitrary diagonal positive definite matrix. Let $\underline{K_P}$ and $\alpha > 0$ be a $n \times n$ diagonal positive definite matrix and a constant scalar, respectively, which have to satisfy conditions to be defined later. Define $R = r + r_a$ and choose:

$$\overline{K_P} = \underline{K_P} + LR^{-1}\overline{K_I} + \alpha K_m + \alpha L\underline{K_D} \tag{19}$$
$$\overline{K_D} = L\underline{K_D} + LR^{-1}\underline{K_P} \tag{20}$$

Using this, defining:

$$\rho = i - R^{-1}[\underline{K_P}\,\tilde{q} + \overline{K_I}\int_0^t \tilde{q}(s)ds]$$

and replacing (18) in (1) we obtain:

$$L\dot{\rho} = -R\rho - K_b\dot{q} - L\underline{K_D}\dot{q} + \alpha(K_m + L\underline{K_D})\tilde{q} \quad (21)$$

Due to the diagonal property of all the involved matrices we can write this as:

$$\rho_j = \frac{1}{p + \frac{R_j}{L_j}}\left(-\frac{K_{bj}}{L_j}\dot{q}_j + \frac{\alpha K_{mj}}{L_j}\,\tilde{q}_j\right) + \\ + \frac{1}{p + \frac{R_j}{L_j}}\left(-\underline{K_{Dj}}\,\dot{q}_j + \alpha\underline{K_{Dj}}\,\tilde{q}_j\right), \quad j = 1, \ldots, n$$

where subindex j represents the j-th diagonal entry if a matrix or the $j-$th component if a vector. Defining $\rho = \varrho + \sigma$, where:

$$\varrho_j = \frac{1}{p + \frac{R_j}{L_j}}\left(-\frac{K_{bj}}{L_j}\dot{q}_j + \frac{\alpha K_{mj}}{L_j}\,\tilde{q}_j\right)$$

$$\sigma_j = \frac{1}{p + \frac{R_j}{L_j}}\left(-\underline{K_{Dj}}\,\dot{q}_j + \alpha\underline{K_{Dj}}\,\tilde{q}_j\right), \quad j = 1, \ldots, n$$

we obtain:

$$L\dot{\varrho} = -R\varrho - K_b\,\dot{q} + \alpha K_m\tilde{q}$$
$$L\dot{\sigma} = -R\sigma - L\underline{K_D}\dot{q} + \alpha L\underline{K_D}\tilde{q}$$

These expressions are equivalent to (21). Using this we obtain the following closed loop system:

$$\frac{d}{dt}\begin{bmatrix}\tilde{q}\\ \dot{q}\\ \\ \\ z\\ \varrho\\ \sigma\end{bmatrix} = \begin{bmatrix} -\dot{q} \\ D(q)^{-1}[-C(q,\dot{q})\dot{q} - g(q) + \\ +K_m\varrho + K_m\sigma + K_P'\tilde{q} + K_I'z + \\ +g(q_d) - K_v\dot{q}] \\ \alpha\tilde{q} - \dot{q} \\ -L^{-1}R\varrho - L^{-1}K_b\,\dot{q} + \alpha L^{-1}K_m\,\tilde{q} \\ -L^{-1}R\sigma - \underline{K_D}\,\dot{q} + \alpha\underline{K_D}\,\tilde{q}\end{bmatrix} \quad (22)$$

where:

$$K_I' = \frac{K_I}{\alpha}$$
$$K_P' = K_P - \frac{K_I}{\alpha}$$
$$z = \int_0^t (\alpha\tilde{q}(s) - \dot{q}(s))ds - (K_I')^{-1}g(q_d)$$
$$K_P = K_m R^{-1}\underline{K_P}, \quad K_I = K_m R^{-1}\overline{K_I} \quad (23)$$

The closed loop system (22) has the unique equilibrium point $(\tilde{q}, \dot{q}, z, \varrho, \sigma) = (0, 0, 0, 0, 0)$. Consider the following scalar function:

$$V(\tilde{q}, \dot{q}, z) = V_1(\tilde{q}) + V_2(\tilde{q}, \dot{q}) + V_3(z)$$
$$V_1(\tilde{q}) = \frac{1}{2}(1-\beta)\tilde{q}^T K_P'\tilde{q} + U(q) - U(q_d) + \tilde{q}^T g(q_d)$$
$$V_2(\tilde{q}, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - \alpha\tilde{q}^T D(q)\dot{q} + \frac{\beta}{2}\tilde{q}^T K_P'\tilde{q}$$
$$V_3(z) = \frac{1}{2}z^T K_I' z$$

for some constant $0 < \beta < 1$. Recalling $K_P' = K_P - \frac{K_I}{\alpha}$ and proceeding as Tomei [1991b] and Kelly [1997] we find that V_1 is positive definite if:

$$\lambda_{min}(K_P) - \frac{\lambda_{max}(K_I)}{\alpha} > \frac{k_g}{1-\beta} \quad (24)$$

In order to prove that V_2 is positive definite we use the fact that this function can be lower bounded as:

$$V_2 \geq \frac{1}{2}\lambda_{min}(D(q))\|\dot{q}\|^2 + \frac{\beta}{2}\lambda_{min}(K_P')\|\tilde{q}\|^2 - \\ -\alpha\lambda_{max}(D(q))\|\tilde{q}\|\,\|\dot{q}\|$$

i.e.:

$$V_2 \geq \frac{\alpha}{2}\begin{bmatrix}\|\dot{q}\|\\ \|\tilde{q}\|\end{bmatrix}^T \begin{bmatrix}\frac{1}{\alpha}\lambda_{min}(D(q)) & -\lambda_{max}(D(q)) \\ -\lambda_{max}(D(q)) & \frac{\beta}{\alpha}\lambda_{min}(K_P')\end{bmatrix}\begin{bmatrix}\|\dot{q}\|\\ \|\tilde{q}\|\end{bmatrix}$$

Recalling $K_P' = K_P - \frac{K_I}{\alpha}$ we realize that matrix in last expression in positive definite if:

$$\lambda_{min}(K_P) - \frac{\lambda_{max}(K_I)}{\alpha} > \frac{\alpha^2}{\beta}\frac{\lambda_{max}^2(D(q))}{\lambda_{min}(D(q))} \quad (25)$$

i.e. V_2 is positive definite under this condition. Hence, V is positive definite and radially unbounded if (24), (25) are satisfied and K_I' is positive definite. Thus, we propose the following scalar function as Lyapunov function candidate:

$$W(\tilde{q}, \dot{q}, z, \varrho, \sigma) = V(\tilde{q}, \dot{q}, z) + \frac{1}{2}\,\varrho^T L\varrho + \frac{1}{2}\,\sigma^T K_m\underline{K_D}^{-1}\sigma$$

Straightforward computations including use of (3), (4) and $K_b = K_m$ show that the time derivative of W along the trajectories of (22) is given as:

$$\dot{W} = -\varrho^T R\varrho - \sigma^T K_m\underline{K_D}^{-1}L^{-1}R\sigma - \dot{q}^T K_v\dot{q} - \\ -\alpha\tilde{q}^T C(q,\dot{q})^T\dot{q} + \alpha\dot{q}^T D(q)\dot{q} - \alpha\tilde{q}^T(g(q_d) - g(q)) - \\ -\alpha\tilde{q}^T(K_P - K_I/\alpha)\tilde{q} + \alpha\tilde{q}^T K_v\dot{q}$$

Using (5), (6), (7), (8), (9), (10) we obtain:

$$-\alpha\tilde{q}^T(K_P - K_I/\alpha)\tilde{q} \leq -\alpha[\lambda_{min}(K_P) - \lambda_{max}(K_I)/\alpha]\|\tilde{q}\|^2$$
$$-\dot{q}^T K_v\dot{q} \leq -\lambda_{min}(K_v)\|\dot{q}\|^2$$
$$-\alpha\tilde{q}^T C(q,\dot{q})^T\dot{q} \leq \alpha k_c\|\tilde{q}\|\|\dot{q}\|^2$$
$$\alpha\dot{q}^T D(q)\dot{q} \leq \alpha\lambda_{max}(D(q))\|\dot{q}\|^2$$
$$-\alpha\tilde{q}^T(g(q_d) - g(q)) \leq \alpha k_g\|\tilde{q}\|^2$$
$$\alpha\tilde{q}^T K_v\dot{q} \leq \alpha\lambda_{max}(K_v)\|\dot{q}\|\|\tilde{q}\|$$

Thus, we can write:

$$\dot{W} \leq -\begin{bmatrix}\|\dot{q}\|\\ \|\tilde{q}\|\end{bmatrix}^T Q\begin{bmatrix}\|\dot{q}\|\\ \|\tilde{q}\|\end{bmatrix} - \varrho^T R\varrho - \\ -\sigma^T K_m\underline{K_D}^{-1}L^{-1}R\sigma$$

where:

$$Q = \alpha \begin{bmatrix} Q_{11} & -\frac{1}{2}\lambda_{max}(K_v) \\ -\frac{1}{2}\lambda_{max}(K_v) & Q_{22} \end{bmatrix}$$

$$Q_{11} = \frac{\lambda_{min}(K_v)}{\alpha} - [\lambda_{max}(D(q)) + k_c\|\tilde{q}\|]$$

$$Q_{22} = [\lambda_{min}(K_P) - k_g] - \frac{\lambda_{max}(K_I)}{\alpha}$$

Note that Q is positive definite if and only if:

$$Q_{11}Q_{22} > \frac{1}{4}\lambda_{max}^2(K_v) \qquad (26)$$
$$Q_{11} > 0, \quad Q_{22} > 0$$

From these conditions we obtain:

$$\frac{\lambda_{min}(K_v)}{\lambda_{max}(D(q)) + k_c\|\tilde{q}\|} > \alpha > \frac{\lambda_{max}(K_I)}{\lambda_{min}(K_P) - k_g} \qquad (27)$$

which implies that $\eta > \|\tilde{q}\|$ where:

$$\eta = \frac{1}{k_c}\left[\frac{\lambda_{min}(K_v)[\lambda_{min}(K_P) - k_g]}{\lambda_{max}(K_I)} - \lambda_{max}(D(q))\right] \qquad (28)$$

Note that Q can always be rendered positive definite as follows. Choose a (small) $\alpha > 0$ satisfying the left hand inequality in (27). This ensures that $Q_{11} > 0$. Using this value of α we can always satisfy the right hand inequality in (27), i.e. to ensure $Q_{22} > 0$, by choosing simultaneously a large K_P and a small K_I. From (23) we realize that K_P and K_I are adjusted through $\underline{K_P}$ and $\overline{K_I}$ which are used in (19), (20) to compute $\overline{K_P}$ and $\overline{K_D}$. Finally, such a selection also renders the product $Q_{11}Q_{22}$ arbitrarily large to satisfy (26). Further, (25) and (24) are also satisfied through such a selection. Hence, we conclude that Q is positive definite, i.e. $\dot{W} \leq 0$, in a ball centered at the origin of $\mathcal{R}^{5n}$ with radius η, given in (28). Moreover, we also conclude that W is a positive definite radially unbounded function. Thus, application of the LaSalle invariance principle ensures asymptotic stability of the equilibrium point $(\tilde{q}, \dot{q}, z, \varrho, \sigma) = (0, 0, 0, 0, 0)$. This completes the proof of proposition 5.

Remark 6. Conditions that matrices R, i.e. r_a, $\overline{K_P}$, $\overline{K_D}$ and $\overline{K_I}$ have to satisfy are summarized in (19), (20),(24), (25), (26), (27), (28) together with (23). Note that $\overline{K_D}$ is an arbitrary positive definite matrix whereas $\overline{K_P}$ is arbitrary as long as it is larger than a lower bound satisfying the fore mentioned conditions. Hence, (19), (20) can be satisfied without requiring the exact knowledge of any of R, L or K_m by choosing large enough matrices $\overline{K_P}$ and $\overline{K_D}$. Further, all of the fore mentioned conditions can be tested as follows. From (19), (20) we can write:

$$\underline{K_P} = (I - \alpha LR^{-1})^{-1}[\,\overline{K_P} - LR^{-1}\overline{K_I} - \alpha K_m - \alpha\overline{K_D}\,]$$

where I stands for the $n \times n$ identity matrix. Note that $(I - \alpha LR^{-1})^{-1}$ can always be rendered positive definite, no matter the value of L, by choosing a large R, i.e. a large r_a, whereas factor in brackets is ensured to be positive definite by choosing a large $\overline{K_P}$. This means that $\underline{K_P}$ is positive definite given the diagonal property of all the involved matrices. We can proceed as follows. Propose diagonal positive definite matrices $\overline{K_P}$, $\overline{K_D}$ and $\overline{K_I}$. Use bounds on the remaining matrices to compute a minimum value for $\underline{K_P}$ by using the previous expression. Using this value, the proposed $\overline{K_I}$, bounds on matrices R^{-1} and K_m and (23) we can obtain $\lambda_{min}(K_P)$ and $\lambda_{max}(K_I)$. With these values we can test conditions (24), (25), (26), (27), (28). If they are not satisfied repeat the procedure using new values for $\overline{K_P}$, $\overline{K_D}$ and $\overline{K_I}$. On the other hand, a value for α can be proposed just to verify that such constant exists satisfying the forementioned conditions. Thus, controller (18) can be tuned without requiring the exact knowledge of any parameter of neither the robot nor the actuators.

Remark 7. Although the previous proof is based on the ideas presented by Kelly et al. [2005] Ch. 9 and Meza et al. [2007] however an important modification is introduced in the present note to improve performance. We use term $\frac{\beta}{2}\tilde{q}^T K_P'\tilde{q}$ in $V_2(\tilde{q}, \dot{q})$ instead of term $\frac{\alpha}{2}\tilde{q}^T K_v\tilde{q}$ used in those works. The tuning procedure proposed in the cited works requires:

$$\lambda_{min}(K_v) > \frac{\lambda_{max}(K_I)}{\lambda_{min}(K_P) - k_g}\frac{\lambda_{max}^2(D(q))}{\lambda_{min}(D(q))} \qquad (29)$$

Being K_v only due to the viscous friction of robot in the present note, $\lambda_{min}(K_v)$ is a small positive constant. This together with factor $\frac{\lambda_{max}^2(D(q))}{\lambda_{min}(D(q))}$ in (29) impose either a very large K_P (i.e. $\overline{K_P}$) or a very small K_I (i.e. $\overline{K_I}$). Both of these possibilities degrade closed loop performance. A large $\overline{K_P}$ produces very large initial voltages to be applied to DC-motor actuators which result in very large peak torques applied to robot links. On the other hand a small $\overline{K_I}$ results in very large settling times. In the cited works (29) is imposed by the condition that $V_2(\tilde{q}, \dot{q})$ (in terms of $\frac{\alpha}{2}\tilde{q}^T K_v\tilde{q}$ in those works) has to satisfy in order to be a positive definite function. In the present note the counterpart of such a condition is given in (25) which does not involve K_v.

Remark 8. According to $\tau = K_m\, i$, the torque applied by brushed DC-motors to robot joints is proportional to current. This fact motivates the industrial practice of designing drives for these motors which include some current controllers ensuring the generation of the desired torque. This is known as torque control (see Parker Automation [1998], for instance). This means that voltage applied to motors is computed as:

$$u = \gamma(i^* - i) \qquad (30)$$

where γ is a $n \times n$ diagonal positive definite matrix and i^* represents the value of the electric current i necessary to generate the desired torque τ^*, i.e.:

$$i^* = K_m^{-1}\tau^* \qquad (31)$$

When a PD controller is used as the desired torque we have:

$$\tau^* = \kappa_p\,\tilde{q} - \kappa_d\,\dot{q} + g(q_d) \qquad (32)$$

Note that controller (11) is retrieved from (30), (31), (32) by setting:

$$r_a = \gamma$$
$$\overline{K_P} = \gamma K_m^{-1}\kappa_p$$
$$\overline{K_D} = \gamma K_m^{-1}\kappa_d$$

We stress that $r_a = \gamma$ is chosen to be large (see Chiasson [2005] pp. 76), i.e. $r_a \gg r$, which means that $R \approx r_a$, i.e.:

$$RK_m^{-1} \approx \gamma K_m^{-1}$$

On the other hand, when a PID controller is used as the desired torque we have:

$$\tau^* = \kappa_p\, \tilde{q} - \kappa_d\, \dot{q} + \kappa_i \int_0^t \tilde{q}(s) ds \tag{33}$$

Note that controller (18) is retrieved from (30), (31), (33) by setting:

$$\begin{aligned} r_a &= \gamma \\ \overline{K_P} &= \gamma K_m^{-1} \kappa_p \\ \overline{K_D} &= \gamma K_m^{-1} \kappa_d \\ \overline{K_I} &= \gamma K_m^{-1} \kappa_i \end{aligned}$$

Thus, an important contribution of our results is the presentation, for the first time, of a theoretical justification for torque control when used to control rigid robot manipulators which are actuated by brushed DC motors. We stress that such a strategy is a common practice in industrial applications.

4. CONCLUSIONS

We have presented two controllers for robot manipulators whose design takes into account the electric dynamics of the brushed DC-motors used as actuators. Contrary to the common assumption inductance is not required to be small. A PD controller with desired gravity compensation is proposed which needs knowledge of the armature circuits resistance and the torque constant because of feedforward compensation of gravity. It is clear that this requirement disappears in robots without any effect of gravity. The classical PID controller is proposed to deal with the effect of gravity. Although this controller does not require the exact knowledge of neither robot nor actuator parameters, however it relies on viscous friction present at the robot joints. In spite of this, nonzero PID gains always exist ensuring asymptotic stability of the desired equilibrium point. We have presented, for the first time, a formal justification for torque control which is a strategy commonly used in industrial practice to control brushed DC motors.

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